MATHEMATICS 271 L01 FALL 2003
ASSIGNMENT 3 SOLUTION

1. Prove the following statements by induction on n.
(a) $3^n + 1$ is divisible by 2 for all integers $n \geq 1$.
(b) $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 for all integers $n \geq 1$.
Solution:
(a) **Base case**: $(n = 1)$ When $n = 1$, $3^n + 1 = 3^1 + 1 = 4 = 2 \times 2$, so $3^n + 1$ is divisible by 2 when $n = 1$.
Induction Step: Let $k \geq 1$ be an integer and suppose that

$$3^k + 1 \text{ is divisible by 2.} \qquad \text{(IH)}$$

We want to show that $3^{k+1} + 1$ is divisible by 2.
Now, from (IH), we have $3^k + 1 = 2m$ for some integer m, and so
$3^{k+1} + 1 = 3 \times 3^k + 1 = 3 \times \left(3^k + 1\right) - 2 = 3 \times 2m - 2 = 2\,(3m - 1)$,
which implies that $3^{k+1} + 1$ is divisible by 2 (note that $3m - 1$ is an integer).
Thus, by the Principle of Mathematical Induction, we have proved that $3^n + 1$ is divisible by 2 for all integers $n \geq 1$.
(b) **Base case**: $(n = 1)$ When $n = 1$, $5^{n+1} + 2 \times 3^n + 1 = 5^2 + 2 \times 3^1 + 1 = 32 = 8 \times 4$, so $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 when $n = 1$.
Induction Step: Let $k \geq 1$ be an integer and suppose that

$$5^{k+1} + 2 \times 3^k + 1 \text{ is divisible by 8.} \qquad \text{(IH)}$$

We want to show that $5^{k+2} + 2 \times 3^{k+1} + 1$ is divisible by 8.
Now, from (IH), we have $5^{k+1} + 2 \times 3^k + 1 = 8s$ for some integer s, and also from (a) we have $3^k + 1$ is divisible by 2, so $3^k + 1 = 2t$ for some integer t. Thus,

$$\begin{aligned}
5^{k+2} + 2 \times 3^{k+1} + 1 &= 5 \times 5^{k+1} + 2 \times 3 \times 3^k + 1 \\
&= 5 \times 5^{k+1} + 6 \times 3^k + 1 \\
&= 5\left(5^{k+1} + 2 \times 3^k + 1\right) - 4 \times 3^k - 4 \\
&= 5\left(5^{k+1} + 2 \times 3^k + 1\right) - 4\left(3^k + 1\right) \\
&= 5 \times 8s - 4 \times 2t \\
&= 8\,(5s - t)\,,
\end{aligned}$$

where $5s - t$ is an integer, which means $5^{k+2} + 2 \times 3^{k+1} + 1$ is divisible by 8.
Thus, by the Principle of Mathematical Induction, we have proved that $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 for all integers $n \geq 1$.

2. For part (a) and (b), $n \geq k \geq 0$ are natural numbers. For part (c) and (d), $n \geq k \geq 1$ are natural numbers.
(a) Prove that $\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} = 2^n$ using the Binomial Theorem (Theorem 14.8).
(b) Prove that $\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} = 2^n$ by a combinatorial proof.
(c) Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(d) Prove that $1\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+...+n\binom{n}{n}=n2^{n-1}$.

Solution:

(a) From the Binomial Theorem, we have

$$\begin{aligned} 2^n=(1+1)^n &= \sum_{i=0}^{n}\binom{n}{i}1^{n-i}1^i \\ &= \binom{n}{0}+\binom{n}{1}+...+\binom{n}{n} \end{aligned}$$

(b) Let A be a set of n elements. We count the number of subsets of A. First, as done in the text, the number of subsets of A is 2^n because there are 2 choices (choose or not to choose the element for the subset) for each of the n elements of A. Secondly, the number of subsets of A is $k_0+k_1+k_3+...+k_n$ where k_i is the number of subsets of A of size i for $i=0,1,2,...,n$. It is clear that $k_i=\binom{n}{i}$ for $i=0,1,2,...,n$. Thus, the number of subsets of A is $\binom{n}{0}+\binom{n}{1}+...+\binom{n}{n}$ and so $\binom{n}{0}+\binom{n}{1}+...+\binom{n}{n}=2^n$.

(c) We show that $k\binom{n}{k}=n\binom{n-1}{k-1}$ by a combinatorial proof. We count the number of ways of choosing a team of k members one of which is the captain from a group of n people. First, this number is $k\binom{n}{k}$ because we have $\binom{n}{k}$ ways of choosing a team of k members, and then we have k ways of choosing the captain from the chosen k members. However, if we choose a captain first (there are n choises from n people), and then we choose $k-1$ members from the remaining $n-1$ people to make the team (there are $\binom{n-1}{k-1}$ of doing this). Thus, the number of ways of choosing a team of k members one of which is the captain from a group of n people is also $n\binom{n-1}{k-1}$. Thus, $k\binom{n}{k}=n\binom{n-1}{k-1}$.

Another way to prove this is to do it algebraically.

$$\begin{aligned} k\binom{n}{k} &= k\frac{n!}{k!\,(n-k)!} &&= \frac{kn\,(n-1)!}{k\,(k-1)!\,(n-k)!} \\ &= \frac{n\,(n-1)!}{(k-1)!\,(n-k)!} &&= n\frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} \\ &= n\binom{n-1}{k-1} \end{aligned}$$

(d) Using the identity $k\binom{n}{k}=n\binom{n-1}{k-1}$ from (c), we have

$$\begin{aligned} 1\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+...+n\binom{n}{n} &= n\binom{n-1}{0}+n\binom{n-1}{1}+n\binom{n-1}{2}+...+n\binom{n-1}{n-1} \\ &= n\left[\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+...+\binom{n}{n}\right] \\ &= n2^{n-1} \qquad \text{by (a)} \end{aligned}$$

3. For positive integers n, let S_n be the number of ways to display flags on an n foot tall flagpole using red flags (which are 1 foot tall), blue flags (which are 2 feet tall) and green flags (which are 2 feet tall).

(a) Find S_1 and S_2. Explain how you get the answers.

(b) Show that for $n\geq 3$, $S_n=2S_{n-2}+S_{n-1}$.

(c) Prove by induction on n that $S_n=\frac{2}{3}2^n+\frac{1}{3}(-1)^n$.

Solution:

(a) $S_1=1$ because there is only one way to display flags on a one-foot pole (using a red flag).

$S_2=3$ because there are three ways to display flags on a one-foot pole (using two red flags or one blue flag or one green flag).

(b) For integers $n \geq 3$, we count the number of ways to display the flags on an n-foot tall pole as follow. We consider the top flag. Either it is red or it is not.

Case 1: The top flag is red. We first display the flags on the first $n-1$ feet (this can be done in S_{n-1} ways) and then we add the red flag on top (this can be done in 1 ways). Thus, there are S_{n-1} ways to display the flags on an n-foot tall pole so that the top flag is red.

Case 2: The top flag is not red. We first display the flags on the first $n-2$ feet (this can be done in S_{n-2} ways) and then we add a two foot flag on top (this can be done in 2 ways, either a blue flag or a green flag). Thus, there are $2S_{n-2}$ ways to display the flags on an n-foot tall pole so that the top flag is not red.

Thus, the number of ways to display the flags on an n-foot tall pole is $S_n = 2S_{n-2}+S_{n-1}$.

(c) **Base cases:** $(n = 1, 2)$

$S_1 = 1 = \frac{4}{3} - \frac{1}{3} = \frac{2}{3}2^1 + \frac{1}{3}(-1) = \frac{2}{3}2^1 + \frac{1}{3}(-1)^1$ and

$S_2 = 3 = \frac{8}{3} + \frac{1}{3} = \frac{2}{3}4 + \frac{1}{3}(-1)^2 = \frac{2}{3}2^2 + \frac{1}{3}(-1)^2$.

Thus, the equality $S_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n$ holds when $n = 1, 2$.

Induction Step: Let $k > 2$ be an integer and suppose that

$S_m = \frac{2}{3}2^m + \frac{1}{3}(-1)^m$ for all integer m, where $1 \leq m < k$. (IH)

We want to show that $S_k = \frac{2}{3}2^k + \frac{1}{3}(-1)^k$.

Since $k > 2$, we have $1 \leq k-1 < k$ and $1 \leq k-2 < k$, so from (IH) we have

$S_{k-2} = \frac{2}{3}2^{k-2} + \frac{1}{3}(-1)^{k-2}$ and $S_{k-1} = \frac{2}{3}2^{k-1} + \frac{1}{3}(-1)^{k-1}$ $(\star)$

Since $k > 2$, from (b) we have

$$\begin{aligned}
S_k &= 2S_{k-2} + S_{k-1} \\
&= 2\left[\tfrac{2}{3}2^{k-2} + \tfrac{1}{3}(-1)^{k-2}\right] + \left[\tfrac{2}{3}2^{k-1} + \tfrac{1}{3}(-1)^{k-1}\right] && \text{by } (\star) \\
&= \tfrac{2^2}{3}2^{k-2} + \tfrac{2}{3}(-1)^{k-2} + \tfrac{2}{3}2^{k-1} + \tfrac{1}{3}(-1)^{k-1} \\
&= \tfrac{1}{3}2^k + \tfrac{2}{3}(-1)^{k-2}(-1)^2 + \tfrac{1}{3}2^k + \tfrac{1}{3}(-1)^k(-1)^{-1} \\
&= \tfrac{1}{3}2^k + \tfrac{1}{3}2^k + \tfrac{2}{3}(-1)^{k-2}(-1)^2 + \tfrac{1}{3}(-1)^k(-1)^{-1} \\
&= \tfrac{2}{3}2^k + \tfrac{2}{3}(-1)^k - \tfrac{1}{3}(-1)^k \\
&= \tfrac{2}{3}2^k + \tfrac{1}{3}(-1)^k && \text{as required.}
\end{aligned}$$

Thus, by the Principle of Mathematical Induction, we have proved that $S_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n$ for all integers $n \geq 1$.