MATHEMATICS 271 L01 FALL 2003
ASSIGNMENT 1 SOLUTION

1. In this question, prove your answers using **only** the definitions of rational and irrational numbers and the fact that $\sqrt{2}$ is irrational. Let $\mathcal{P}$ be the statement : "If $x+y$ is rational and $x-y$ is irrational then x is irrational and y is irrational."

(a) Is $\mathcal{P}$ true for all real numbers x and y? Prove your answer.

(b) State the *converse* of $\mathcal{P}$. Is the *converse* of $\mathcal{P}$ true for all real numbers x and y? Prove your answer.

(c) State the the *contrapositive* of $\mathcal{P}$. Is the *contrapositive* of $\mathcal{P}$ true for all real numbers x and y? Explain.

(d) State the the *negation* of $\mathcal{P}$. Is the *negation* of $\mathcal{P}$ true for some real numbers x and y? Explain.

Solution:

(a) $\mathcal{P}$ is true for all real numbers x and y.

Proof: Let $x, y \in \mathbb{R}$ so that $x+y$ is rational and $x-y$ is irrational. Since $x+y$ is rational, there are $m, n \in \mathbb{Z}$ so that $x+y=\frac{m}{n}$ and $n \neq 0$.

First, we show that x is irrational by a contradiction proof. Suppose that x is rational. Then $x=\frac{s}{t}$ for some $s, t \in \mathbb{Z}$ where $t \neq 0$. It follows that, $x-y=2x-(x+y)=2\frac{s}{t}-\frac{m}{n}=\frac{2sn-mt}{nt}$ which implies that $x-y$ is rational (note that $2sn-mt$ and nt are integers because $2, m, n, s, t \in \mathbb{Z}$, and $nt \neq 0$ because $n \neq 0$ and $t \neq 0$). This contradicts the assumption that $x-y$ is irrational. Thus, x is irrational.

Similarly, we show that y is irrational by a contradiction proof. Suppose that y is rational. Then $x=\frac{p}{q}$ for some $p, q \in \mathbb{Z}$ where $q \neq 0$. It follows that, $x-y=(x+y)-2y=\frac{m}{n}-2\frac{p}{q}=\frac{mq-2pn}{nq}$ which implies that $x-y$ is rational (note that $mq-2pn$ and nq are integers because $2, m, n, p, q \in \mathbb{Z}$, and $nq \neq 0$ because $n \neq 0$ and $q \neq 0$). This contradicts the assumption that $x-y$ is irrational. Thus, y is irrational.

(b) The *converse* of $\mathcal{P}$ is: "If x is irrational and y is irrational then $x+y$ is rational and $x-y$ is irrational."

The *converse* of $\mathcal{P}$ is false for some real numbers x and y. For example, when $x=y=\sqrt{2}$, we see that x is irrational and y is irrational. However, $x-y=\sqrt{2}-\sqrt{2}=0$, which is not rational.

(c) The *contrapositive* of $\mathcal{P}$ is: "If x is rational or y is rational then $x+y$ is irrational or $x-y$ is rational."

The *contrapositive* of $\mathcal{P}$ is true for all real numbers x and y, because it is logically equivalent to $\mathcal{P}$ which is true all real numbers x and y as proven in (a).

(d) The *negation* of $\mathcal{P}$ is: " $x+y$ is rational and $x-y$ is irrational, but x is rational or y is rational."

The *negation* of $\mathcal{P}$ false for all real numbers x and y. This is because $\mathcal{P}$ is true all real numbers x and y as proven in (a).

2. In this question, a, b and c are integers. Let $\mathcal{P}$ be the statement : "if $a \mid b$ and $a \mid c$ then $a \mid 2b + c$ and $a \mid b + 2c$." and let $\mathcal{Q}$ be the statement : "if $a \mid b$ and $a \mid c$ then $a \mid 2b + c$ and $a \mid 3b + 2c$.".

(a) Is $\mathcal{P}$ true? Prove your answer.

(b) State the *converse* of $\mathcal{P}$. Is the *converse* of $\mathcal{P}$ true? Prove your answer.

(c) Is $\mathcal{Q}$ true? Prove your answer.

(d) State the *converse* of $\mathcal{Q}$. Is the *converse* of $\mathcal{Q}$ true? Prove your answer.

Solution:

(a) $\mathcal{P}$ is true and here is a proof.

Let a, b and c be integers so that $a \mid b$ and $a \mid c$. Since $a \mid b$ and $a \mid c$, there are $x, y \in \mathbb{Z}$ so that $b = ax$ and $c = ay$. Thus, $2b + c = 2ax + ay = a(2x + y)$ and $b + 2c = ax + 2ay = a(x + 2y)$, which imply $a \mid 2b + c$ and $a \mid b + 2c$ (We note that $2x + y$ and $x + 2y$ are integers because $2, x, y \in \mathbb{Z}$).

(b) The *converse* of $\mathcal{P}$ is: "if $a \mid 2b + c$ and $a \mid b + 2c$ then $a \mid b$ and $a \mid c$.".

The *converse* of $\mathcal{P}$ not true. For example, when $a = 3$ and $b = c = 1$, we have $2b + c = 3 = 3 \times 1 = a \times 1$ and $b + 2c = 3 = 3 \times 1 = a \times 1$, which mean $a \mid 2b + c$ and $a \mid b + 2c$. However, $3 \nmid 1$, that is $a \nmid b$.

(c) $\mathcal{Q}$ is true and here is a proof.

Let a, b and c be integers so that $a \mid b$ and $a \mid c$. Since $a \mid b$ and $a \mid c$, there are $x, y \in \mathbb{Z}$ so that $b = ax$ and $c = ay$. Thus, $2b + c = 2ax + ay = a(2x + y)$ and $3b + 2c = 3ax + 2ay = a(3x + 2y)$, which imply $a \mid 2b + c$ and $a \mid 3b + 2c$ (We note that $2x + y$ and $3x + 2y$ are integers because $2, 3, x, y \in \mathbb{Z}$).

(d) The *converse* of $\mathcal{Q}$ is: "if $a \mid 2b + c$ and $a \mid 3b + 2c$ then $a \mid b$ and $a \mid c$.".

The *converse* of $\mathcal{Q}$ is true and here is a proof.

Let a, b and c be integers so that $a \mid 2b+c$ and $a \mid 3b+2c$. Since $a \mid 2b+c$ and $a \mid 3b+2c$, there are $m, n \in \mathbb{Z}$ so that $2b + c = am$ and $3b + 2c = an$. Now, $b = 2(2b + c) - 3b + 2c = 2am - an = a(2m - n)$ and $c = 2(3b + 2c) - 3(2b + c) = 2an - 3am = a(2n - 3m)$, which imply $a \mid b$ and $a \mid c$ (We note that $2m - n$ and $2n - 3m$ are integers because $2, 3, m, n \in \mathbb{Z}$).

3. For each of the following statements, determine whether the statement is true or false and **prove your answer**.

(a) For all integers y, there is an integer x so that $x^3 + x = y$.

(b) For all integers x and y, if $2x^2 + x = 2y^2 + y$ then $x = y$.

(c) For all integers m and n, if $m \mid n$ and $n \mid m$ then $n = m$.

(d) For all natural numbers m and n, if $m \mid n$ and $n \mid m$ then $n = m$.

Solution:

(a) This statement is false when $y = 1$. Let $x \in \mathbb{Z}$. We show $x^3 + x \neq 1$ by a contradiction proof. Suppose that $x^3 + x = 1$, that is, $x(x^2 + 1) = 1$ which implies that $x = x^2 + 1 = 1$ or $x = x^2 + 1 = -1$ (this is because x and $x^2 + 1$ are integers). However, these are impossible because when $x = 1$, we have $x^2 + 1 = 2 \neq 1$, and $x^2 + 1$ is positive and so $x^2 + 1 \neq -1$. Thus, $x^3 + x \neq 1$.

(b) This statement is true and here is a proof.

Let $x, y \in \mathbb{Z}$ and suppose that $2x^2 + x = 2y^2 + y$, which implies $2x^2 + x - (2y^2 + y) = 0$. Equivalently, we have $2(x^2 - y^2) + x - y = 0$ and so we get,

$$(x - y)[2(x + y) + 1] = 0 \qquad (1)$$

Note that $2(x + y) + 1$ is an odd number because $x + y$ is an integer, so $2(x + y) + 1 \neq 0$ and from (1), we get that $x - y = 0$ which implies that $x = y$.

(c) This statement is false because $1, -1 \in \mathbb{Z}$, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$.

(d) This statement is true and here is a proof.

Let $m, n \in \mathbb{N}$ and suppose that $m \mid n$ and $n \mid m$. Since $m \mid n$ and $n \mid m$, there are $p, q \in \mathbb{Z}$ so that $n = mp$ and $m = nq$. Thus, $n = mp = (nq)p = n(pq)$ so we have

$$n = n(pq) \qquad (2)$$

We consider two cases:

Case 1: $n = 0$

Then $m = nq = 0 \times q = 0 = n$. Thus $m = n$.

Case 2: $n \neq 0$

In this case we have $n > 0$ because n is a natural number. By dividing both sides of (2), we have $pq = 1$ and so either $p = q = 1$ or $p = q = -1$ (for p and q are integers). We note that the second case can not happen because if $p = q = -1$, we have $m = nq = n(-1) = -n < 0$ (because $n > 0$). This contradicts the assumtion that m is a natural number (natural numbers are non-negative). Thus, the first case must happen, that is, $p = q = 1$ and so $n = mp = m \times 1 = m$.