MATHEMATICS 271 L01 FALL 2003
ASSIGNMENT 2 SOLUTION

1. Prove or disprove each of the following statements.
(a) For all sets A and B, $2^A \cup 2^B \subseteq 2^{A \cup B}$.
(b) For all sets A and B, $2^{A \cup B} \subseteq 2^A \cup 2^B$.
(c) For all sets A and B, $2^A - 2^B \subseteq 2^{A-B}$.
(d) For all sets A and B, $2^{A-B} \subseteq 2^A - 2^B$.

Solution:
(a) This statement is true and here is a proof. Let A and B be sets. Suppose that $X \in 2^A \cup 2^B$; that is, $X \in 2^A$ or $X \in 2^B$.
Case 1: $X \in 2^A$. Then $X \subseteq A$ and so $X \subseteq A \cup B$ which means $X \in 2^{A \cup B}$.
Case 1: $X \in 2^B$. Then $X \subseteq B$ and so $X \subseteq A \cup B$ which means $X \in 2^{A \cup B}$.
Thus, $2^A \cup 2^B \subseteq 2^{A \cup B}$.

(b) This statement is false. For example, when $A = \{1\}$ and $B = \{2\}$, we have $\{1, 2\} = A \cup B \in 2^{A \cup B}$, but $\{1, 2\} \notin 2^A$ and $\{1, 2\} \notin 2^B$ and so $2^{A \cup B} \nsubseteq 2^A \cup 2^B$.

(c) This statement is false. For example, when $A = \{1, 2\}$ and $B = \{2\}$, we have $\{1, 2\} = A \in 2^A$, but $\{1, 2\} \notin 2^B$ because $\{1, 2\} \nsubseteq \{2\} = B$. Thus, $\{1, 2\} \in 2^A - 2^B$. However, $\{1, 2\} \notin 2^{A-B}$ because $\{1, 2\} \nsubseteq \{1\} = A - B$. Therefore, $2^A - 2^B \nsubseteq 2^{A-B}$ in this case.

(d) This statement is false. In fact, for any sets A and B, we see that $\emptyset \in 2^{A-B}$ and $\emptyset \notin 2^A - 2^B$ (because $\emptyset \subseteq A - B$ and $\emptyset \in 2^B$). Thus, $2^{A-B} \nsubseteq 2^A - 2^B$.

2. Let $\mathcal{R}$ be a relation on $\mathbb{N} \times \mathbb{N}$ defined by: "for all $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$, $(a, b) \mathcal{R} (c, d)$ if and only if $ab = cd$."
(a) Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.
(b) Is there an equivalence class of $(\mathbb{N} \times \mathbb{N}, \mathcal{R})$ that has exactly one element?
(c) Is there an equivalence class of $(\mathbb{N} \times \mathbb{N}, \mathcal{R})$ that has exactly two elements?
(d) List all elements of the equivalence class $[(1, 8)]$ have?
(e) List all elements of the equivalence class $[(1, 81)]$ have?
(f) Is there an equivalence class of $(\mathbb{N} \times \mathbb{N}, \mathcal{R})$ that has exactly 271 elements?

Solution:
(a) We need to prove that $\mathcal{R}$ is reflexive, symmetric and transitive.
$\mathcal{R}$ **is reflexive**: Let $(a, b) \in \mathbb{N} \times \mathbb{N}$. Since $ab = ab$, we have $(a, b) \mathcal{R} (a, b)$.
$\mathcal{R}$ **is symmetric**: Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$ and suppose that $(a, b) \mathcal{R} (c, d)$, that is, $ab = cd$ and hence $cd = ab$ and so $(c, d) \mathcal{R} (a, b)$.
$\mathcal{R}$ **is transitive**: Let $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$ and suppose that $(a, b) \mathcal{R} (c, d)$ and $(c, d) \mathcal{R} (e, f)$,that is, $ab = cd$ and $cd = ef$, hence $ab = ef$ and so $(a, b) \mathcal{R} (e, f)$.
Since $\mathcal{R}$ is reflexive, symmetric and transitive on $\mathbb{N} \times \mathbb{N}$,$\mathcal{R}$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

(b) Yes, there is an equivalence class of $(\mathbb{N} \times \mathbb{N}, \mathcal{R})$ that has exactly one element. For example, $[(1, 1)] = \{(1, 1)\}$ which has exactly one element.

(c) Yes, there is an equivalence class of $(\mathbb{N}\times\mathbb{N},\mathcal{R})$ that has exactly two element. For example, $[(1,2)]=\{(1,2),(2,1)\}$ which has exactly two element.

(d) The elements of $[(1,8)]$ are $(1,8),(2,4),(4,2),(8,1)$.

(e) The elements of $[(1,81)]$ are $(1,81),(3,27),(9,9),(27,3),(81,1)$.

(f) Yes, there an equivalence class of $(\mathbb{N}\times\mathbb{N},\mathcal{R})$ that has exactly 271 elements. For example, the equivalence class $[(1,2^{270})]=\{(1,2^{270}),(2,2^{269}),(2^2,2^{268}),...,(2^{270},1)\}$ has exactly 271 elements.

3. Let $A=\{x\in\mathbb{N}:x\leq 1000\}$Let $\mathcal{S}$ be a relation on $2^A-\{\emptyset\}$ defined by: "for all $X,Y\in 2^A$, $(X,Y)\in\mathcal{S}$ if and only if the maximum element of X equals the maximum element of Y."

(a) Prove that $\mathcal{S}$ is an equivalence relation on $2^A-\{\emptyset\}$.
(b) Is there an equivalence class of $\left(2^A-\{\emptyset\},\mathcal{S}\right)$ that has exactly one element?
(c) Is there an equivalence class of $\left(2^A-\{\emptyset\},\mathcal{S}\right)$ that has exactly two elements?
(d) List all elements of the equivalence class $[\{3\}]$.
(e) How many elements does the equivalence class $[\{271\}]$ have? Explain.
(f) Is there an equivalence class of $\left(2^A-\{\emptyset\},\mathcal{S}\right)$ that has exactly 271 elements?

Solution:

(a) We need to prove that $\mathcal{S}$ is reflexive, symmetric and transitive.

$\mathcal{S}$ **is reflexive**: Let $X\in 2^A-\{\emptyset\}$. Since the maximum element of X equals the maximum element of X, we have $X\mathcal{S}X$.

$\mathcal{S}$ **is symmetric**: Let $X,Y\in 2^A-\{\emptyset\}$ and suppose that $X\mathcal{S}Y$, that is, the maximum element of X equals the maximum element of Y, and so the maximum element of Y equals the maximum element of X,and hence so $Y\mathcal{S}X$.

$\mathcal{S}$ **is transitive**: Let $X,Y,Z\in 2^A-\{\emptyset\}$ and suppose that $X\mathcal{S}Y$ and $Y\mathcal{S}Z$,that is, the maximum element of X equals the maximum element of Y, and the maximum element of Y equals the maximum element of Z,and hence the maximum element of X equals the maximum element of Z, which means $X\mathcal{S}Z$.

Since $\mathcal{S}$ is reflexive, symmetric and transitive on $2^A-\{\emptyset\}$,$\mathcal{S}$ is an equivalence relation on $2^A-\{\emptyset\}$.

(b) Yes, there an equivalence class of $\left(2^A-\{\emptyset\},\mathcal{S}\right)$ that has exactly one element. For instance, $[\{0\}]=\{\{0\}\}$ which has exactly one element.

(c) Yes, there an equivalence class of $\left(2^A-\{\emptyset\},\mathcal{S}\right)$ that has exactly two elements. For instance, $[\{1\}]=\{\{0,1\},\{1\}\}$ which has exactly two elements.

(d) The elements of the equivalence class $[\{3\}]$ are:

$\{3\},\{0,3\},\{1,3\},\{2,3\},\{0,1,3\},\{0,2,3\},\{1,2,3\},\{0,1,2,3\}$.

(e) The equivalence class $[\{271\}]$ has 2^{271} elements. This is because $X\mathcal{S}\{271\}$ if and only if $X=\{271\}\cup Y$ where Y is a subset of $\{0,1,2,3,...,270\}$. Thus the number of such X is the number of subsets of $\{0,1,2,3,...,270\}$, which is 2^{271}.

(f) No, no equivalence class of $\left(2^A-\{\emptyset\},\mathcal{S}\right)$ has exactly 271 elements. This is because given any equivalence class, its elements must have the same maximum elements, say M, where M is a natural number. A similar argument as in (e) tells us that the number of elements of this equivalence class is 2^M, which is a power of 2, and so it can not be 271.