The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #4

1. Find an equation of the tangent line to

$$\sqrt{x^2 - y} = \frac{9x}{y} - 1$$

at the point P $(5, 9)$.

2. Find a general antiderivative of $f(x) = \dfrac{5\sqrt{x} - 6x^3 - 8x^2 + 3}{x^2}$ for $x > 0$.

3. Solve $y'' = 2\sin(\pi - 2x)$ with $y'(\pi) = 0$ and $y(\pi) = 3$.

4. Solve for x : $\dfrac{1}{2^{x+1}} = \dfrac{5}{4^x}$.

5. Find y' in terms of x and y if $2x + 3y = \dfrac{y^2}{x+1}$.

6. Find a general antiderivative of $f(x) = \dfrac{1}{\cos^2(3x-1)}$ in the domain (find the domain).

7. Solve $y'' = \dfrac{3}{\sqrt{x}} - 6x$, $y'(4) = 2, y(4) = 0$.

8. Solve for x:

   (a) $\frac{1}{2}\ln(x+3) + 1 = 0$
   (b) $3^{x^2} = 9^{x-3}$.

9. Find an equation of the tangent line at the point $(6, \pi)$ to

$$2\cos\frac{y}{x} + \frac{xy}{6} = \sqrt{3} + \pi.$$

10. Solve (i.e. find y including an interval)

$$y' = \frac{1}{(5-x)^3}$$

with $y(4) = 1$

11. Solve for x: $\log_4(x+4) - 2\log_4(x+1) = \frac{1}{2}$.

12. Find $\displaystyle\int \left(3\sqrt{x} - \frac{1}{3x}\right)^2 dx$ for $x > 0$.

SOLUTIONS

For 1)

Use implicit differentiation and Chain Rule on the left ,Quotient Rule on right:

$\frac{1}{2}\left(x^2-y\right)^{-\frac{1}{2}}\cdot\left(x^2-y\right)'=9\cdot\left(\dfrac{x}{y}\right)'$

$\frac{1}{2}\left(x^2-y\right)^{-\frac{1}{2}}\cdot\left(2x-y'\right)=9\cdot\dfrac{1\cdot y-x\cdot y'}{y^2}$

now,$x=5, y=9, y'=m$

$\frac{1}{2}\left(25-9\right)^{-\frac{1}{2}}\cdot\left(10-m\right)=9\cdot\dfrac{9-5m}{9^2}$ so $\frac{1}{8}\cdot(10-m)=\frac{1}{9}\cdot(9-5m)$

multiply by $9\cdot 8$

$90-9m=72-40m$ thus $31m=-18$ and $m=-\frac{18}{31}$and an equation is

$y=-\frac{18}{31}\left(x-5\right)+9.$

For 2)

$\int f(x)dx=5\int\frac{\sqrt{x}}{x^2}dx-6\int\dfrac{x^3}{x^2}dx-8\int\dfrac{x^2}{x^2}dx+3\int x^{-2}dx=$

$5\int x^{-\frac{3}{2}}dx-6\int xdx-8\displaystyle\int dx+3\cdot\frac{x^{-1}}{-1}+c=5\cdot(-2)\,x^{-\frac{1}{2}}-6\cdot\frac{x^2}{2}-8x-\frac{3}{x}+c$

$=-\dfrac{10}{\sqrt{x}}-3x^2-8x-\frac{3}{x}+c$ for $x>0$.

For 3)

$y'=\int y''dx=2\int\sin\left(\pi-2x\right)dx=2\cdot\frac{-\cos(\pi-2x)}{-2}+c_1=\cos\left(\pi-2x\right)+c_1$

now use the condition $y'=0$ for $x=\pi$

$0=\cos\left(-\pi\right)+c_1=-1+c_1$ so $c_1=1$ and $y'=\cos\left(\pi-2x\right)+1$

$y=\int y'dx=\int\cos\left(\pi-2x\right)dx+\int 1dx+c_2=\frac{\sin(\pi-2x)}{-2}+x+c_2=-\frac{1}{2}\sin\left(\pi-2x\right)+x+c_2$

use the second condition $y=3$ for $x=\pi$

$3=-\frac{1}{2}\sin\left(-\pi\right)+\pi+c_2=\pi+c_2$ so $c_2=3-\pi$ and

the solution is $y=-\frac{1}{2}\sin\left(\pi-2x\right)+x+3-\pi$

For 4)

cross multiply first,so $4^x=5\cdot 2^{x+1}$, then apply ln to both sides

$\ln 4^x=\ln\left(5\cdot 2^{x+1}\right)=\ln 5+\ln 2^{x+1}$

thus $x\ln 4=\ln 5+(x+1)\ln 2$,and $x\ln 4-x\ln 2=\ln 5+\ln 2$

So $x(\ln 4-\ln 2)=\ln\left(5\cdot 2\right)$ and finally $x\ln\frac{4}{2}=\ln 10, x=\dfrac{\ln 10}{\ln 2}$

For 5)

use implicit differentiation,Quotient and Chain Rules:

$2+3y'=\dfrac{2yy'\left(x+1\right)-y^2}{(x+1)^2}$ multiply both side by $\left(x+1\right)^2$

$2(x+1)^2+3y'\left(x+1\right)^2=2yy'(x+1)-y^2$ all terms with y'

$y'\left[3\left(x+1\right)^2-2y\left(x+1\right)\right]=-y^2-2(x+1)^2$

so

$$y'=\frac{-y^2-2(x+1)^2}{3\left(x+1\right)^2-2y\left(x+1\right)}\text{ if the denominator is not 0.}$$

OR we can first multiply by $(x+1)$ to get

$2x^2+2x+3xy+3y=y^2$ then differemtiate

$4x+2+3y+3xy'+3y'=2yy'$

$4x+2+3y=y'(2y-3x-3)$

then $\quad y' = \dfrac{4x+2+3y}{2y-3x-3}$ if the denominator is not 0.

For 6)

$\displaystyle\int \frac{1}{\cos^2(3x-1)}dx = \frac{1}{3}\tan(3x-1)+c$

since $(\tan x)' = \sec^2 x = \dfrac{1}{\cos^2 x}$

if $3x-1 \neq \frac{\pi}{2}+k\pi$ so $x \neq \frac{1}{3}+\frac{\pi}{6}+k\frac{\pi}{3}, k = 0, \pm1, \pm2, \pm3, ...$

For 7)

$y'' = \dfrac{3}{\sqrt{x}} - 6x,\ y'(4) = 2, y(4) = 0.$

$y' = \displaystyle\int y''dx = 3\int x^{-\frac{1}{2}}dx - 6\int xdx + c_1 = 6\sqrt{x} - 3x^2 + c_1$

now $x = 4, y' = 2$

$2 = 6\cdot 2 - 3\cdot 16 + c_1 \qquad c_1 = 38$

again

$y = \displaystyle\int y'dx = 6\int \sqrt{x}dx - 3\int x^2dx + 38\int dx = 4x^{\frac{3}{2}} - x^3 + 38x + c_2$

now $x = 4, y = 0$

$0 = 4\cdot 2^3 - 4^3 + 38\cdot 4 + c_2 \qquad c_2 = -32+64-152 = -120$

thus the solution of the given problem is

$y = 4x^{\frac{3}{2}} - x^3 + 38x - 120$ for any x.

For 8a)

$\frac{1}{2}\ln(x+3) = -1, \ln(x+3) = -2$ then exp.f. to both sides and $(x+3) = e^{-2}$ and so $x = e^{-2} - 3$.

b)

Take log of both sides: $x^2\ln 3 = (x-3)\ln 9 = (x-3)\cdot 2\ln 3$, cancel $\ln 3$ and $x^2 = 2x-6$, everything on one side :$x^2 - 2x + 6 = 0$, discriminant of this polynomial is $D = (-2)^2 - 4\cdot 1\cdot 6 = -20$ so no real roots exist and the problem has NO solution. Also we can change both sides to the same base: $3^{x^2} = (3^2)^{x-3} = 3^{2x-6}$ and by comparing the exponents we get the same quadratic polynomial.

For 9)

Use implicit differentiation:

$2\left(-\sin\frac{y}{x}\right)\left(\dfrac{y}{x}\right)' + \frac{1}{6}(x\cdot y)' = 0$

$-2\sin\frac{y}{x}\cdot\dfrac{y'\cdot x - y\cdot 1}{x^2} + \frac{1}{6}(1\cdot y + x\cdot y') = 0$

Now $x = 6, y = \pi$,and $y' = m$:

$-2\cdot\sin\frac{\pi}{6}\cdot\dfrac{6m-\pi}{36} + \frac{1}{6}\cdot(\pi + 6m) = 0$, multiply both sides by 36 and use $\sin\frac{\pi}{6} = \frac{1}{2}$

thus

$-(6m-\pi) + 6(\pi+6m) = 0$ and the equation is now:$-6m+\pi+6\pi+36m = 0$,thus $30m = -7\pi$ and $m = -\frac{7\pi}{30}$.The equation of the tangent line is :

$$y = -\frac{7\pi}{30}(x-6)+\pi$$

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For 10)

For $y \neq 5$ $\quad y = \int y' dx = \int (5-x)^{-3}\, dx = \dfrac{(5-x)^{-2}}{(-2)\cdot(-1)} + c$

using $\int (ax+b)^r\, dx = \dfrac{(ax+b)^{r+1}}{a(r+1)} + c$

now if $x = 4, y = 1$ solve for c: $\quad 1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$.

Together the solution is $y = \frac{1}{2}(5-x)^{-2} + \frac{1}{2}$ for $x \in (-\infty, 5)$

For 11)

Simplify the left-hand side

$\log_4(x+4) - 2\log_4(x+1) = \log_4(x+4) - \log_4(x+1)^2 =$

$= \log_4 \frac{x+4}{(x+1)^2} = \frac{1}{2}$ apply exp.function $4^{\#}$ to cancel $\log_4$

then $\dfrac{x+4}{(x+1)^2} = 4^{\frac{1}{2}} = 2$ get rid of fraction $\quad x+4 = 2(x+1)^2$

we got a quadratic equation

$0 = 2x^2 + 4x + 2 - x - 4 = 2x^2 + 3x - 2 = (2x-1)(x+2)$

with two roots $x = -2, \frac{1}{2}$ but **only** $x = \frac{1}{2}$ is the solution of the original equation since $x, x+1$ must be positive.

OR you can change to ln:

$\dfrac{\ln(x+4)}{\ln 4} - \dfrac{2\ln(x+1)}{\ln 4} = \frac{1}{2} \longrightarrow \ln(x+4) - 2\ln(x+1) = \frac{1}{2}\ln 4$

and $\ln(x+4) - \ln(x+1)^2 = \ln 4^{\frac{1}{2}}$ finally $\ln \dfrac{x+4}{(x+1)^2} = \ln 2$

aplly exp.function $e^{\#}$,then again $\dfrac{x+4}{(x+1)^2} = 2$ ...

For 12)

$\displaystyle\int \left(3\sqrt{x} - \frac{1}{3x}\right)^2 dx$(get rid of the power using $(A-B)^2 = A^2 - 2AB + B^2$j

$= \displaystyle\int \left[(3\sqrt{x})^2 - 2\cdot 3\sqrt{x}\cdot\frac{1}{3x} + \left(\frac{1}{3x}\right)^2\right] dx =$

$= 9\int x dx - 2\int x^{-\frac{1}{2}} dx + \frac{1}{9}\int x^{-2} dx = 9\cdot\frac{1}{2}x^2 - 2\cdot 2x^{\frac{1}{2}} + \frac{1}{9}\cdot\frac{x^{-1}}{-1} + c = \dfrac{9}{2}x^2 - 4\sqrt{x} - \dfrac{1}{9x} + c$

for $x > 0$.