The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #2

1. For $f(x) = \dfrac{1}{1-x}\left(1 - \dfrac{4}{x+3}\right)$ find $\lim f(x)$

   (a) as $x \to 1$ and
   (b) as $x \to -3^+$
   (c) as $x \to +\infty$.

   Solution.

   For 1a)

   the type is "$\frac{0}{0}$" so we have to simplify

   $f(x) = \dfrac{1}{1-x} \cdot \dfrac{x+3-4}{x+3} = \dfrac{-(1-x)}{(1-x)(x+3)} = \dfrac{-1}{x+3}$ for any $x \neq 1, -3$.

   As $x \to 1$ the limit is $\frac{-1}{4}$.

   For1b)

   We can use the simplification from above and the type of the limit is "$\frac{-1}{0^+}$" so the limit is $-\infty$.

   OR

   from the original formula the limit is $L = \frac{1}{4} \cdot \left(1 - \text{"}\frac{4}{0^+}\text{"}\right) = -\infty$

   For 1c)

   From the simplified formula the type is "$\frac{-1}{\infty}$" so the limit is 0.

   OR

   from the original the type is "$\frac{1}{-\infty}$" $\cdot \left(1 - \text{"}\frac{4}{\infty}\text{"}\right) = 0 \cdot (1-0) = 0$.

2. For $f(x) = \sqrt{9-x^2}$ and $g(x) = \dfrac{3}{x-1}$ find the compositions $g \circ g$ and $f \circ g$ and their domains.

   Solution.

   For 2)

   First the domains of the given functions for D_f solve $9 - x^2 \geq 0, (3-x)(3+x) \geq 0$

   split points are $x = \pm 3$, testing $--^{neg}--_{-3}---^{pos}---_{3}--^{neg}---$

   so $D_f = [-3, 3]$ $\qquad D_g = \{x \neq 1\}$ since $x - 1 \neq 0$.

   $g \circ g(x) = g(g(x)) = \dfrac{3}{(...) - 1} = \dfrac{3}{\frac{3}{x-1} - 1} = \dfrac{3}{\frac{3-(x-1)}{x-1}} = \dfrac{3(x-1)}{4-x}$

we must start in D_g i.e. $x \neq 1$ and we have to guarantee that

$4 - x \neq 0$ so $x \neq 4$ $D_{g \circ g} = \{x \neq 1 \wedge x \neq 4\}$

$f \circ g(x) = \sqrt{9 - (..)^2} = \sqrt{9 - \frac{9}{(x-1)^2}} = \sqrt{9 \cdot \left(1 - \frac{1}{(x-1)^2}\right)} = 3 \cdot \sqrt{\frac{x^2-2x+1-1}{(x-1)^2}} = 3\sqrt{\frac{x(x-2)}{(x-1)^2}}$

we must start in $\mathrm{D}_g, x \neq 1$ and quarantee that $\dfrac{x(x-2)}{(x-1)^2} \geq 0$,split points are $x = 0, 2, 1$

testing $--^{pos}-_0--neg--_1--^{neg}--_2-^{pos}---$so $\quad D_{f \circ g} = (-\infty, 0] \cup [2, \infty)$.

3. For $g(x) = \dfrac{4}{2x-8}$ and $f(x) = \sqrt{x^2 - 9}$ find $g \circ g$ and $g \circ f$ and their domains

Solution

For 3)

$D_g = \{x \neq 4\}, D_f = (-\infty, -3] \cup [3, +\infty)$ since we have to solve: $x^2 - 9 \geq 0, x^2 \geq 9, \sqrt{x^2} = |x| \geq 3$

Now, $g \circ g(x) = \dfrac{4}{2\,() - 8} = \dfrac{2 \cdot 2}{2\left[\left(\frac{4}{2x-8}\right) - 4\right]} = \dfrac{2}{\frac{4-8x+32}{2x-8}} = 2 \cdot \dfrac{2x-8}{36-8x} = \dfrac{4\,(x-4)}{4\,(9-2x)} = \dfrac{x-4}{9-2x}$

for $x \neq 4$ and $x \neq \frac{9}{2}$,so $D_{g \circ g} = (-\infty, 4) \cup (4, 4.5) \cup (4.5, +\infty)$.

For $g \circ f(x) = \dfrac{4}{2\,() - 8} = \dfrac{4}{2\sqrt{x^2-9} - 8} = \dfrac{2}{\sqrt{x^2-9} - 4} \cdot \dfrac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4} =$

$= \dfrac{2\left(\sqrt{x^2-9}+4\right)}{x^2-9-4^2} = \dfrac{2\left(\sqrt{x^2-9}+4\right)}{x^2-25}$

we know that $x \in D_f$ and that new denominator must be non-zero

$\sqrt{x^2-9} - 4 \neq 0, \sqrt{x^2-9} \neq 4$, so $x^2 - 9 \neq 4^2, x^2 \neq 25$

OR after simplification $x^2 - 25 \neq 0$ i.e. $x \neq \pm 5$,together

$D_{g \circ f} = (-\infty, -5) \cup (-5, -3] \cup [3, 5) \cup (5, +\infty)$.

4. Find : $\lim \dfrac{1-4x^2}{6x^2-5x+1}$

(a) as $x \to -\infty$ (b) as $x \to \frac{1}{2}$ (c) as $x \to \frac{1}{3}^-$.

For 4a)

divide top and bottom by x^2 : $\lim\limits_{x \to -\infty} \dfrac{\frac{1}{x^2} - 4}{6 - \frac{5}{x} + \frac{1}{x^2}} = \dfrac{0-4}{6-0+0} = -\frac{4}{6} = -\dfrac{2}{3}$ (since "$\frac{1}{\pm\infty}$" $= 0$)

For 4b)

the type is "$\frac{0}{0}$" and we have polynomials

$\lim\limits_{x \to \frac{1}{2}} \dfrac{(1-2x)(1+2x)}{(2x-1)(3x-1)} = \lim\limits_{x \to \frac{1}{2}} \dfrac{-(1+2x)}{3x-1} = \frac{-2}{\frac{1}{2}} = -4.$

For 4c)

we can use the simplification from above but the type is "$\frac{neg\#}{0^-}$" since $x < \frac{1}{3}$ so $3x-1 < 0$

$$\lim_{x \to \frac{1}{3}^-} \frac{-(1+2x)}{3x-1} = \text{"}\frac{-\frac{5}{3}}{0^-}\text{"} = \text{"}\frac{1}{0^+}\text{"} = +\infty$$

5. Find : $\lim \dfrac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}$

   (a) as $x \to 3^+$, (b) as $x \to +\infty$, (c) as $x \to 0$

   For 5a)

$$\lim_{x \to 3^+} \frac{\sqrt{3x}-3}{\sqrt{2x^2-6x}} = \text{"}\frac{0}{0}\text{"} = \lim_{x \to 3^+} \frac{\sqrt{3x}-3}{\sqrt{2x^2-6x}} \cdot \frac{\sqrt{3x}+3}{\sqrt{3x}+3} = \lim_{x \to 3^+} \frac{3x-3^2}{\sqrt{2x^2-6x}} \cdot \frac{1}{\sqrt{3x}+3} =$$

$$= \lim_{x \to 3^+} \frac{3(x-3)}{\sqrt{2x(x-3)}} \cdot \frac{1}{\sqrt{3x}+3} = \lim_{x \to 3^+} \frac{3}{\sqrt{2x}} \cdot \frac{x-3}{\sqrt{x-3}} \cdot \frac{1}{\sqrt{3x}+3} =$$

$$= \lim_{x \to 3^+} \frac{3}{\sqrt{2x}} \cdot \sqrt{x-3} \cdot \frac{1}{\sqrt{3x}+3} = \frac{3}{\sqrt{6}} \cdot 0 \cdot \frac{1}{6} = 0.$$

   For 5b)

   the type is "$\frac{\infty}{\infty}$" so we have to divide by the highest power in the denominator

   in the original from by $x = \sqrt{x^2}$ $(x > 0)$:

$$\lim_{x \to +\infty} \frac{\sqrt{3x}-3}{\sqrt{2x^2-6x}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x \to +\infty} \frac{\sqrt{\frac{3}{x}} - \frac{3}{x}}{\sqrt{2-\frac{6}{x}}} = \frac{0-0}{\sqrt{2}} = 0$$

   OR

   we can used the simplified expresion from b)

$$\lim_{x \to +\infty} \frac{3}{\sqrt{2x}} \cdot \sqrt{x-3} \cdot \frac{1}{\sqrt{3x}+3} = \lim_{x \to +\infty} 3\sqrt{\tfrac{x-3}{2x}} \cdot \tfrac{1}{\sqrt{3x}+3} =$$

$$= \lim_{x \to +\infty} 3\sqrt{\tfrac{1}{2} - \tfrac{3}{2x}} \cdot \lim_{x \to +\infty} \tfrac{1}{\sqrt{3x}+3} = \tfrac{3}{\sqrt{2}} \cdot 0 = 0 \text{ since } \text{"}\tfrac{1}{\infty}\text{"} = 0.$$

   For 5c)

   the limit DNE (does not exist neither as a number nor as $\pm\infty$)

   since the function is not defined for small negative $x(\sqrt{neg})$

6. For $f(x) = \dfrac{\sqrt{3-x}}{x^2-4x+3}$ find $\lim f(x)$

   (a) as $x \to 3^-$ and

   (b) as $x \to 1^+$

   (c) as $x \to +\infty$.

For 6 a)

the type is "$\frac{0}{0}$" and the function is defined for $x < 3$ and $x \neq 1$ we can simplify

$$f(x) = \frac{\sqrt{3-x}}{x^2-4x+3} = \frac{\sqrt{3-x}}{(x-3)(x-1)} = \frac{\sqrt{3-x}}{-(3-x)(x-1)} = \frac{\sqrt{3-x}}{-\sqrt{3-x}\sqrt{3-x}(x-1)} = -\frac{1}{\sqrt{3-x}\,(x-1)}$$

Now the type is "$\frac{-1}{0^+\cdot(2)}$" = "$\frac{1}{0^-}$" and the limit is $-\infty$.

For 6b)

We can use the simplification from above or at least identify the type "$\frac{\sqrt{2}}{0}$" so we have to investigate the sign of the bottom

Since $x > 1$ and $f(x) = \dfrac{\sqrt{3-x}}{(x-3)(x-1)}$ we can see that the type is :"$\frac{\sqrt{2}}{(-2)\cdot 0^+}$" = "$\frac{1}{0^-}$" and the limit is $-\infty$.

For 6c)

the limit DNE (does not exists) since the function is not defined for big positive x .

7. For $g(x) = \sqrt{3+x}$ and $f(x) = \sqrt{x-5}$ find the compositions $g \circ g$ and $f \circ g$ and their domains.

For 7)

First the domains of the given functions $D_g = [-3, +\infty)$ since $3 + x \geq 0$;

$D_f = [5, +\infty)$ since $x - 5 \geq 0$.

$f \circ g(x) = f(g(x)) = \sqrt{(..) - 5} = \sqrt{\sqrt{3+x} - 5}$

we must start in D$_g$ i.e. $x \in [-3, +\infty)$ and we have to guarantee that

$\sqrt{3+x} - 5 \geq 0$,so $\sqrt{3+x} \geq 5$

both sides are positive so we can square $(3 + x) \geq 25$, and $x \geq 22$,together

$D_{f\circ g} = [22, +\infty[$

$g \circ g(x) = \sqrt{3 + (..)} = \sqrt{3 + \sqrt{3+x}}$

we must start in D$_g = [-3, +\infty)$ and quarantee that $3 + \sqrt{3+x} \geq 0$

but it is always true for any $x \in [-3, +\infty)$ so $D_{g\circ g} = [-3, +\infty)$.

8. For $f(x) = \dfrac{|x-3| - |x+3|}{x}$ find $\lim f(x)$

   (a) as $x \to 0$ and

   (b) as $x \to -\infty$

   (c) as $x \to +\infty$.

For 8a)

the type is "$\frac{0}{0}$" and if x is a small # ,neg. or pos,$x-3$ is close to -3 so negative and $|x-3|=-(x-3)=3-x$, $x+3$ is close to 3 so $x+3$ is positive and $|x+3|=x+3$ therefore

$f(x)=\dfrac{|x-3|-|x+3|}{x}=\dfrac{3-x-(x+3)}{x}=\dfrac{-2x}{x}=-2$ for $x\neq 0$

so the limit is -2.

ALSO

$f(x)=\dfrac{|x-3|-|x+3|}{x}\cdot\dfrac{|x-3|+|x+3|}{|x-3|+|x+3|}=\dfrac{|x-3|^2-|x+3|^2}{x\cdot(|x-3|+|x+3|)}=\dfrac{(x-3)^2-(x+3)^2}{x\cdot(|x-3|+|x+3|)}=$

$=\dfrac{x^2-6x+3^2-(x^2+6x+3^2)}{x\cdot(|x-3|+|x+3|)}=\dfrac{-12x}{x\cdot(|x-3|+|x+3|)}=\dfrac{-12}{(|x-3|+|x+3|)}$ for any $x\neq 0$

Now the limit is

$L=\frac{-12}{3+3}=-2.$

For8b)

For x big negative number $x-3$ is negative and $|x-3|=-(x-3)=3-x$,also $x+3$ is negative and

$|x+3|=-(x+3)=-3-x,$

$f(x)=\dfrac{|x-3|-|x+3|}{x}=\dfrac{3-x+x+3}{x}=\dfrac{6}{x}$ so the type of the limit is "$\frac{1}{\infty}$" and $L=0.$

ALSO for both b) and c)

using the siplification from above $f(x)=\dfrac{-12}{(|x-3|+|x+3|)}$ so the type is "$\frac{-12}{\infty}$" and the limit is 0.

For 8 c)

For x big positive both expressions $x-3$ and $x+3$ are positive so we can ignore absolute values and

$f(x)=\dfrac{x-3-(x+3)}{x}=\frac{-6}{x}$ and the type of the limit is "$\frac{-6}{\infty}$" and the limit is 0.

9. For $g(x)=\sqrt{3-x}$ and $f(x)=\dfrac{6}{3x-1}$ find the compositions $g\circ f$ and $f\circ f$ and their domains

For 9)

First the domains of the given functions $D_f=\left\{x\neq\frac{1}{3}\right\}$ since $3x-1\neq 0$;
$D_g=(-\infty,3]$ since $3-x\geq 0$.

$f\circ f(x)=f(f(x))=\dfrac{6}{3(...)-1}=\dfrac{6}{3\cdot\frac{6}{3x-1}-1}=\dfrac{6}{\frac{18-(3x-1)}{3x-1}}=\dfrac{6\,(3x-1)}{19-3x}$

we must start in D_f i.e. $x \neq \frac{1}{3}$ and we have to guarantee that $19 - 3x \neq 0$ so $x \neq \frac{19}{3}$ and $D_{f \circ f} = \left\{ x \neq \frac{1}{3} \wedge x \neq \frac{19}{3} \right\}$

$g \circ f(x) = \sqrt{3 - (..)} = \sqrt{3 - \frac{6}{3x-1}} = \sqrt{\frac{3(3x-1)-6}{3x-1}} = \sqrt{\frac{9x-9}{3x-1}} = 3\sqrt{\frac{x-1}{3x-1}}$

we must start in $\mathrm{D}_f, x \neq \frac{1}{3}$ and quarantee that $\dfrac{x-1}{3x-1} \geq 0$,split points are $x = 1, \frac{1}{3}$

testing $-^{pos} - -_{\frac{1}{3}} - -^{neg} - -_1 - -^{pos} -$

so the domain is $D_{g \circ f} = \left(-\infty, \frac{1}{3} \right) \cup [1, +\infty)$