MATHEMATICS 221 L04/12/22 FALL 2003
QUIZ 5 SOLUTIONS
Thursday, December 4, 2003 at 11:00

1. Does the plane $2x - 3y + z = 1$ contain the line $[x, y, z]^T = [3, 2, 1]^T + t\,[2, 0, -4]^T$? Explain.

Solution: Yes, it does. This is because the point $(3, 2, 1)$ of the line is also on the plane (for $2 \times 3 - 3 \times 2 + 1 = 1$), and the direction $\overrightarrow{d} = [2, 0, -4]^T$ is orthogonal to the normal $\overrightarrow{n} = [2, -3, 1]^T$ of the plane (It is clear that $\overrightarrow{d} \cdot \overrightarrow{n} = 0$).

2. Find an equation of the line that passes through $P\,(1, 2, 3)$ and is parallel to both of the planes $x + y + z = 1$ and $x - y + z = 1$.

Solution: A point of the line is $P\,(1, 2, 3)$. Since the line is parallel to both of the planes, a direction of the line is a scalar multiple of the cross products of the two normals of the plane. Thus, we can choose

$$\overrightarrow{d} = \tfrac{1}{2}\,[1, 1, 1]^T \times [1, -1, 1]^T = \tfrac{1}{2}\,[2, 0, -2]^T = [1, 0, -1]^T.$$

Thus, an equation of the line is $[x, y, z]^T = [1, 2, 3]^T + t\,[1, 0, -1]^T$.

3. Let L_1 be the line with equation $[x, y, z]^T = [2, -5, 1]^T + t\,[0, 1, 1]^T$ and L_2 be the line with equation $[x, y, z]^T = [5, 0, 0]^T + s\,[2, 1, 0]^T$.

(a) Are L_1 and L_2 parallel? Explain.

Solution: No, they are not because the directions of the lines are not parallel (they are not multiple of each other).

(b) Find the shortest distance between L_1 and L_2, and find a point A on L_1 and a point B on L_2 so that $\left\|\overrightarrow{AB}\right\|$ is the shortest distance between L_1 and L_2.

Solution: Let A and B be the points on L_1 and L_2 respectively so that they are closest together. Then the coordinates of them are $A\,(2, -5 + t, 1 + t)$ and $B\,(5 + 2s, s, 0)$ for some real numbers s and t. Since $\overrightarrow{AB}$ is orthogonal to the directions $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$ of the lines where $\overrightarrow{d}_1 = [0, 1, 1]^T$ and $\overrightarrow{d}_2 = [2, 1, 0]^T$ we have $\overrightarrow{d}_1 \cdot \overrightarrow{AB} = 0$ and $\overrightarrow{d}_2 \cdot \overrightarrow{AB} = 0$. Now,

$$\begin{cases} \overrightarrow{d}_1 \cdot \overrightarrow{AB} = 0 \\ \overrightarrow{d}_2 \cdot \overrightarrow{AB} = 0 \end{cases} \Leftrightarrow \begin{cases} [0, 1, 1]^T \cdot [2s + 3, s - t + 5, -1 - t]^T = 0 \\ [2, 1, 0]^T \cdot [2s + 3, s - t + 5, -1 - t]^T = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (s - t + 5) + (-1 - t) = 0 \\ 2\,(2s + 3) + (s - t + 5) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} s - 2t = -4 \\ 5s - t = -11 \end{cases}$$

Solve the above system, we get $s = -2$ and $t = 1$, so the coordinates of A and B are $A\,(2, -4, 2)$ and $B\,(1, -2, 0)$ and the shortest distance between L_1 and L_2 is

$$\left\|\overrightarrow{AB}\right\| = \left\|[-1, 2, -2]^T\right\| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3.$$

Thursday, December 20, 2003 at 16:00

1. Consider the point $P\,(1,2,3)$ and the line L whose equation is
$[x,y,z]^T = [6,1,4]^T + t\,[4,1,-1]^T$.
(a) Find the shortest distance from the point P and the line L, and find the coordinates of the point on line L that is closest to P.
Solution:
(b) Find an equation of the plane containing the point P and the line L.
Solution:

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2. Does the line through $P(1,2,-3)$ with direction $\overrightarrow{d} = [1,1,-1]^T$ lie in the plane $2x - y + z = -3$? Explain.

Solution: Yes, it does. This is because the point $P(1,2,-3)$ of the line is also on the plane (for $2 \times 1 - 2 + (-3) = -3$), and the direction $\overrightarrow{d} = [1,1,-1]^T$ is orthogonal to the normal $\overrightarrow{n} = [2,-1,1]^T$ of the plane (It is clear that $\overrightarrow{d} \cdot \overrightarrow{n} = 0$).

3. Do the lines with equations $[x,y,z]^T = [2,3,1]^T + t[1,1,0]^T$ and $[x,y,z]^T = [3,0,5]^T + s[2,-1,3]^T$ intersect? Show all work.

Solution: Let (x,y,z) be a point of the intersection of the two lines, this means must satisfy both equations of the lines, so there are numbers s and t so that $(x,y,z) = (2+t, 3+t, 1) = (3+2s, -s, 5+3s)$. This is equivalent to

$$\begin{cases} 2+t &= 3+2s \\ 3+t &= -s \\ 1 &= 5+3s \end{cases} \Leftrightarrow \begin{cases} 2s-t &= -1 \\ s+t &= -3 \\ 3s &= -4 \end{cases}$$

Solve this system,

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -3 \\ 3 & 0 & -4 \end{bmatrix} \overset{R_1+R_2}{\rightarrow} \begin{bmatrix} 3 & 0 & -4 \\ 1 & 1 & -3 \\ 3 & 0 & -4 \end{bmatrix} \underset{R_3-R_1}{\rightarrow} \begin{bmatrix} 3 & 0 & -4 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{3}R_1}{\rightarrow}$$

$$\begin{bmatrix} 1 & 0 & \frac{-4}{3} \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2-R_1}{\rightarrow} \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $s = -\frac{4}{3}$ and $t = -\frac{5}{3}$, and the lines intersect at one point $P\left(\frac{1}{3}, \frac{4}{3}, 1\right)$.