FINAL EXAMINATION MATH221 (L 02/10)

FALL 2000 Time: 3 hours

[5] 1. Solve the system:

$$\begin{array}{rcrcrcrcrcr} -x & & & + & z & + & u & - & 2w & = & 2 \\ 3x & + & y & - & z & - & u & + & 6w & = & -1 \\ -5x & - & y & + & 3z & + & 2u & - & 11w & = & 2 \end{array}$$

[10] 2. Let $A = \begin{bmatrix} 3 & 5 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

(a) Find an invertible matrix U so that $UA = R$ where R is the reduced row-echelon form of A.

(b) Express U^{-1} as a product of elementary matrices.

[10] 3. Let A, B and C be 3×3 matrices with $\det A = -2$, $\det B = 2$ and $\det C = 3$.

(a) Find $\det\left(-A^2\left(B^T\right)^{-1}C\right)$.

(b) Compute $\det\left(-2A^2\left(adj A\right)^{-1}\right)$.

[10] 4. Consider the system $AX = B$ where

$A = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Solve the system $AX = B$.

(c) Can the above system $AX = B$ be solved by Cramer's Rule? Explain. (Do not solve the system again)

[10] 5. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Determine whether A is diagonalizable. If A is diagonalizable, find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

[7] 6. Show that $\|\vec{\mathbf{u}} + \vec{\mathbf{v}}\|^2 + \|\vec{\mathbf{u}} - \vec{\mathbf{v}}\|^2 = 2\left(\|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2\right)$ for all vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

[8] 7. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation so that $T\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and $T\left[\begin{smallmatrix} -3 \\ -2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right]$

(a) Find the matrix A so that $T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

(b) Find $T^{-1}\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$.

[10] 8. Consider the points $A(3,1,1)$, $B(2,1,2)$, $C(3,5,3)$ and $D(5,0,3)$.

(a) Find an equation of the plane containing the three points A, B and C.

(b) Find the distance between the point D and the plane in part (a).

[10] 9. Consider the points $A(3,1,-2)$, $B(5,2,-1)$ and $C(4,3,-3)$.

(a) Find the area of the triangle with the vertices A, B and C.

(b) Find the three internal angles of the triangle with the vertices A, B and C.

[10] 10. Consider the lines

$L_1 : \quad [x,y,z]^T = [5,-1,2]^T + t\,[2,2,1]^T \quad$ and $\quad L_2 : \quad [x,y,z]^T = [2,-5,5]^T + t\,[1,-3,2]^T$

(a) Find an equation of the line which passes through $P(1,2,1)$ and perpendicular to both of the lines L_1 and L_2.

(b) Find the distance between the lines L_1 and L_2.

[10] 11. Solve for z if $z^2 - iz + (1+3i) = 0$.

[5] 12. Solve the system:

$$\begin{array}{rcrcrcrcrcr} -x & & & + & z & + & u & - & 2w & = & 2 \\ 3x & + & y & - & z & - & u & + & 6w & = & -1 \\ -5x & - & y & + & 3z & + & 2u & - & 11w & = & 2 \end{array}$$

[10] 13. Let $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$. Express A as a product of elementary matrices.

[10] 14. Let A, B and C be 3×3 matrices with $\det A = -2$, $\det B = 2$ and $\det C = 3$.

(a) Compute $\det\left(A^2\left(2B^T\right)^{-1}C\right)$.

(b) Compute $\det\left(-2A^{-1} + adj A\right)$.

[10] 15. Consider the matrices

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$$

(a) Find $adj A$.

(b) Compute $A.adj A$.

(c) Can the above system $AX = B$ be solved by Cramer's Rule? Explain. (Do not solve the system)

16. Let $A = \begin{bmatrix} 3 & 6 \\ -2 & -4 \end{bmatrix}$.

[5] (a) Find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$

[5] (b) Compute A^{10}

[7] 17. Show that if λ is an eigenvalue of an $n \times n$ matrix A then $\lambda^2 - 2\lambda$ is an eigenvalue of $A^2 - 2A$.

[10] 18. Consider the points $A(-1, 1, 2)$, $B(2, -1, 3)$ and $C(3, 2, -2)$.

(a) Find an equation of the plane containing the three points A, B and C.

(b) Find the distance between the origin and the plane in part (a).

[10] 19. .Consider the lines

$L_1 :\quad (x, y, z) = (5, -1, 2) + t(2, 2, 1)$ and $L_2 :\quad (x, y, z) = (2, -5, 5) + s(1, -3, 2)$

(a) Find an equation of the line which passes through $P(1, 2, 1)$ and perpendicular to both of the lines L_1 and L_2.

(b) Find the distance between the lines L_1 and L_2.

[8] 20. .Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation so that $T\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and $T\left[\begin{smallmatrix} -3 \\ -2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right]$

(a) Find the matrix A so that $T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

(b) Find $T^{-1}\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$.

[10] 21. Find all complex numbers z that satisfy the equation $z^4 = 2\left(i\sqrt{3} - 1\right)$.