MATHEMATICS 221 L04 FALL 2003 MIDTERM SOLUTION

1. Solve the system:

$$\begin{array}{rcrcrcrcl} x & - & 2y & - & z & + & 3w & = & 1 \\ 2x & - & 4y & + & z & & & = & 5 \\ x & - & 2y & + & 2z & - & 3w & = & 4 \end{array}$$

Solution:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \begin{array}{c} \longrightarrow \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{bmatrix} \begin{array}{c} \longrightarrow \\ \frac{1}{3}R_2 \\ \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -6 & 3 \end{bmatrix} \begin{array}{c} R_1 + R_2 \\ \longrightarrow \\ R_3 - 3R_2 \end{array} \begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $x = 2s - t + 2,\ y = s,\ z = 2t + 1$ and $w = t$ where s and t are any real numbers.

2. Given that $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 5$. Find $\det \begin{bmatrix} a+2x & b+2y & c+2z \\ 3x+4p & 3y+4q & 3z+4r \\ -2p & -2q & -2r \end{bmatrix}$.

Solution:

$$\det \begin{bmatrix} a+2x & b+2y & c+2z \\ 3x+4p & 3y+4q & 3z+4r \\ -2p & -2q & -2r \end{bmatrix} = \det \left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 3 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 0 & -2 & 0 \end{bmatrix} \det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

Now, by expanding along first column,

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 0 & -2 & 0 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 \\ -2 & 0 \end{bmatrix} = 6. \text{ Thus, } \det \begin{bmatrix} a+2x & b+2y & c+2z \\ 3x+4p & 3y+4q & 3z+4r \\ -2p & -2q & -2r \end{bmatrix} =$$

$6 \times 5 = 30$.

3. Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix}$. Find an invertible matrix U so that $UA = R$ where R is the reduced row-echelon form of A and express U as a product of elementary matrices.

Solution:

$$\left[\begin{array}{rr|rr} 3 & -1 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{array}\right] \underset{E_1}{\overset{R_1+R_2}{\longrightarrow}} \left[\begin{array}{rr|rr} 1 & -1 & 1 & 1 \\ -2 & 0 & 0 & 1 \end{array}\right] \underset{R_2+2R_1}{\overset{E_2}{\longrightarrow}} \left[\begin{array}{rr|rr} 1 & -1 & 1 & 1 \\ 0 & -2 & 2 & 3 \end{array}\right] \underset{-\frac{1}{2}R_2}{\overset{E_3}{\longrightarrow}}$$

$$\left[\begin{array}{rr|rr} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & \frac{-3}{2} \end{array}\right] \underset{E_4}{\overset{R_1+R_2}{\longrightarrow}} \left[\begin{array}{rr|rr} 1 & 0 & 0 & \frac{-1}{2} \\ 0 & 1 & -1 & \frac{-3}{2} \end{array}\right]. \text{ Thus, } U = \begin{bmatrix} 0 & \frac{-1}{2} \\ -1 & \frac{-3}{2} \end{bmatrix}$$

and $U = E_4E_3E_2E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

4. Find all values of x so that the matrix $\begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$ is **not** invertible.

Solution: A square matrix is not invertible when its determinant is zero. Now,

$$\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \begin{matrix} = \\ R_2 - xR_1 \\ R_3 - xR_1 \end{matrix} \begin{vmatrix} 1 & x & x \\ 0 & 1-x^2 & x-x^2 \\ 0 & x-x^2 & 1-x^2 \end{vmatrix} = (1-x)^2 \begin{vmatrix} 1 & x & x \\ 0 & 1+x & x \\ 0 & x & 1+x \end{vmatrix}$$

$$= (1-x)^2 \begin{vmatrix} 1+x & x \\ x & 1+x \end{vmatrix} = (1-x)^2 \left[(1+x)^2 - x^2\right] = (1-x)^2 \left[1+2x+x^2-x^2\right]$$

$$= (1-x)^2 (1+2x).$$

Thus, the above matrix is not invertible when ,$(1-x)^2(1+2x) = 0$; that is, $x = 1$ or $x = \frac{-1}{2}$.

5. Prove that if $A^3 = 0$ then $I - A$ is invertible and $(I-A)^{-1} = I + A + A^2$.

Solution: Suppose that $A^3 = 0$. Then since we can compute A^3, A must be a square matrix and
$(I-A)(I+A+A^2) = I + A + A^2 - A - A^2 - A^3 = I - A^3 = I - 0 = I$. Thus, $I - A$ is invertible and $(I-A)^{-1} = I + A + A^2$.

6. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Is A diagonalizable? If A is diagonalizable, find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$.

Solution:
$c_A(x) = \begin{vmatrix} x-1 & -3 \\ -2 & x-2 \end{vmatrix} = (x-1)(x-2) - 6 = x^2 - 3x - 4 = (x+1)(x-4).$

Solve $c_A(x) = 0$, we get that the eigenvalues of A are -1 and 4. Since A is 2×2 and A has two eigenvalues, A is diagonalizable.

To find eigenvectors of A corresponding to the eigenvalue -1, we solve $(-I-A)X = 0$:
$\begin{bmatrix} -2 & -3 & 0 \\ -2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we get $X = t\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ where $t \neq 0$.

To find eigenvectors of A corresponding to the eigenvalue 4, we solve $(4I - A)X = 0$:
$\begin{bmatrix} 3 & -3 & 0 \\ -2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we get $X = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $t \neq 0$.

Put $P = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$, we have $A = PDP^{-1}$ where $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

7. For each of the following statements, determine whether the statement is true (T) or false (F). No explanation is needed.

(a) If $A^2 = A$ then $A = 0$ or $A = I$. F

(b) If $A^3 = 3I$ then A is invertible. T

(c) $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ is an elementary matrix. T

(d) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $adj A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$. T

(e) If A is a square matrix and A is not invertible then 0 is an eigenvalue of A. $\boxed{\text{T}}$

8. Find all complex numbers z so that $z^3 = -27i$. Express your answers in the form $a+bi$ where a and b are real numbers.

Solution: We note that $-27i = 27e^{i\left(-\frac{\pi}{2}+2k\pi\right)}$. Now, let $z = re^{i\theta}$. Then, $z^3 = -27i$ becomes $\left(re^{i\theta}\right)^3 = 27e^{i\left(-\frac{\pi}{2}+2k\pi\right)}$, and so by DeMoivre Theorem we have , $r^3e^{i3\theta} = 27e^{i\left(-\frac{\pi}{2}+2k\pi\right)}$. This implies that $r^3 = 27$ and $3\theta = -\frac{\pi}{2} + 2k\pi$, and so $r = 3$ and $\theta = -\frac{\pi}{6} + 2k\frac{\pi}{3}$. Thus, $z = 3e^{i\left(-\frac{\pi}{6}+2k\frac{\pi}{3}\right)}$ where k is any integer.

When $k = 0$, $z = 3e^{i\left(-\frac{\pi}{6}\right)} = 3\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$.

When $k = 1$, $z = 3e^{i\pi} = 3\left(\cos(\pi) + i\sin(\pi)\right) = 3(0+i) = 3i$.

When $k = 2$, $z = 3e^{i\left(\frac{7\pi}{6}\right)} = 3\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = 3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$.