MATH 211 L01

Quiz 1a
February 2
15:00-15:50

Problem 1. (10 marks)
Find conditions on the numbers a and b that the system

$$\begin{aligned} x_1 + ax_2 &= 1, \\ ax_1 + x_2 &= b, \end{aligned}$$

has no solution, a unique solution, or infinitely many solutions.

Solution.
Reduction of the augumented coefficient matrix:

$$\begin{pmatrix} 1 & a & 1 \\ a & 1 & b \end{pmatrix} \to \begin{pmatrix} 1 & a & 1 \\ 0 & 1-a^2 & b-a \end{pmatrix}.$$

Case 1: [**3 marks**] If $1 - a^2 \neq 0$, we can divide row 2 by $1 - a^2$ obtaining

$$\begin{pmatrix} 1 & a & 1 \\ 0 & 1-a^2 & b-a \end{pmatrix} \to \begin{pmatrix} 1 & a & 1 \\ 0 & 1 & \frac{b-a}{1-a^2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 - a\frac{b-a}{1-a^2} \\ 0 & 1 & \frac{b-a}{1-a^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1-ab}{1-a^2} \\ 0 & 1 & \frac{b-a}{1-a^2} \end{pmatrix}.$$

In this case we have a unique solution

$$x_1 = \frac{1-ab}{1-a^2},\ x_2 = \frac{b-a}{1-a^2}.$$

Note: students can present results without simplifying them.

Case 2: [**3 marks**] If $1 - a^2 = 0$, then $a = \pm 1$ and our matrix is

$$\begin{pmatrix} 1 & a & 1 \\ 0 & 0 & b-a \end{pmatrix}.$$

Case 2a [**2 marks**] If $a = \pm 1$ and $b \neq a$, then there are no solutions.

Case 2b [**2 marks**] If $b = a = \pm 1$, then the reduced eauation is $x_1 + ax_2 = 1$.We have infinitely many solutions $x_1 = 1 - ax_2$, where x_2 is arbitrary.

Problem 2

A/ (6 marks). Find all solutions of the system of linear equations.

$$\begin{aligned} x + y - z &= 4, \\ 3x - y + z &= 2 \\ x - y + z &= -1 \end{aligned}$$

B/ (2 marks) Write down the homogeneous system associated to the system in **A**. Deduce the general solution of this system from the results of the part **A.**

C/ (2 mark) Find the ranks of the coefficient matrix and the augumented coefficient matrix in **A**.

Solution.

A. Reduction of the augumented coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 3 & -1 & 1 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & -4 & 4 & 4 \\ 0 & -2 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding system of linear equations is

$$\begin{aligned} x &= 0 \\ y - z &= 0 \\ 0 &= 1. \end{aligned}$$

No solutions.

B The homogeneous system associated to the system in **A** is

$$\begin{aligned} x + y - z &= 0 \\ 3x - y + z &= 0 \\ x - y + z &= 0 \end{aligned}$$

The reduced homogeneous system is

$$\begin{aligned} x &= 0 \\ y - z &= 0. \end{aligned}$$

Its general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ t \end{pmatrix}, \text{ where } t \text{ is arbitrary.}$$

C. Rank of A is 3.

Quiz 1b

February 3,

17:00-17:50

Problem 1. (10 marks)

Find conditions on the numbers a and b that the system

$$\begin{aligned} x_1 + bx_2 &= 1, \\ ax_1 + x_2 &= b, \end{aligned}$$

has no solution, a unique solution, or infinitely many solutions.

Solution.

Reduction of the augumented coefficient matrix:

$$\begin{pmatrix} 1 & b & 1 \\ a & 1 & b \end{pmatrix} \to \begin{pmatrix} 1 & a & 1 \\ 0 & 1-ab & b-a \end{pmatrix}.$$

Case 1: [**3 marks**] If $1 - ab \neq 0$, we can divide row 2 by $1 - ab$ obtaining

$$\begin{pmatrix} 1 & a & 1 \\ 0 & 1-ab & b-a \end{pmatrix} \to \begin{pmatrix} 1 & a & 1 \\ 0 & 1 & \frac{b-a}{1-ab} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 - a\frac{b-a}{1-ab} \\ 0 & 1 & \frac{b-a}{1-ab} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1-2ab+a^2}{1-ab} \\ 0 & 1 & \frac{b-a}{1-ab} \end{pmatrix}.$$

In this case we have a unique solution

$$x_1 = \frac{1 - 2ab + a^2}{1 - a^2}, \; x_2 = \frac{b-a}{1-ab}.$$

Note: students can present results without simplifying them.

Case 2: [**3 marks**] If $1 - ab = 0$, then our matrix is

$$\begin{pmatrix} 1 & a & 1 \\ 0 & 0 & b-a \end{pmatrix}.$$

Case 2a [**2 marks**] If $1 - ab = 0$ and $a - b \neq 0$ then there are no solutions.

Case 2b [**2 marks**] If $1 - ab = 0$ and $a - b = 0$, that is $a = b = \pm 1$, then the reduced eauation is $x_1 + ax_2 = 1$.We have infinitely many solutions $x_1 = 1 - ax_2$, where x_2 is arbitrary.

Problem 2

A/ (6 marks) Find the general solution of the system of linear equations

$$\begin{aligned} x + y - z &= 1, \\ 3x - y &= 3, \\ 2x - 2y + z &= 2. \end{aligned}$$

B/ (2 marks) Write down the homogeneous system associated to the system in **A**. Deduce the general solution of this system from the results of the part **A.**

C/ (2 mark) Find the ranks of the coefficient matrix and the augumented coefficient matrix in **A**.

Solution.

A. Reduction of the augumented coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 3 & -1 & 0 & 3 \\ 2 & -2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -4 & 3 & 0 \\ 0 & -4 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{4} & 1 \\ 0 & 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding system of linear equations is

$$\begin{aligned} x - \frac{1}{4}z &= 1 \\ y - \frac{3}{4}z &= 0. \end{aligned}$$

Genral solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{4}t \\ \frac{3}{4}t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{pmatrix}, \text{ where } t \text{ is arbitrary.}$$

Note: students can present the solution in any other form.

B. **B** The homogeneous system associated to the system in **A** is

$$\begin{aligned} x + y - z &= 0, \\ 3x - y &= 0, \\ 2x - 2y + z &= 0. \end{aligned}$$

The reduced homogeneous system is

$$\begin{aligned} x - \frac{1}{4}z &= 0 \\ y - \frac{3}{4}z &= 0. \end{aligned}$$

Its general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{pmatrix}, \text{ where } t \text{ is arbitrary.}$$

C. Rank of A is 2.