Noether Theorem

March 3, 2010

Definition 1 *A diffeomorphism φ of M is a symmetry of a Lagrangian $L : TM \to \mathbb{R}$ if $(T\varphi)^* L = L$.*

Note that $T\varphi : TM \to TM$ is defined as follows. For every $u \in T_x M$ and $f \in C^\infty(M)$,

$$(T\varphi(u))(f) = u(\varphi^* f) = u(f \circ \varphi).$$

Consider local coordinates (q^i) on M. If $x \in M$ has coordinates $(q^1(x), ..., q^n(x))$, then $\varphi(x)$ has coordinates

$$(q^1(\varphi(x)), ..., q^n(\varphi(x))) = (F^1((q^1(x), ..., q^n(x)), ..., F^n(q^1(x), ..., q^n(x))).$$

In other words, φ maps a point with coordinates $(q^1, ..., q^n)$ to a point with coordinates $(F^1((q^1, ..., q^n), ..., F^n(q^1, ..., q^n))$. Treating as a point in M we can write

$$\varphi(q^1, ..., q^n) = (F^1((q^1, ..., q^n), ..., F^n(q^1, ..., q^n))$$

Let $\gamma : [a, b] \to M : t \mapsto \gamma(t)$ be a curve in M with coordinates $(q^1(t), ..., q^n(t))$. The tangent vector to the curve γ at t is

$$\dot{\gamma}(t) = \dot{q}^1 \frac{\partial}{\partial q^1} + ... + \dot{q}^n \frac{\partial}{\partial q^n}.$$

The vector $T\varphi(\dot{\gamma}(t))$ is tangent at t to the curve $\varphi \circ \gamma : [a, b] \to M : t \mapsto \varphi(\gamma(t))$. But

$$(q^1(\varphi(\gamma(t))), ..., q^n(\varphi(\gamma(t)))) = (F^1(q^1(\gamma(t)), ..., q^n(\gamma(t))), ..., F^n(q^1(\gamma(t)), ..., q^n(\gamma(t)))).$$

Differentiating with respect to t we get

$$T\varphi(\dot{\gamma}(t)) = \frac{d}{dt}\varphi(\gamma(t)) = \sum_{i,j=1}^{n} \frac{\partial F^i}{\partial q^j} \frac{dq^j}{dt} \frac{\partial}{\partial q^i} = \sum_{i,j=1}^{n} \frac{\partial F^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial q^i}.$$

Thus, $T\varphi$ takes a vector $\dot{q}^1 \frac{\partial}{\partial q^1} + ... + \dot{q}^n \frac{\partial}{\partial q^n}$ at $x \in M$ to the vector $\sum_{i,j=1}^{n} \frac{\partial F^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial q^i}$ at $\varphi(x)$.

Consider now a 1-parameter group $\varphi_t = \exp tX$ of diffeomorphisms of M generated by a vector field

$$X = f^1(q^1, ..., q^n)\frac{\partial}{\partial q^1} + ... + f^n(q^1, ..., q^n)\frac{\partial}{\partial q^n}.$$

Then,

$$F^i(q^1, ..., q^n) = q^i + tf^i(q^1, ..., q^n) + \text{ terms of order } h \text{ and higher.}$$

Therefore, setting

$$\delta^i_j = \frac{\partial q^i}{\partial q^j}$$

we get

$$\begin{aligned} T\varphi(\dot{q}) &= T(\exp tX)(\dot{q}) = \sum_{i,j=1}^{n} \frac{\partial F^i}{\partial q^j}\dot{q}^j\frac{\partial}{\partial q^i} = \sum_{i,j=1}^{n}(\delta^i_j + t\frac{\partial f^i}{\partial q^j})\dot{q}^j\frac{\partial}{\partial q^i} + \text{ higher order} \\ &= \sum_{i=1}^{n}\dot{q}^i\frac{\partial}{\partial q^i} + t\sum_{i,j=1}^{n}\frac{\partial f^i}{\partial q^j}\frac{\partial}{\partial q^i} + \text{ higher order.} \end{aligned}$$

We are going to compute

$$\begin{aligned} (T\exp(tX))^*L(q,\dot{q}) &= L((\exp(tX)(q), T\exp(tX)(\dot{q})) \\ &= L(q + tf(q), \sum_{i=1}^{n}\dot{q}^i\frac{\partial}{\partial q^i} + t\sum_{i,j=1}^{n}\frac{\partial f^i}{\partial q^j}\dot{q}^j\frac{\partial}{\partial q^i}) + \text{ higher order.} \\ &= L(q + tf(q), \dot{q} + tDf(\dot{q})) + \text{ higher order.} \end{aligned}$$

Consider now the action integral $I = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))dt$. Since $= L$, it follows that

$$\begin{aligned} I &= \int_{t_0}^{t_1} L(q(t), \dot{q}(t))dt = \int_{t_0}^{t_1}(T(\exp sX)^*L(q(t), \dot{q}(t))dt \\ &= \int_{t_0}^{t_1} L(q(t) + sf(q(t)), \dot{q}(t) + s(Df)(\dot{q}(t)))dt + \text{higher order in } s. \end{aligned}$$

Differentiating with respect to s and setting $s = 0$, we get

$$0 = \int_{t_0}^{t_1}\left(\frac{\partial}{\partial q}L(q(t), \dot{q}(t))f(q(t)) + \frac{\partial}{\partial \dot{q}}L(q(t), \dot{q}(t))Df(\dot{q}(t)))\right)dt.$$

However,

$$(Df)(\dot{q}(t)) = \frac{d}{dt}f(q(t)).$$

Hence,

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} \left(\frac{\partial}{\partial q} L(q(t),\dot{q}(t)) f(q(t)) + \frac{\partial}{\partial \dot{q}} L(q(t),\dot{q}(t)) \frac{d}{dt} f(q(t)) \right) dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial}{\partial q} L(q(t),\dot{q}(t)) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}} L(q(t),\dot{q}(t)) \right) f(q(t)) dt + \\ & \quad + \frac{\partial}{\partial \dot{q}} L(q(t_1),\dot{q}(t_1)) f(q(t_1)) - \frac{\partial}{\partial \dot{q}} L(q(t_0),\dot{q}(t_0)) f(q(t_0)). \end{aligned}$$

The integrand vanishes if the curve $t \longmapsto q(t)$ satisfies the Lagrange equations. In this case

$$\frac{\partial}{\partial \dot{q}} L(q(t_1),\dot{q}(t_1)) f(q(t_1)) = \frac{\partial}{\partial \dot{q}} L(q(t_0),\dot{q}(t_0)) f(q(t_0)).$$

Therefore, $\frac{\partial}{\partial \dot{q}} L(q(t),\dot{q}(t)) f(q(t))$ is constant along any curve $t \longmapsto q(t)$ satisfying the Lagrange equations. Thus, is a constant of motion. It is denoted by $J_X(q,\dot{q})$.

It remains to write the obtained constant of motion in geometric terms. Introducing the Legendre transformation $\mathfrak{L} : TM \to T^*M : (q,\dot{q}) \longmapsto (q,p)$ where $p = \frac{\partial}{\partial \dot{q}} L(q,\dot{q})$, we can pull-back the canonical 1-form $\theta = pdq$ of T^*M to TM obtaining

$$\mathfrak{L}^*\theta = \frac{\partial}{\partial \dot{q}} L(q,\dot{q}) dq.$$

Clearly,

$$J_X = \mathfrak{L}^*\theta(X) = \langle \mathfrak{L}^*\theta \mid X \rangle.$$