# Chapter 5

## Riemannian Manifolds

## 1 Tensors

Let $M$ be a manifold of dimension $n$. $C^\infty(M)$ the ring of smooth functions on $M$, $= \mathrm{Der}C^\infty(M)$ Recall that the space $\mathfrak{X}(M)$ of vector fields and the space $\Omega^1(M)$ of 1-forms on $M$ are modules over $C^\infty(M)$. This means that vector fields and 1-forms can be multiplied by functions in $C^\infty(M)$.

**Definition 1** *A covariant $k$-tensor field on $M$ is a $k$-linear map $T$ from $\mathfrak{X}(M)$ to $C^\infty(M)$ such that, for every $X_1, ..., X_k \in \mathfrak{X}(M)$, $i = 1, ..., k$, and $f \in C^\infty(M)$,*

$$T(X_1, ..., fX_i, ..., X_k) = fT(X_1, ..., X_i, ..., X_k).$$

**Definition 2** *A contravariant $l$-tensor field on $M$ is a $l$-linear map $T$ from $\Omega^1(M)$ to $C^\infty(M)$ such that, for every $\omega_1, ..., \omega_l \in \Omega^1(M)$, $i = 1, ..., l$, and $f \in C^\infty(M)$,*

$$T(\omega_1, ..., f\omega_i, ..., \omega_l) = fT(\omega_1, ..., \omega_i, ..., \omega_l). \tag{1}$$

**Definition 3** *A mixed tensor field of type (k,l) is a map*

$$T : (X^1, ..., X^k, \omega_1, ..., \omega_l) \mapsto T(X^1, ..., X^k, \omega_1, ..., \omega_l) \tag{2}$$

*which is $k$-linear in vector fields $X^1, ..., X^k$, $l$-linear in forms $\omega_1, ..., \omega_l$ and such that,*

$$T(X^1, ..., f_iX_i, ..., X^k, \omega_1, ..., f_j\omega_j, ...\omega_l) = f_if_jT(X^1, ..., X_i, ..., X^k, \omega_1, ..., \omega_j, ..., \omega_l) \tag{3}$$

Differential $k$-forms on $M$ are examples of covariant $k$-tensor field on $M$. Vector fields on $M$ are examples of contravariant 1-tensors on $M$. A vector field $X$ on $M$ associates ot a 1-form $\omega$ on $M$ a function $\omega(X)$ on $M$. Mixed tensor fields of type $(k, 1)$ are often referred to as covariant $k$-tensor fields with values in $\mathfrak{X}(M)$. Examples of mixed tensors will be given in subsequent sections.

The condition (1) implies that, for every $x \in M$, a covariant $k$-tensor field $T$ on $M$ defines a $k$-linear map $T(x)$ from $T_xM$ to $\mathbb{R}$ such that

$$T(X_1, ..., X_k)(x) = T(x)(X_1(x), ..., X_k(x)).$$

Similarly, a contravariant $l$-tensor field $T$ defines an $l$-linear map from $T_x^*M$ to $\mathbb{R}$ such that

$$T(\omega_1, ..., \omega_l)(x) = T(x)(\omega_1(x), ..., \omega_l(x)).$$

Moreover, a mixed tensor field $T$ of type $(k, l)$ gives a map $T(x)$ such that

$$T(x) : (X^1(x), ..., X^k(x), \omega_1(x), ..., \omega_l(x))) \mapsto T(X^1, ..., X^k, \omega_1, ..., \omega_l)(x).$$

In terms of local coordinates $q^1, ..., q^k$ on $U$, a tensor field of type $(k, l)$ is given as

$$T_{|U} = \sum_{i_1,...,i_l=1}^{n} \sum_{j_1,...,j_k=1}^{n} T^{i_1...i_l}_{j_1...j_k} dq^{j_1}...dq^{j_k} \frac{\partial}{\partial q^{i_1}} ... \frac{\partial}{\partial q^{i_l}},$$

where the components $T^{i_1...i_l}_{j_1...j_k}$ are functions on $U$. If $\psi : U \to \mathbb{R}^n$ is another chart with coordinates $p^i$, for $i = 1..., n$, then

$$T_{|U} = \sum_{i'_1,...,i'_l=1}^{n} \sum_{j'_1,...,j'_k=1}^{n} T^{i'_1...i'_l}_{j'_1...j'_k} dp^{j'_1}...dp^{j'_k} \frac{\partial}{\partial p^{i'_1}} ... \frac{\partial}{\partial p^{i'_l}}$$

$dq^i = \sum_{j=1}^{n} \frac{\partial q^i}{\partial p^j} dp^j$ and

$$T^{i'_1...i'_l}_{j'_1...j'_k} = \sum_{j'_1,...,j'_k=1}^{n} T^{i_1...i_l}_{j_1...j_k} \frac{\partial q^{j_1}}{\partial p_{j'_1}} ... \frac{\partial q^{j_k}}{\partial p_{j'_k}} \frac{\partial p^{i'_1}}{\partial q^{i_1}} ... \frac{\partial p^{i'_l}}{\partial q^{i_l}}.$$

## 2 Riemannian metric

The Euclidean space $\mathbb{R}^n$ is endowed with the Euclidean scalar product, which we will denote by $g_{\mathbb{R}^n}$, such that, for $\vec{u} = (u^1, ..., u^n)$ and $\vec{v} = (v^1, ..., v^n)$

$$g_{\mathbb{R}^n}(\vec{u}, \vec{v}) = \sum_{i=1}^{n} u^i v^i.$$

Generalization of the notion of the scalar product on $\mathbb{R}^n$ leads to the notion of a Riemannian metric on a manifold.

**Definition 4** *A Riemannian metric on $M$ is a covariant 2-tensor $g$ on $M$ such that, for every $X \in \mathfrak{X}(M)$ and $x \in MT$,*

$$g(X, X) \geq 0 \text{ and } g(X, X)(x) = 0 \text{ only if } X(x) = 0. \quad (4)$$

The property (4) is usually expressed by saying that $g$ is positive definite. The number

$$\|X(x)\| = \sqrt{g(X, X)(x)} = \sqrt{g(x)(X(x), X(x))}$$

is interpreted as the length of the vector $X(x)$. In terms of coordinates $q^1, ..., q^n$ on $U$,

$$g = \sum_{i,j=1}^{n} g_{ij} dq^i dq^j,$$

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial q^i},$$

and

$$\|X\| = \sqrt{g(X,X)} = \sqrt{\sum_{i,j=1}^{n} g_{ij} X^i X^j}.$$

A manifold $M$ endowed with a Riemannian metric is called a *Riemannian manifold*.

Let $\gamma : [a,b] \to M$ be a curve in $M$. The derived map of $\gamma$ is $T\gamma : T[a,b] \to TM$. Since $[a,b]$ is an interval in $\mathbb{R}$, $T[a,b] = [a,b] \times \mathbb{R}$. The map $\dot{\gamma} : [a,b] \to TM$,such that

$$\dot{\gamma}(t) = T\gamma(t,1)$$

for every $t \in [a,b]$, is called the tangent vector of the curve $\gamma$.

**Definition 5** *The length of a curve $\gamma : [a,b] \to M$ is given by*

$$L[\gamma] = \int_a^b \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

If $\gamma[a,b]$ is contained in the domain $U$ of a coordinate chart with coordinates $q^1, ..., q^n$, then we write $q^i(t) = q^i(\gamma(t))$ and $\dot{q}^i(t) = \frac{d}{dt} q^i(t)$. With this notation, $\dot{\gamma}(t) = (\dot{q}^1(t), ..., \dot{q}^n(t))$ and

$$L[\gamma] = \int_a^b \sqrt{\sum_{i,j=1}^{n} g_{ij}(q^1(t), ..., q^n(t)) \dot{q}^i(t) \dot{q}^j(t)} dt.$$

**Definition 6** *A curve $\gamma : [a,b] \to M$ is a geodesics of the Riemannian metric $g$ on $M$ if it is a critical point of the integral $L[\gamma]$ under variations of $\gamma$ which preserve the endpoints.*

We are going to derive differential equations for a geodesics contained in the domain $U$ of a coordinate chart on $M$. As we have said $\gamma(t) = (q^1(t), ..., q^n(t))$. We denote by

$$\gamma(t) + \delta\gamma(t) = (q^1(t) + \delta q^1(t), ..., q^n(t) + \delta q^n(t))$$

a variation of $\gamma$. The assumption that the endpoints are fixed implies that

$$\delta q^1(a) = ... = \delta q^n(a) = 0 \text{ and } \delta q^1(b) = ... = \delta q^n(b) = 0.$$

Moreover, we have

$$\begin{aligned}
\delta(\dot{q}^1(t), ..., \dot{q}^n(t)) &= \delta\dot{\gamma}(t) = \frac{d}{dt}(\gamma(t) + \delta\gamma(t)) - \dot{\gamma}(t) = \frac{d}{dt}(\delta q^1(t), ..., \delta q^n(t)) \\
&= (\frac{d}{dt}\delta q^1(t), ..., \frac{d}{dt}\delta q^n(t)).
\end{aligned}$$

In other words,

$$\delta\dot{q}^i(t) = \frac{d}{dt}\delta q^i(t), \text{ for } i = 1, ..., n.$$

Hence,

$$\begin{aligned}
\delta L[\gamma] &= \int_a^b \delta\left(\sqrt{\sum_{i,j=1}^n g_{ij}(q^1(t), ..., q^n(t))\dot{q}^i(t)\dot{q}^j(t)}\right) dt \\
& The \\
&= \int_a^b \left(2\sqrt{\sum_{i,j=1}^n g_{pq}(q^1(t), ..., q^n(t))\dot{q}^p(t)\dot{q}^q(t)}\right)^{-1} \left(\delta \sum_{i,j=1}^n g_{ij}(q^1(t), ..., q^n(t))\dot{q}^i(t)\dot{q}^j(t)\right) dt \\
&= \int_a^b \left(2\sqrt{\sum_{i,j=1}^n g_{pq}\dot{q}^p\dot{q}^q}\right)^{-1} \sum_{i,j=1}^n \left(2g_{ij}\dot{q}^i\delta\dot{q}^j + \sum_{k=1}^n \frac{\partial g_{ij}}{\partial q^k}\delta q^k\dot{q}^i\dot{q}^j\right) dt \\
&= \int_a^b \left(2\sqrt{\sum_{i,j=1}^n g_{pq}\dot{q}^p\dot{q}^q}\right)^{-1} \sum_{i,j=1}^n \left(2g_{ij}\dot{q}^i\frac{d}{dt}\delta q^j + \sum_{k=1}^n \frac{\partial g_{ij}}{\partial q^k}\delta q^k\dot{q}^i\dot{q}^j\right) dt \\
&= \int_a^b \frac{d}{dt}\left[\left(2\sqrt{\sum_{i,j=1}^n g_{ij}\dot{q}^i\dot{q}^j}\right)^{-1} \sum_{i,j=1}^n 2g_{ij}\dot{q}^i\delta q^j\right] dt + \\
&\quad - \int_a^b \left\{\frac{d}{dt}\left[\left(2\sqrt{\sum_{i,j=1}^n g_{ij}\dot{q}^i\dot{q}^j}\right)^{-1} \sum_{i,j=1}^n 2g_{ij}\dot{q}^i\right]\delta q^j - \left(2\sqrt{\sum_{i,j=1}^n g_{pq}\dot{q}^p\dot{q}^q}\right)^{-1} \sum_{i,j,k=1}^n \frac{\partial g_{ij}}{\partial q^k}\delta q^k\dot{q}^i\dot{q}^j\right\} \\
&= \left[\left(2\sqrt{\sum_{i,j=1}^n g_{ij}\dot{q}^i\dot{q}^j}\right)^{-1} \sum_{i,j=1}^n 2g_{ij}\dot{q}^i\delta q^j\right]_{t=a}^{t=b}. \\
&\quad - \int_a^b \left\{\frac{d}{dt}\left[\left(2\sqrt{\sum_{i,j=1}^n g_{pq}\dot{q}^p\dot{q}^q}\right)^{-1} \sum_{i,j=1}^n 2g_{ij}\dot{q}^i\right]\delta q^j - \left(2\sqrt{\sum_{i,j=1}^n g_{pq}\dot{q}^p\dot{q}^q}\right)^{-1} \sum_{i.j.k=1}^n \frac{\partial g_{ij}}{\partial q^k}\delta q^k\dot{q}^i\dot{q}^j\right\}
\end{aligned}$$

The boundary term vanishes because $\delta q^i(a) = \delta q^i(b)$ for $i = 1, ..., n$. Hence,

$$\delta L[\gamma] = -\int_a^b \left\{ \frac{d}{dt} \left[ \left( 2\sqrt{\sum_{i,j=1}^{n} g_{pq}\dot{q}^p\dot{q}^q} \right)^{-1} \sum_{i,j=1}^{n} 2g_{ij}\dot{q}^i \right] \delta q^j - \sum_{i,j,k=1}^{n} \frac{\partial g_{ij}}{\partial q^k} \delta q^k \dot{q}^i \dot{q}^j \right\} dt$$

$$= -\int_a^b \left\{ \frac{d}{dt} \left[ \left( \sqrt{\sum_{i,j=1}^{n} g_{pq}\dot{q}^p\dot{q}^q} \right)^{-1} \sum_{i,k=1}^{n} g_{ik}\dot{q}^i \right] - \sum_{i,j,k=1}^{n} \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \dot{q}^j \right\} \delta q^k dt.$$

The requirement that $\gamma$ is a critical point of $L$ with respect to all variations of $\gamma$ vanishing at the endpoints implies that the integrand must vanish. Hence, we get a differential equations

$$\frac{d}{dt} \left[ \left( \sqrt{\sum_{i,j=1}^{n} g_{pq}\dot{q}^p\dot{q}^q} \right)^{-1} \sum_{i,j=1}^{n} g_{ik}\dot{q}^i \right] - \left( 2\sqrt{\sum_{i,j=1}^{n} g_{pq}\dot{q}^p\dot{q}^q} \right)^{-1} \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \dot{q}^j = 0. \quad (5)$$

This equation takes a very simple form if we assume that the curve $\gamma$ is parametrized by its arc-length which we denote by

$$s(t) = \int_a^t \sqrt{\sum_{i,j=1}^{n} g_{ij}(q^1(t'), ..., q^n(t'))\dot{q}^i(t')\dot{q}^j(t')dt'}.$$

Differentiating this equation with respect to $s$ we get

$$\frac{ds}{dt} = \sqrt{\sum_{i,j=1}^{n} g_{ij}(q^1(t), ..., q^n(t))\dot{q}^i(t)\dot{q}^j(t)}.$$

In particular,

$$\sqrt{\sum_{i,j=1}^{n} g_{ij} \frac{dq^i}{ds} \frac{dq^j}{ds}} = 1.$$

Hence,

$$\frac{d}{ds} \left[ \sum_{i=1}^{n} g_{ik} \frac{dq^i}{ds} \right] - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial q^k} \frac{dq^i}{ds} \frac{dq^j}{ds} = 0. \quad (6)$$

Let $(g^{ij})$ be a symmetric matrix such that

$$\sum_{j=1}^{n} g^{ij} g_{jk} = \delta^j_k.$$

The collection $(g^{ij})$ gives components of a contravariant symmetric 2-tensor called the *contravariant metric tensor* associated to the metric $g$. We can rewrite eqauation (6) as follows.

$$\begin{aligned}
&\frac{d}{ds}\left[\sum_{i=1}^{n} g_{ik}\frac{dq^i}{ds}\right] - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial g_{ij}}{\partial q^k}\frac{dq^i}{ds}\frac{dq^j}{ds} \\
= &\sum_{i=1}^{n} g_{ik}\frac{d^2q^i}{ds^2} + \sum_{i,j=1}^{n}\frac{\partial g_{ik}}{\partial x^j}\frac{dq^i}{ds}\frac{dq^j}{ds} - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial g_{ij}}{\partial q^k}\frac{dq^i}{ds}\frac{dq^j}{ds} \\
= &\sum_{i=1}^{n} g_{ik}\frac{d^2q^i}{ds^2} + \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial q^k}\right)\frac{dq^i}{ds}\frac{dq^j}{ds} = 0.
\end{aligned}$$

Contracting with $g^{lk}$ we get

$$\frac{d^2q^l}{ds^2} + \frac{1}{2}\sum_{i,j,k=1}^{n} g^{lk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial q^k}\right)\frac{dq^i}{ds}\frac{dq^j}{ds} = 0.$$

The quantities

$$\Gamma^l_{ij} = \frac{1}{2}\sum_{k=1}^{n} g^{lk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial q^k}\right)$$

are called the *Christoffel symbols* (of the second type) corresponding to the metric $g$. It should be noted that the Christoffel symbols $\Gamma^l_{ij}$ do not define a tensor on $M$. The equation of geodesics corresponding to the metric $g$ are

$$\frac{d^2q^l}{ds^2} + \sum_{i,j=1}^{n}\Gamma^l_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds} = 0.$$

# 3 Linear connection.

We have no natural way of comparing vectors at different points of a manifold. Hence, if we want to write expressions involving derivatives of vector fields we need to introduce the notion of a linear connection on $M$.

**Definition 7** *A linear connection on $M$ is defined by a bilinear mapping*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y$$

*such that*

$$\nabla_{fX}Y = f\nabla_X Y \text{ and } \nabla_X(fY) = (X(f))Y + f\nabla_X Y. \tag{7}$$

*for every $f \in C^\infty(M)$.*

The vector field $\nabla_X Y$ is called the *covariant derivative* of $Y$ in the direction of $X$. The appearance of the derivative of $f$ on the right hand side of equation (7) shows that $\nabla$ is not a tensorial quantity.

**Proposition 8** *The map*

$$\Theta : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$$

*is an antisymmetric covariant 2-tensor on $M$.*

**Proof.** Clearly, $\Theta$ is bilinear and antisymmetric. For every $f \in C^\infty(M)$,

$$\begin{aligned}\Theta(fX, Y) &= \nabla_{fX} Y - \nabla_Y fX - [fX, Y] = f\nabla_X Y - (Y(f))X - f\nabla_Y X - f[X, Y] + Y(f)X \\ &= f\Theta(X, Y).\end{aligned}$$

Hence, $\Theta$ is a covariant 2-tensor. ∎

The 2-tensor $\Theta$ is called the torsion of the connection $\nabla$. A connection $\nabla$ such that $\Theta = 0$ is called symmetric or torsion-free.

Given a linear connection $\nabla$ on $M$ we can define covariant derivatives of any tensorial quantities by requirement that the covariant derivative preserves the evaluation. In particular, if $\omega$ is a 1-form on $M$, then $\nabla_X \omega$ is a 1-form defined by the condition that

$$X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$$

for every $Y \in \mathfrak{X}(M)$. Hence,

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega((\nabla_X Y) \ \forall \ Y \in \mathfrak{X}(M).$$

Extensions to multi-tensors is given by multi-linearity. In particular, if $g$ is a metric on $M$, then

$$(\nabla_X g)(Y_1, Y_2) = X(g(Y_1, Y_2)) - g(\nabla_X Y_1, Y_2) - g(Y_1, \nabla_X Y_2). \tag{8a}$$

In order to describe the linear connection $\nabla$ in terms of a chart with coordinates $q^1, ..., q^n$, it suffices to exibit the covariant derivatives of the basic vector fields $\frac{\partial}{\partial q^1}, ..., \frac{\partial}{\partial q^n}$. Let

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \sum_{k=1}^{n} C_{ij}^k \frac{\partial}{\partial q^k}.$$

The functions $C_{ij}^k$ are called the connection coefficients. If $X = \sum_{i=1}^n X^i \frac{\partial}{\partial q^i}$ and $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial q^i}$ then

$$\begin{aligned}
\nabla_X Y &= \nabla_{\sum_{i=1}^n X^i \frac{\partial}{\partial q^i}} \left( \sum_{j=1}^n Y^j \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i=1}^n X^i \nabla_{\frac{\partial}{\partial q^i}} \left( \sum_{j=1}^n Y^j \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i,j=1}^n X^i \nabla_{\frac{\partial}{\partial q^i}} \left( Y^j \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i,j=1}^n \left( X^i \frac{\partial Y^j}{\partial q^i} \frac{\partial}{\partial q^j} + X^i Y^J \nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i,j=1}^n \left( X^i \frac{\partial Y^j}{\partial q^i} \frac{\partial}{\partial q^j} + X^i Y^j \sum_{k=1}^n C_{ij}^k \frac{\partial}{\partial q^k} \right).
\end{aligned}$$

It follows that

$$\Theta(X, Y) = \sum_{i,j,k} (C_{ij}^k - C_{ji}^k) X^i Y^j \frac{\partial}{\partial q^k}.$$

In other words, the components of the torsion tensor are given by

$$\Theta_{ij}^k = C_{ij}^k - C_{ji}^k.$$

Suppose $\nabla$ is a symmetric connection. That is $C_{ij}^k = C_{ji}^k$ in every coordinate system. We want to compute the covariant derivative of the metric tensor $g = \sum_{i,j=1}^n g_{ij} dq^i dq^j$. Equation (8a) with $X = \frac{\partial}{\partial q^k}$, $Y_1 = \frac{\partial}{\partial q^i}$ and $Y_2 = \frac{\partial}{\partial q^j}$ gives .

$$\begin{aligned}
(\nabla_{\frac{\partial}{\partial q^k}} g)_{ij} &= (\nabla_{\frac{\partial}{\partial q^k}} g)(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) \\
&= \frac{\partial}{\partial q^k}(g_{ij}) - g(\sum_{l=1}^n C_{ki}^l \frac{\partial}{\partial q^l}, \frac{\partial}{\partial q^j}) - g(\frac{\partial}{\partial q^i}, \sum_{l=1}^n C_{kj}^l \frac{\partial}{\partial q^l}) \\
&= \frac{\partial g_{ij}}{\partial q^k} - \sum_{l=1}^n C_{ki}^l g_{lj} - \sum_{l=1}^n C_{kj}^l g_{il}.
\end{aligned}$$

**Theorem 9** *Let $g$ be a Riemannian metric on $M$. There exists a unique symmetric linear connection $\nabla$ on $M$ such that $\nabla g = 0$.*

**Proof.** $\nabla g = 0$ implies that

$$\frac{\partial g_{ij}}{\partial q^k} - \sum_{l=1}^n C_{ki}^l g_{lj} - \sum_{l=1}^n C_{kj}^l g_{il} = 0.$$

Using the symmetry of $C^l_{ij}$, we can write

$$\begin{aligned}\frac{\partial g_{ik}}{\partial q^j} - \sum_{l=1}^n C^l_{ji} g_{lk} - \sum_{l=1}^n C^l_{kj} g_{il} &= 0,\\ \frac{\partial g_{kj}}{\partial q^i} - \sum_{l=1}^n C^l_{ki} g_{lj} - \sum_{l=1}^n C^l_{ij} g_{kl} &= 0.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} &= \sum_{l=1}^n C^l_{ki} g_{lj} + \sum_{l=1}^n C^l_{ij} g_{kl} + \sum_{l=1}^n C^l_{ji} g_{lk} + \sum_{l=1}^n C^l_{kj} g_{il} - \sum_{l=1}^n C^l_{ki} g_{lj} - \sum_{l=1}^n C^l_{kj} g_{il}\\ &= 2\sum_{l=1}^n C^l_{ij} g_{kl}.\end{aligned}$$

Contracting with $g^{km}$ we get

$$C^m_{ij} = \frac{1}{2}\sum_{k=1}^n \left( \frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right) = \Gamma^m_{ij}.$$

■

**Remark 10** *Remark that the connection coefficients given above are precisely the Chtistoffel symbols obtained when we considered the equation of geodesics. The connection defined here is called the Levi-Civitta connection of the Riemannian metric $g$.*

Let $\nabla$ be a linear connection on $M$ and $\gamma : [a, b] \to M$ a curve in $M$. Consider a curve $\tilde{\gamma} : [a, b] \to TM$ such that $\tau(\tilde{\gamma}(t)) = \gamma(t)$. We say that the curve $\tilde{\gamma}$ in $TM$ covers $\gamma$. For each $t \in [a, b]$, we have a vector $\tilde{\gamma}(t)$ attached $\gamma(t) \in M$. We can compute $\nabla_{\dot{\gamma}(t)}\tilde{\gamma}(t)$. In terms of coordinates $q^1, ..., q^n$ in $M$, $\gamma(t) = (q^1(t), ..., q^m(t))$, and

$$\tilde{\gamma}(t) = \sum_{i=1}^n u^i(t) \frac{\partial}{\partial q^i} \text{ and } \dot{\gamma}(t) = \sum_{i=1}^n \dot{q}^i(t) \frac{\partial}{\partial q^i},$$

where $\dot{q}^i(t) = \frac{d}{dt} q^i(t)$. Then,

$$\begin{aligned}\nabla_{\dot{\gamma}(t)}\tilde{\gamma}(t) &= \nabla_{\left(\sum_{i=1}^n \dot{q}^i(t)\frac{\partial}{\partial q^i}\right)} \sum_{j=1}^n u^j(t) \frac{\partial}{\partial q^j}\\ &= \sum_{j=1}^n \frac{du^j(t)}{dt} \frac{\partial}{\partial q^j} + \sum_{i,j=1}^n \dot{q}^i u^j \nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j}\\ &= \sum_{j=1}^n \frac{du^j(t)}{dt} \frac{\partial}{\partial q^j} + \sum_{i,j,k=1}^n \dot{q}^i u^j \Gamma^k_{ij} \frac{\partial}{\partial q^k}.\end{aligned}$$

**Definition 11** *If $\nabla_{\dot\gamma(t)}\tilde\gamma(t) = 0$ for all $t \in [a,b]$, we say that the curve $\tilde\gamma(t)$ gives the parallel transport along the curve $t \mapsto \gamma(t)$ of the vector $\tilde\gamma(a) \in T_{\gamma(a)}M$.*

Consider now the equation of geodesics $\frac{d^2q^l}{ds^2} + \sum_{i,j=1}^n \Gamma^l_{ij}\frac{dq^i}{ds}\frac{dq^j}{ds} = 0$. In this case we take $\tilde\gamma = \dot\gamma$, and parametrize the curve $\gamma$ by arc-length $s$. Therefore,

$$\nabla_{\dot\gamma(t)}\dot\gamma(s) = \sum_{j=1}^n \frac{d\dot q(s)}{ds}\frac{\partial}{\partial q^j} + \sum_{i,j,k=1}^n \dot q^i \dot q^j \Gamma^k_{ij}\frac{\partial}{\partial q^k}.$$

Therefore the equation of geodesics is equivalent to

$$\nabla_{\dot\gamma(t)}\dot\gamma(s) = 0.$$

In other words, a curve $t \mapsto \gamma(t)$ is a geodesics if its tangent vector $\dot\gamma(t)$ is parallely transported along $\gamma$.

# 4 Curvature

The operators $\nabla_X$ and $\nabla_Y$ need not commute even if the vector fields $X$ and $Y$ commute. Consider the map

$$R : (\mathfrak{X}(M))^3 \to \mathfrak{X}(M) : (X,Y,Z) \mapsto R(X,Y,Z) = \nabla_X(\nabla_Y(Z)) - \nabla_Y(\nabla_X(Z)) - \nabla_{[X,Y]}Z. \tag{9}$$

**Proposition 12** *$R$ is a covariant 3-tensor field with values in $\mathfrak{X}(M)$. In other words, $R$ is a mixed tensor field of type $(3,1)$.*

**Proof.** We need to show that, for each $X$, $Y$, $Z$ in $\mathfrak{X}(M)$ and $f_1, f_2, f_3 \in C^\infty(M)$,

$$\nabla_{f_1X}(\nabla_{f_2Y}(f_3Z)) - \nabla_{f_2Y}(\nabla_{f_1X}(f_3Z)) - \nabla_{[f_1X,f_2Y]}f_3Z = f_1f_2f_3\left(\nabla_Y(\nabla_XZ) - \nabla_X(\nabla_YZ) - [X,Y]Z\right).$$

We consider each factor separately.

$$\begin{aligned} & \nabla_{f_1X}(\nabla_Y(Z)) - \nabla_Y(\nabla_{f_1X}(Z)) - \nabla_{[f_1X,Y]}Z \\ = \; & f_1\nabla_X(\nabla_Y(Z)) - \nabla_Y(f_1\nabla_X(Z)) - \nabla_{f_1[X,Y]-Y(f_1)X}Z \\ = \; & f_1\nabla_X(\nabla_Y(Z)) - f_1\nabla_Y(\nabla_X(Z)) - Y(f_1)\nabla_XZ - f_1\nabla_{[X,Y]} + Y(f_1)\nabla_XZ \\ & f_1\left(\nabla_Y(\nabla_XZ) - \nabla_X(\nabla_YZ) - [X,Y]Z\right). \end{aligned}$$

By symmetry we have

$$\nabla_X(\nabla_{f_2Y}(Z)) - \nabla_{f_2Y}(\nabla_X(Z)) - \nabla_{[X,f_2Y]}Z = f_2\left(\nabla_Y(\nabla_XZ) - \nabla_X(\nabla_YZ) - [X,Y]Z\right).$$

Finally

$$
\begin{aligned}
& \nabla_X(\nabla_Y(f_3 Z)) - \nabla_Y(\nabla_X(f_3 Z)) - \nabla_{[X,Y]} f_3 Z \\
= \; & \nabla_X(f_3 \nabla_Y(Z) + Y(f_3)Z) - \nabla_Y(f_3 \nabla_X(Z) + X(f_3)Z) - f_3 \nabla_{[X,Y]} Z - [X,Y](f_3)Z \\
= \; & f_3 \nabla_X(\nabla_Y(Z)) + X(f_3)\nabla_Y Z + Y(f_3)\nabla_X Z + X(Y(f_3)) - f_3 \nabla_Y(\nabla_X(Z)) - Y(f_3)\nabla_X Z + \\
& -Y(X(f_3))Z - X(f_3)\nabla_Y Z - f_3 \nabla_{[X,Y]} Z - [X,Y](f_3)Z \\
= \; & f_3 \left(\nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) - [X,Y]Z\right).
\end{aligned}
$$

■

**Definition 13** *The tensor field R defined in equation (9) is called the Riemann tensor of the Levi-Civita connection of the metric g.*

A Riemannian manifold $(M, g)$ of dimension $n$ is said to be flat if there is a diffeomorphism $\varphi$ of $M$ onto an open subset of $\mathbb{R}^n$ such that

$$\varphi^* g_{\mathbb{R}^n} = g. \tag{10}$$

Equation (10) mean that

$$g(u, v) = g_{\mathbb{R}^n}(T\varphi(u), T\varphi(v))$$

for every $x \in M$ and $u, v \in T_x M$.

We say that $(M, g)$ is *locally flat* if, for every $x \in M$ there is a neighbourhood $U$ of $x$ such that the restriction $g_{|U}$ of the metric $g$ to $U$ is flat. Thus, $(M, g)$ is locally flat if, for each $x \in M$, there is a chart $\varphi : U \to V$, where $U$ is a neighbourhood of $x$ in $M$ and $V$ is an open set in $\mathbb{R}^n$, such that the components of the metric in this chart are

$$g_{ij} = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} .$$

**Theorem 14** *A Riemannian manifold is locally flat if and only if the Riemann tensor vanishes identically.*

**Proof.** See Mike Spivak, *Differential Geometry*, vol 2, Chapter 4, Theorem 13. ■

**Problem 15** *Find the Riemann tensor of the 2-sphere of radius r*

$$S^2_r = \{(x, y, z) \in R^3 \mid x^2 + y^2 + z^2 = r^2\}$$

*with metric g obtained as the pull-back of the Euclidean metric* $g_{\mathbb{R}^3}$ *by the inclusion map* $S^2 \hookrightarrow \mathbb{R}^3$.