Chapter 4

Differential Forms

1 Differential Forms

Let M be a manifold, $C^\infty(M)$ the differential structure of M. We denote by $\mathfrak{X}(M)$ the space of vector fields on M. Since M is a manifold, $\mathfrak{X}(M)$ is the same as the space of derivations of $C^\infty(M)$. In other words, $X(M) = \text{Der}C^\infty(M)$.

The space $\mathfrak{X}(M)$ is a Lie algebra with the Lie bracket $[X, Y]$ defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

for every $f \in C^\infty(M)$. Observe that the Lie bracket $[X, Y]$ is bilinear, antisymmetric and it satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X.Y]] = 0$$

for every $X, Y, Z \in \mathfrak{X}(M)$.

If $X_1, X_2 \in \mathfrak{X}(M)$ and $f_1, f_2 \in C^\infty(M)$ then $(f_1X_1 + f_2X_2) \in \mathfrak{X}(M)$ is defined by

$$(f_1X_1 + f_2X_2)h = f_1 \cdot X(f) + f_2 \cdot X(f)$$

for each $f \in C^\infty(M)$. We say that $\mathfrak{X}(M)$ is a *module* over $C^\infty(M)$. Moreover,

$$\begin{aligned} [f_1X_1, f_2X_2](f) &= (f_1X_1)(f_2 \cdot X_2(f)) - (f_2X_2)(f_1 \cdot X_1(f)) \\ &= f_1 \cdot X_1(f_2) \cdot X_2(f) + f_1 \cdot f_2 \cdot X_1(X_2(f)) + \\ &\quad -f_2 \cdot X_2(f_1) \cdot X_1(f) - f_2 \cdot f_1 \cdot X_2(X_1(f)). \end{aligned}$$

Hence,

$$[f_1X_1, f_2X_2] = (f_1 \cdot f_2)[X_1, X_2] + (f_1 \cdot X_1(f_2))X_2 - (f_2 \cdot X_2(f_1))X_1$$

for every $X_1, X_2 \in \mathfrak{X}(M)$ and $f_1, f_2 \in C^\infty(M)$.

Definition 1 *A differential k-form ω on M is a k-linear over $C^\infty(M)$ antisymmetric map from $\mathfrak{X}(M)$ to $C^\infty(M)$.*

In other words, for every $X_1, ..., X_k \in \mathfrak{X}(M)$, $\omega(X_1, ..., X_k) \in C^\infty(M)$. Moreover,

$$\omega(X_1, ..., X_i, ..., X_j, ...X_k) = -\omega(X_1, ..., X_j, ..., X_i, ...X_k), \quad (1)$$

and

$$\omega(X_1, ..., fX_i, ...X_k) = f \cdot \omega(X_1, ..., X_i, ...X_k). \quad (2)$$

We denote by $\Omega^k(M)$ the space of differential k-forms on M. It is a module over $C^\infty(M)$. For $\omega_1, \omega_2 \in \Omega^k(M)$ and $f_1, f_2 \in C^\infty(M)$, $f_1\omega_1 + f_2\omega_2$ is a k-form such that

$$(f_1\omega_1 + f_2\omega_2)(X_1, ..., X_k) = f_1 \cdot \omega_1(X_1, ..., X_k) + f_2 \cdot \omega_2(X_1, ..., X_k)$$

for every $X_1, ..., X_k \in \mathfrak{X}(M)$.

Proposition 2 *If $X_1, ..., X_k$ are linearly dependent then $\omega(X_1, ..., X_k) = 0$ for every $\omega \in \Omega^k(M)$.*

Proof. $X_1, ..., X_k$ are linearly dependent if the equation

$$\lambda_1 X_1 + ... + \lambda_k X_k = 0$$

has solutions $\lambda_1, ..., \lambda_k$ not all equal zero. If $\lambda_1 \neq 0$, we can solve locally for X_1 obtaining

$$X_1 = -(\lambda_2/\lambda_1)X_2 - ... - (\lambda_k/\lambda_1)X_k.$$

Hence,

$$\begin{aligned} \omega(X_1, X_2, ..., X_k) &= \omega((-(\lambda_2/\lambda_1)X_2 - ... - (\lambda_k/\lambda_1)X_k), X_2, ..., X_k) \\ &= -(\lambda_2/\lambda_1)\omega(X_2, X_2, ..., X_k) + ... + \\ &\quad -(\lambda_k/\lambda_1)\omega(X_k, X_2, ..., X_k) \\ &= 0. \end{aligned}$$

■

Corollary 3 *If $k > \dim M$ then $\Omega^k(M) = \{0\}$.*

A permutation of indices $1, ..., k$ is one-to-one map π from the set $\{1, ..., k\}$ onto itself. Permutations of $\{1, ..., k\}$ form of a group denoted $\mathfrak{S}_k$. A permutation π is a transposition of it interchanges two elements of the set $\{1, ..., k\}$ keeping all other elements fixed.A transposition interchanging i and j is denoted (i, j). Every permutation in $\mathfrak{S}_k$ can be presented as the composition of transpositions. For example $\pi : \{1, 2.3, 4\} \to \{2, 3, 4, 1\}$ can be written as $\pi = (4, 1)(3, 1)(2, 1)$. A permutation π is *even* if it can be written as an even number of transpositions. If π is the product of an odd number of transpositions, we say that π is *odd*. The signature of a permutation is defined by

$$\begin{aligned} \text{sign } \pi &= 1 \text{ if } \pi \text{ is even,} \\ \text{sign } \pi &= -1 \text{ if } \pi \text{ is odd.} \end{aligned}$$

Let $\omega_1 \in \Omega^k(M)$ and $\omega_2 \in \Omega^l(M)$. We can define the product $\omega_1 \wedge \omega_2 \in \Omega^{k+l}(M)$, called the *wedge product* or *exterior product*, as follows. For $X_1, ..., X_{k+l} \in \mathfrak{X}(M)$,

$$(\omega_1 \wedge \omega_2)(X_1, ..., X_{k+l}) = \sum_{\substack{\pi(1) < ... < \pi(k) \\ \pi(k+1) < ... < \pi(k+l)}} (\text{sign } \pi)\omega_1(X_{\pi(1)}, ..., X_{\pi(k)}) \cdot \omega_2(X_{\pi(k+1)}, ..., X_{\pi(k+l)}).$$

For example

$$\begin{aligned} (df \wedge dg)(X_1, X_2) &= (df)(X_1) \cdot (dg)(X_2) - (df)(X_2) \cdot dg(X_1) \\ &= X_1(f) \cdot X_2(g) - X_2(f) \cdot X_1(g). \end{aligned}$$

Remark 4 *If $\omega_1 \in \Omega^k(M)$ and $\omega_2 \in \Omega^l(M)$, then*

$$\omega_2 \wedge \omega_1 = (-1)^{kl} \omega_1 \wedge \omega_2$$

In particular, if k is odd, then

$$\omega_1 \wedge \omega_1 = 0.$$

The Lie bracket $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ induces an *exterior differential* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by

$$\begin{aligned} d\omega(X_0, X_1, ..., X_k) &= \frac{1}{k+1} \sum_{i=0}^{k} X_i(\omega(X_0, ..., \widehat{X}_i, ..., X_k)) + \\ &\quad + \frac{1}{k+1} \sum_{0 \le i < j \le k}^{k} (-1)^{i+j} (\omega([X_i, X_j], X_0, ..., \widehat{X}_i, ..., \widehat{X}_j, ..., X_k)), \end{aligned}$$

where the symbol $\widehat{\ }$ means that the term is omitted. In particular, if $\omega \in \Omega^1(M)$, then

$$d\omega(X_1, X_2) = \frac{1}{2} \left(X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2]\right).$$

Proposition 5 *For every $\omega \in \Omega^k(M)$,*

$$d(d\omega) = 0.$$

Proof. by calculation from the definition. One has to use the Jacobi identity of the Lie bracket ■

Jf $d\omega = 0$, we say that ω is *closed.* If ω there is a form ω_1 such that $\omega = d\omega$, we say that ω is *exact.* The proposition above states that an exact form is closed. The converse is true only locally, see Theorem 8 below.

If ω is a k-form on M and X is a vector field on M, we can define the left interior product $X \lrcorner \omega$ of ω by X to be the $(k-1)$-form such that

$$(X \lrcorner \omega)(X_1, ..., X_{k-1}) = \omega(X, X_1, ..., X_{k-1})$$

for every $X_1, ..., X_{k-1} \in \mathfrak{X}(M)$.

Notation 6 *In literature the left interior product* $X \lrcorner \omega$ *is also denoted* $i_X \omega$.

Recall that a vector field X on M can be identified with a map from M to the tangent bundle space TM of M associating to each $x \in M$ the derivation $X(x) \in T_xM$. Conversely, every vector field $X \in \mathfrak{X}(M)$ corresponds to a smooth map $\sigma : M \to TM$ such that $\tau \circ \sigma = identity$, where $\tau : TM \to M$ is the tangent bundle projection. Equation (2) implies that, for every $x \in M$, $\omega(X_1, X_2, ..., X_k)(x)$ depends only on $X_1(x), ..., X_k(x)$. Hence, there is a k-linear antisymmetric map $\omega(x)$ from T_xM to R such that

$$\omega(x)(X_1(x), X_2(x), ..., X_k(x)) = \omega(X_1, X_2, ..., X_k)(x).$$

Hence, we may identify $\omega \in \Omega^k(M)$ with the map $x \mapsto \omega(x)$. This identification allows for another operation on forms.

Suppose that $\varphi : N \to M$ is smooth and $T\varphi : TN \to TM$ is the tangent map of φ. Consider a k-form ω on M as the map associating to each $x \in M$ the k-linear map $\omega(x)$ on T_xM. Consider now $y \in N$, and let $x = \varphi(x)$. Then $T\varphi(T_yN)$ is a vector subspace of T_xM. Hence, we the restriction $\omega(x)_{|T\varphi(T_yN)}$ of $\omega(x)$ to $T\varphi(T_yN)$ is a k-linear map of $T\varphi(T_yN)$ to $\mathbb{R}$. Since the restriction $T\varphi_{|T_yN}$ of $T_y\varphi$ to T_yN is a linear isomorphism of T_yN onto $T\varphi(T_yN) \subseteq T_xM$ we get a k-linear map $\varphi^*\omega(y)$ from T_yN to $\mathbb{R}$ such that

$$\varphi^*\omega(y)(Y_1(y), ..., Y_k(y)) = \omega(x)(T\varphi(Y_1(y), ..., T\varphi(Y_k(y)))$$

for every $Y_1(y), ..., Y_k(y) \in T_yY$. The map $y \mapsto \varphi^*\omega(y)$ corresponds to a k-form $\varphi^*\omega$ on N called the *pull-back* of ω by φ.

Proposition 7 *Let* $\varphi : N \to M$ *be a smooth map and* $\omega \in \Omega^k(M)$. *Then*

$$\varphi^* d\omega = d\varphi^*\omega.$$

Let ω be a k-form on M and U be an open subset of M with the inclusion map $\iota : U \to M$. The pull-back $\iota^*\omega$ is a k-form on U called the restriction of ω to U and denoted $\omega_{|U}$.

Theorem 8 *Let* ω *be a closed* k*-form on* M. *Then, for each point* $x \in M$ *there exists a neighbourhood* U *of* x *and a form* ω_1 *on* U *such that* $\omega_{|U} = d\omega_1$.

Proof. is a consequence of the Poincaré Lemma in the next section. ■

2 Coordinate representation.

We consider first differential forms on $\mathbb{R}^n$. Vector fields on $\mathbb{R}^n$ are spanned over $C^\infty(\mathbb{R}^n)$ by the partial derivatives $\partial_1, ..., \partial_n$. In other words a vector field on R^n is of the form

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i},$$

where $X^1, ..., X^n$ are smooth functions on $\mathbb{R}^n$ called the components of X with reepect to the natural coordinates $(x^1, ..., x^n)$. In order to determine the value of a k-form ω on vector fields on R^n it suffices to know the value of ω on the basic vector fields $\partial_1, ..., \partial_n$.

Proposition 9 *Let ω be a k-form on $\mathbb{R}^n$. For every choice $i_1, ..., i_k$ of k distinct indices from $(1, ..., n)$ we set*

$$f_{i_1 ... i_k} = \omega \left(\frac{\partial}{\partial x^{i_1}}, ..., \frac{\partial}{\partial x^{i_k}} \right).$$

Then,

$$\omega = \sum_{1 \le i_1 < ... < i_k \le n} f_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}. \tag{3}$$

Moreover,

$$d\omega = \sum_{1 \le i_1 < ... < i_k \le n} \sum_{j=1}^{n} \frac{\partial f_{i_1 ... i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}. \tag{4}$$

A domain $D \subseteq R^n$ is *star-shaped* with respect to the origin O if, for every $(x^1, ..., x^n) \in D$ and $t \in [0, 1]$, the point $(tx^1, ..., tx^n)$ is in D.

Theorem 10 *(Poincaré Lemma). If D is open in R^n and star-shaped with respect to the origin O then every closed form on D is exact.*

Proof. We define a function I from k-forms to $(k-1)$-forms, for each k, such that $I(0) = 0$ and

$$\omega = I(d\omega) + d(I(\omega))$$

for every ω. This shows that, $d\omega = 0$ implies that $\omega = d(I(\omega))$.

Let

$$\omega = \sum_{1 \le i_1 < ... < i_k \le n} f_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

Since D is star-shaped, we can define

$$(I(\omega))(x) = \sum_{1 \le i_1 < ... < i_k \le n} \sum_{\alpha=1}^{k} (-1)^{\alpha - 1} \left(\int_0^1 t^{k-1} f_{i_1 ... i_k}(tx) dt \right) x^{i_\alpha} dx^{i_1} \wedge ... \wedge \widehat{dx^{i_\alpha}} \wedge ... \wedge dx^{i_k}$$

where $\widehat{dx^{i_\alpha}}$ means that dx^{i_α} is omitted. The proof that $\omega = I(d\omega)+d(I(\omega))$ is an elaborate computation (see M. Spivak, *Calculus on Manifolds*, the Benjamin/Cummings Publishing Company, Menlo Park, CA, 1965). ■

Let us introduce the notation

$$\delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} .$$

Then

$$\frac{\partial}{\partial x^j} \lrcorner dx^i = dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$$

and

$$\frac{\partial}{\partial x^j} \lrcorner \left(dx^{i_1} \wedge ... \wedge dx^{i_k}\right) = \sum_{l=1}^{k} (-1)^{l-1} \delta^{i_l}_j (dx^{i_1} \wedge ... \wedge \widehat{dx^{i_l}} \wedge ... \wedge dx^k)$$

where $\widehat{dx^{i_l}}$ means that dx^{i_l} is to be omitted. Further, if $X = \sum_i X^i \partial_i$, then

$$\begin{aligned}
X \lrcorner \omega &= X \lrcorner \sum_{1 \leq i_1 < ... < i_k \leq n} f_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}. \\
&= \sum_{1 \leq i_1 < ... < i_k \leq n} X \lrcorner (f_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}) \\
&= \sum_{1 \leq i_1 < ... < i_k \leq n} X^i \frac{\partial}{\partial x^i} \lrcorner (f_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}) \\
&= \sum_{1 \leq i_1 < ... < i_k \leq n} \sum_{i=1}^{n} X^i f_{i_1 ... i_k} \frac{\partial}{\partial x^i} \lrcorner (dx^{i_1} \wedge ... \wedge dx^{i_k}) \\
&= \sum_{1 \leq i_1 < ... < i_k \leq n} \sum_{i=1}^{n} X^i f_{i_1 ... i_k} \sum_{l=1}^{k} (-1)^{l-1} \delta^{i_l}_i (dx^{i_1} \wedge ... \widehat{dx^{i_l}} ... \wedge dx^{i_k}) \\
&= \sum_{1 \leq i_1 < ... < i_k \leq n} \sum_{l=1}^{k} (-1)^{l-1} X^{i_l} f_{i_1 ... i_k} (dx^{i_1} \wedge ... \widehat{dx^{i_l}} ... \wedge dx^{i_k}).
\end{aligned}$$

Similar reults hold in open subsets of $\mathbb{R}^n$.

Let U be an open subset of an n-dimensional manifold M and $\varphi : U \to V \subseteq R^n$ a chart on M with domain U. This means that V is an open subset of R^n and φ is a diffeomorphism with inverse $\varphi^{-1} : V \to U$. If ω is a k-form on M, then the restriction of ω to U is a k-form $\omega_{|U}$ on U which can be pulled back to V by φ^{-1} giving rise to a k-form $(\varphi^{-1})^* \omega_{|U}$. By Proposition 9

$$(\varphi^{-1})^* \omega_{|U} = \sum_{1 \leq i_1 < ... < i_k \leq n} f_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$$

where $f_{i_1\ldots i_k} \in C^\infty(V)$ and, for $i = 1, ..., k$, $x^i : V \to R$ is the i'th coordinate function on V. Hence,

$$\omega_{|U} = \sum_{1 \le i_1 < \ldots < i_k \le n} \varphi^*(f_{i_1\ldots i_k}) d(\varphi^* x^{i_1}) \wedge \ldots \wedge d(\varphi^* x^{i_k})$$

where $\varphi^*(f_{i_1\ldots i_k}) = f_{i_1\ldots i_k} \circ \varphi \in C^\infty(U)$ and $\varphi^* x^i = x^i \circ \varphi = q^i$ is the i'th coordinate on U.

Proposition 11 *If ω is a k-form on M and $\varphi : U \to V \subseteq R^n$ is a chart on M with coordinate functions $q^i = \varphi^* x^i$, then the restriction of ω to U is given by .*

$$\omega_{|U} = \sum_{1 \le i_1 < \ldots < i_k \le n} h_{i_1\ldots i_k} dq^{i_1} \wedge \ldots \wedge dq^{i_k}$$

where $h_{i_1\ldots i_k} \in C^\infty(U)$. Moreover,

$$d\omega_U = \sum_{1 \le i_1 < \ldots < i_k \le n} (dh_{i_1\ldots i_k}) \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_k}$$

and

$$\begin{aligned} X \lrcorner \omega_U &= \sum_{1 \le i_1 < \ldots < i_k \le n} h_{i_1\ldots i_k} X \lrcorner (dq^{i_1} \wedge \ldots \wedge dq^{i_k}) \\ &= \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{l=1}^{k} (-1)^{l-1} (X \lrcorner dq^{i_l}) f_{i_1\ldots i_k} (dq^{i_1} \wedge \ldots \widehat{dq^{i_l}} \ldots \wedge dq^{i_k}) \\ &= \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{l=1}^{k} (-1)^{l-1} (X \lrcorner dq^{i_l}) f_{i_1\ldots i_k} (dq^{i_1} \wedge \ldots \widehat{dq^{i_l}} \ldots \wedge dq^{i_k}) \\ &= \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{l=1}^{k} (-1)^{l-1} (X(q^{i_l})) f_{i_1\ldots i_k} (dq^{i_1} \wedge \ldots \widehat{dq^{i_l}} \ldots \wedge dq^{i_k}). \end{aligned}$$

If $\psi : U \to \mathbb{R}^n$ is another chart with coordinates $p^i = \psi^* x^i$, for $i = 1..., n$, then $dq^i = \sum_{j=1}^n \frac{\partial q^i}{\partial p^j} dp^j$ and

$$\begin{aligned} \omega_{|U} &= \sum_{1 \le i_1 < \ldots < i_k \le n} h_{i_1\ldots i_k} dq^{i_1} \wedge \ldots \wedge dq^{i_k} = \\ & \sum_{1 \le i_1 < \ldots < i_k \le n} h_{i_1\ldots i_k} \left(\sum_{j_1=1}^{n} \frac{\partial q^{i_1}}{\partial p^{j_1}} dp^{j_1} \right) \wedge \ldots \wedge \left(\sum_{j_k=1}^{n} \frac{\partial q^{i_k}}{\partial p^{j_k}} dp^{j_k} \right) \\ &= \sum_{1 \le i_1 < \ldots < i_k \le n} h_{i_1\ldots i_k} \left(\left(\sum_{j_1=1}^{n} \frac{\partial q^{i_1}}{\partial p^{j_1}} \right) \ldots \left(\sum_{j_k=1}^{n} \frac{\partial q^{i_k}}{\partial p^{j_k}} \right) \right) dp^{j_1} \wedge \ldots \wedge dp^{j_k} \\ &= \sum_{1 \le j_1 < \ldots < j_k \le n} \left(\sum_{i_1, \ldots, i_k = 1}^{n} h_{i_1\ldots i_k} \frac{\partial q^{i_1}}{\partial p^{j_1}} \cdots \frac{\partial q^{i_k}}{\partial p^{j_k}} \right) dp^{j_1} \wedge \ldots \wedge dp^{j_k}. \end{aligned}$$

In the “kernel-index” notation [see Ricci Calculus by Schouten] geometric objects on a manifold M was an assignment to a coordinate system of a collection of functions satisfying a specified transformation law. In this case, to a family $(q^1, ..., q^n)$ of coordinates on M we associate the family $(h_{i_1...i_k})$ of functions. To the famiy $(p^1, ..., p^n)$ we associate the family $(h_{j_1...j_k})$ of functions, where

$$h_{j_1...j_k} = \sum_{i_1,...,i_k=1}^{n} h_{i_1...i_k} \frac{\partial q^{i_1}}{\partial p^{j_1}} \cdots \frac{\partial q^{i_k}}{\partial p^{j_k}}. \tag{5}$$

The coordinate systems are distinguished by the indices $(i_1, ..., i_n)$ for $(q^1, ..., q^n)$ or $(j_1, ..., j_n)$ for $(p^1, ..., p^n)$. This is the “index” part of the notation. The “kernel” is the letter h which identifies the object. The type of the object, in this case a k-form, is identified by the transformation law in equation (5). In particular, if ω is an n-form on M, where $n = \dim M$, then

$$\begin{aligned} \omega_{|U} &= \sum_{1\le i_1<...<i_n\le n} h_{i_1...i_n}(q^1, ..., q^n) dq^{i_1} \wedge ... \wedge dq^{i_n} = h_{12...n}(q^1, ..., q^n) dq^1 \wedge ... \wedge dq^n \\ &= h_{12...n}(q^1(p^1, ..., p^n), ..., q^n(p^1, ..., p^n)) \frac{\partial(q^1, ..., q^n)}{\partial(p^1, ..., p^n)} dp^1 \wedge ... \wedge dp^n, \end{aligned}$$

where

$$\frac{\partial(q^1, ..., q^n)}{\partial(p^1, ..., p^n)} = \det \left(\frac{\partial q^i}{\partial p^j} \right)_{i,j=1,...,n}$$

is the Jacobian of the map $(p^1, ..., p^n) \to (q^1, ..., q^n)$. The transformation rule (5) becomes

$$h_{12...n}(q^1, ..., q^n) \to h_{12...n}(q^1(p^1, ..., p^n), ..., q^n(p^1, ..., p^n)) \frac{\partial(q^1, ..., q^n)}{\partial(p^1, ..., p^n)}.$$

Similarly, a vector field X on M with the restriction X_U,given in terms of the coordinates $(q^1, ..., q^n)$ on M by

$$X_U = \sum_{i=1}^{n} X^i \frac{\partial}{\partial q^i},$$

is given in terms of the coordinates $(p^1, ..., p^n)$ by

$$X_U = \sum_{j=1}^{n} X^j \frac{\partial}{\partial p^j} = \sum_{j=1}^{n} \sum_{i=1}^{n} X^j \frac{\partial q^i}{\partial p^j} \frac{\partial}{\partial q^i}.$$

Hence,

$$X^i = \sum_{i=1}^{n} X^j \frac{\partial q^i}{\partial p^j}. \tag{6}$$

Thus, the vector field X is given in terms of the coordinates $(q^1, ..., q^n)$ by the family (X^i) of functions on U. In terms of the coordinates $(p_1, ..., p_n)$ it is given in terms of the family (X^j) of functions on U. One must remember that the index j represents the coordinate system and $(p_1, ..., p_n)$ and not only an integer from 1 to n. The "kernel" identifying the object is the letter X. The statement that X is a vector field is encoded in the transformation law given by equation (6).

3 Orientable Manifolds

Definition 12 *An n-dimensional manifold M is* orientable *if there exists a nowhere zero differential n-form ω.*

Two nowhere zero n-forms on M are related by

$$\omega' = f\omega.$$

where $f \in C^\infty(M)$ is nowhere zero. Hence, either $f(x) > 0$ for all $x \in M$ or $f(x) < 0$ for all $x \in M$. This gives an equivalence relation ˜ on the space of nowhwere vanishing n-forms on an orientable manifold M such that $\omega \sim \omega'$ if $\omega' = f\omega$ for $f > 0$. Equivalence classes of $\sim$ are called orientations of M.

Definition 13 *An orientable manifold M is oriented if an orientation of M is chosen.*

Example 14 *The Möbius strip.*

4 Integration of forms

Let ω be an n-form on $\mathbb{R}^n$. It can be presented in the form

$$\omega = \sum_{1 \leq i_1 < ... < i_n \leq n} f_{i_1 ... i_n} dx^{i_1} \wedge ... \wedge dx^{i_n} = f_{12...n} dx^1 \wedge dx^2 \wedge ... \wedge dx^n.$$

A rectangle in $\mathbb{R}^n$ is a set

$$R = \{(x^1, ..., x^n) \in \mathbb{R}^n \mid a^i \leq x^i \leq b^i \text{ for } i = 1, ..., n\}.$$

The integral of ω over R is defined by

$$\int_R \omega = \int_R f_{12...n}(x^1, ...x^n) dx^1 dx^2 ... dx^n = \int_{a^1}^{b^1} dx^1 \int_{a^2}^{b^2} dx^2 ... \int_{a^n}^{b^n} dx^n f_{12...n}(x^1, x^2, ..., x^n).$$

Note that $f_{i_1 ... i_n}$ is completely antisymmetric in its indices. To get the right sign we have to take $i_1, ..., i_n$ to be an even permutation of $1, ..., n$.

Let φ be a diffeomorphism of

$$[0,1]^n = \{(y^1,...,y^n) \in \mathbb{R}^n \mid 0 \leq y^i \leq 1\}$$

onto R. For each $i = 1,...,n$, let $x^i = \varphi^* y^i$. Assume that the Jacobian $\frac{\partial(x^1,...,x^n)}{\partial(y^1,...,y^n)} > 0$. Then, changing the variables of integration from $x^1,...,x^n$ to $y^1,...,y^n$, we get

$$\begin{aligned}
\int_R \omega &= \int_R f_{12...n}(x^1,...,x^n)dx^1...dx^n \\
&= \int_{[0,1]^n} f_{12...n}(\varphi^* y^1,...,\varphi^* y^n)\frac{\partial(x^1,...,x^n)}{\partial(y^1,...,y^n)}dy^1dy^2...dy^n \\
&= \int_{[0,1]^n} \varphi^*\omega.
\end{aligned}$$

Remark 15 *The assumption that the Jacobian* $\frac{\partial(x^1,...,x^n)}{\partial(y^1,...,y^n)} > 0$ *implies that the map* $\varphi : [0,1]^n \to R$ *preserves the orientation given by the ordering of the coordinates.*

Definition 16 $\int_R \omega = \int_{[0,1]^n} \varphi^*\omega$, *where* $\varphi : [0,1]^n \to R$ *is a smooth orientation preserving diffeomorphism.*

We shall generalize this result to manifolds. Let M be a manifold of dimension n.

Definition 17 *A singular* k*-cube on* M *is a smooth map* $\sigma : [0,1]^k \to M$.

Definition 18 *The standard singular cube is the map*

$$I^k : [0,1]^k \to \mathbb{R}^k : \boldsymbol{x} \mapsto \boldsymbol{x}.$$

Definition 19 *A chain on* M *is a formal linear combination of singular cubes with integer coefficients.*

For example, if $\sigma_1, \sigma_2, ..., \sigma_r : [0,1]^k \to M$ are singular k-cubes and $m_1, m_2, ..., m_r \in \mathbb{Z}$ then $C = \sum_{i=1}^r m_i\sigma_i$ is a k-chain on M.

Definition 20 *The integral of a* k*-form* ω *on* M *over a* k*-chain* $C = \sum_{i=1}^r m_i\sigma_i$ *is*

$$\int_C \omega = \sum_{i=1}^r m_i \int_{[0,1]^k} \sigma_i^*\omega.$$

If $k = 0$, then $[0,1]^k$ can be identified with the origin $\{0\}$ in $\mathbb{R}^n$. Then, a 0-cube is a map $\sigma : \{0\} \to M$. Moreover, a 0-form ω is a function in $C^\infty(M)$. We define

$$\int_{[0,1]^0} \sigma^*\omega = \omega(\sigma(0)).$$

Problem 21 *A window has the shape of a square of side 2 surmounted by a semicircle. Find its area. Express the computation in terms of the integral of the area form* $\omega = dx \wedge dy$ *over a 2-chain in* $\mathbb{R}^2$*. Identify the chain.*

5 Stokes Theorem

Consider the standard k-cube $[0,1]^k$ in $\mathbb{R}^k$ given by

$$I^k : [0,1]^k \to \mathbb{R}^k : (x^1, ..., x^k) \mapsto (x^1, ..., x^k).$$

We want to define the boundary of $I^k : [0,1]^k \to \mathbb{R}^k$.

If $k = 0$, we identify $[0,1]^k = [0,1]^0$ with the origin O. In this case, we say that $[0,1]^0$ has empty boundary.

If $k = 1$, $[0,1]^1 = [0,1]$ has boundary consisting of two points $\{0\}$ and $\{1\}$ in $\mathbb{R}^1$ We write it as the difference of two singular 0-cubes $\sigma_1 - \sigma_0$ where $\sigma_0(0) = 0$ and $\sigma_1(0) = 1$.

For general k, $I^k : [0,1]^k \to [0,1]^k$ is given by the identity map. For each $i = 1, ..., k$, define two singular $(k-1)$-cubes $I^k_{(i.0)}$ and $I^k_{(i,1)}$ as follows. For $(x^1, ..., x^{k-1}) \in [0,1]^{k-1}$, set

$$\begin{aligned} I^k_{(i.0)}(x^1, ..., x^{k-1}) &= I^k(x^1, ..., x^{i-1}, 0, x^i, ..., x^{k-1}) \\ &= (x^1, ..., x^{i-1}, 0, x^i, ..., x^{k-1}), \end{aligned}$$

and

$$\begin{aligned} I^k_{(i.1)}(x^1, ..., x^{k-1}) &= I^k(x^1, ..., x^{i-1}, 1, x^i, ..., x^{k-1}) \\ &= (x^1, ..., x^{i-1}, 1, x^i, ..., x^{k-1}). \end{aligned}$$

They are called faces of I^k. We define

$$\partial I^k = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^k_{(i,\alpha)}.$$

For a general singular cube $\sigma : [0,1]^k \to M$ we define

$$\sigma_{(i,\alpha)} = \sigma \circ I^k_{(i,\alpha)}$$

and

$$\partial\sigma = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \sigma_{(i,\alpha)} = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \sigma \circ I^k_{(i,\alpha)}.$$

For any k-chain $\sum_{j=1}^r m_r \sigma^r$,

$$\partial \sum_{j=1}^{r} m_r \sigma^r = \sum_{j=1}^{r} m_r \partial\sigma^r.$$

Proposition 22 *For any chain* $C = \sum_{j=1}^{r} m_r \sigma^r$,

$$\partial \left(\partial \sum_{j=1}^{r} m_r \sigma^r \right) = 0.$$

Proof. see M. Spivak, *Calculus on Manifolds*, the Benjamin/Cummings Publishing Company, Menlo Park, CA, 1965. ■

Theorem 23 *(Stokes Theorem) For any* k*-form* ω *on* M *and any* k*-chain* C *on* M

$$\int_C d\omega = \int_{\partial C} \omega.$$

Proof. Suppose first that C is the standard cube I^k and ω is a $(k-1)$-form on $\mathbb{R}^k$. Then ω is the sum of $(k-1)$-forms of the type

$$f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k,$$

and it suffices to prove the theorem for each of these.

Observe that if $i = j$, then

$$\int_{[0,1]^{k-1}} (I^k_{(j,\alpha)})^* (f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k) = \int_{[0.1]^k} f(x^1, ..., x^{i-1}, \alpha, x^{i+1}, ..., x^k) dx^1 ... dx^k$$

since

$$\int_0^1 f(x^1, ..., x^{i-1}, \alpha, x^{i+1}, ..., x^k) dx^i = f(x^1, ..., x^{i-1}, \alpha, x^{i+1}, ..., x^k)$$

for constant $\alpha = 0$ or 1. For $i \neq j$, we have

$$\int_{[0,1]^{k-1}} (I^k_{(j,\alpha)})^* (f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k) = 0.$$

Therefore,

$$\begin{aligned} & \int_{\partial I^k} f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k \\ = \; & \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} I^k_{(j,\alpha)} \int_{[0,1]^{k-1}} (I^k_{(j,\alpha)})^* (f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k) \\ = \; & (-1)^{i+1} \left(\int_{[0,k]^k} f(x^1, ..., 1, ..., x^k) dx^1 ... dx^k - \int_{[0,k]^k} f(x^1, ..., 0, ..., x^k) dx^1 ... dx^k \right) \end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{I^k} d(f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k) \\
= &\int_{[0.1]^k} \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k) \\
= &(-1)^{i-1} \int_{[0.1]^k} \frac{\partial f}{\partial x^i} dx^1 ... dx^k.
\end{aligned}$$

By Fubini's theorem and the fundamental theorem of calculus (in one dimension), we have

$$\begin{aligned}
&\int_{I^k} d(f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k) \\
= &(-1)^{i-1} \int_0^1 ... \int_0^k \left(\frac{\partial f}{\partial x^i} \right) dx^1 ... dx^k \\
= &(-1)^{i-1} \int_0^1 ... \int_0^1 \left(f(x^1, ..., 1, ..., x^k \right) - f(x^1, ..., 0, ..., x^k)) dx^1 ... \widehat{dx^i} ... dx^k \\
= &(-1)^{i+1} \left(\int_{[0,k]^k} f(x^1, ..., 1, ..., x^k) dx^1 ... dx^k - \int_{[0,k]^k} f(x^1, ..., 0, ..., x^k) dx^1 ... dx^k \right) \\
= &\int_{\partial I^k} f dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k.
\end{aligned}$$

Hence,

$$\int_{I^k} d\omega = \int_{\partial I^k} \omega.$$

If $\sigma : [0,1]^k \to M$ is an arbitrary singular k-cube, then

$$\int_{\partial \sigma} \omega = \int_{\partial I^k} \sigma^* \omega.$$

Therefore,

$$\int_\sigma d\omega = \int_{I^k} \sigma^* d\omega = \int_{I^k} d(\sigma^* \omega) = \int_{\partial I^k} \sigma^* \omega = \int_{\partial \sigma} \omega.$$

Finally, if $C = \sum_{j=1}^r m_r \sigma^r$ is a k-chain, then

$$\int_C d\omega = \sum_{j=1}^r m_r \int_{\sigma^r} d\omega = \sum_{j=1}^r m_r \int_{\partial \sigma^r} \omega = \int_{\partial C} \omega.$$

■