Chapter 3

Integration of Vector Fields

1 Global derivations

Let $C^\infty(M)$ be the differential structure of an n-dimensional manifold M.

Definition 1 *A derivation of* $C^\infty(M)$ *is a linear map* $X : C^\infty(M) \to C^\infty(M)$ *satisfying the Leibniz rule*

$$X(f_1 \cdot f_2) = X(f_1) \cdot f_2 + f_1 \cdot X(f_2).$$

We denote by $\mathrm{Der}(C^\infty(M))$ the space of derivations of $C^\infty(M)$. If X and Y are derivations of $C^\infty(M)$ and $a, b \in R$, then

$$aX + bY : C^\infty(M) \to C^\infty(M) : f \mapsto aX(f) + bY(f)$$

is a linear map of $C^\infty(M)$ to itself. Moreover,

$$\begin{aligned}
(aX + bY)(f_1 \cdot f_2) &= aX(f_1 \cdot f_2) + bY(f_1 \cdot f_2) \\
&= aX(f_1) \cdot f_2 + af_1 \cdot X(f_2) + bY(f_1) \cdot f_2 + bf_1 \cdot Y(f_2) \\
&= (aX(f_1) + bY(f_1)) \cdot f_2 + f_1 \cdot (aX(f_2) + bY(f_2)).
\end{aligned}$$

Hence, $aX + bY$ is a derivation of $C^\infty(M)$. Thus, $\mathrm{Der}(C^\infty(M))$ form a vector space.

If X is a derivation and $h \in C^\infty(M)$, then

$$hX : C^\infty(M) \to C^\infty(M) : f \mapsto hX(f)$$

is a derivation of $C^\infty(M)$ since it is a linear map satisfying the Leibniz' rule. We say that $\mathrm{Der}(C^\infty(M))$ is a *module* over $C^\infty(M)$.

Further, if X and Y are derivations of $C^\infty(M)$, then

$$[X, Y] : C^\infty(M) \to C^\infty(M) : f \mapsto X(Y(f)) - Y(X(f))$$

is a linear map of $C^\infty(M)$ to itself. Moreover,

$$\begin{aligned}
[X, Y](f_1 \cdot f_2) &= X(Y(f_1 \cdot f_2)) - Y(X(f_1 \cdot f_2)) \\
&= X(Y(f_1) \cdot f_2 + f_1 \cdot Y(f_2)) - Y(X(f_1) \cdot f_2 + f_1 \cdot X(f_2)) \\
&= X(Y(f_1) \cdot f_2) + X(f_1 \cdot Y(f_2)) - Y(X(f_1) \cdot f_2) - Y(f_1 \cdot X(f_2)) \\
&= X(Y(f_1)) \cdot f_2 + Y(f_1) \cdot X(f_2) + X(f_1) \cdot Y(f_2)) + f_1 \cdot X(Y(f_2)) \\
&\quad -Y(X(f_1)) \cdot f_2 - X(f_1) \cdot Y(f_2) - Y(f_1) \cdot X(f_2)) - f_1 \cdot Y(X(f_2)) \\
&= [X, Y](f_1) \cdot f_2 + f_1 \cdot [X, Y](f_2).
\end{aligned}$$

Thus, $[X, Y] \in \mathrm{Der}C^\infty(M)$. The derivation $[X, Y]$ is called the Lie bracket of X and Y. We say that $\mathrm{Der}C^\infty(M)$ is a *Lie algebra.*

In the following we shall eplore some properties of derivations.

Proposition 2 *If c is a constant function on M and X is a derivation of $C^\infty(M)$, then $X(c) = 0$.*

Proof. Consider first the case when $c = 1$. Since $1 = 1 \cdot 1$, the Leibniz rule gives

$$X(1) = X(1 \cdot 1) = X(1) \cdot 1 + 1 \cdot X(1) = 2X(1).$$

Hence, $X(1) = 0$. For $c \neq 1$, we have $c = c \cdot 1$. Since X is linear, it follows that

$$X(c) = X(c \cdot 1) = c \cdot X(1) = c \cdot 0 = 0.$$

■

Example 3 *Let V be an subset of $\mathbb{R}^n$ and $x_1, ..., x_n$ the coordinate functions on $\mathbb{R}^n$. Every derivation Y of $C^\infty(U)$ is of the form*

$$Y(F) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} Y(x_i) = \sum_{i=1}^{n} Y(x_i) \frac{\partial F}{\partial x_i} = \sum_{i=1}^{n} Y(x_i) \frac{\partial}{\partial x_i} F.$$

Thus,

$$Y = \sum_{i=1}^{n} Y(x_i) \frac{\partial}{\partial x_i}$$

is a differential operator on $C^\infty(U)$ assigning to each F the directional derivative of F in the direction of the vector field $[Y(x_1), ..., Y(x_n)]$. For this reason derivations of $C^\infty(M)$ are called vector fields on M.

Project 5 Proof the formula

$$Y(F) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} Y(x_i)$$

in the above example.

Consider now an open subset U of M. For $f \in C^\infty(M)$, we denote by $f_{|U}$ the restriction of f to U.

Proposition 4 *For each derivation X of $C^\infty(M)$, the restriction $(X(f))_{|U}$ of $X(f)$ to U depends only on $f_{|U}$ and not on values of f in the complement of U.*

Proof. Let f_1 and f_2 be two smooth functions on M which coincide on U. In other words, $f_{1|U} - f_{2|U} = 0$. Since U is open in M, for each $x \in U$, there exist neighbourhoods V and W of x such that $x \in V \subset W \subset U$ and a function $h \in C^\infty(M)$ such that $h_{|V} = 1$ and $h_{|M\setminus W} = 0$. This implies that $h \cdot (f_1 - f_2) = 0$. Hence, $X(h) \cdot (f_1 - f_2) + h \cdot X(f_1 - f_2) = X(h \cdot (f_1 - f_2)) = 0$. Therefore, $X(h)(x)(f_1(x) - f_2(x)) + h(x)(X(f_1)(x) - X(f_2)(x)) = 0$.Since, $f_1(x) = f_2(x)$ and $h(x) = 1$, it follows that $X(f_1)(x) - X(f_2)(x) = 0$. It holds for every $x \in U$. Hence, $X(f_1)_{|U} = X(f_2)_{|U}$. ■

Proposition 5 *Every derivation X of $C^\infty(M)$ gives rise to a derivation $X_{|U} : C^\infty(U) \to C^\infty(U)$, called the restriction of X to U, such that, for each $f \in C^\infty(M)$*

$$X_{|U}(f_{|U}) = (X(f))_{|U}.$$

Proof. Let $h \in C^\infty(U)$. We define $X_{|U}(h)$ as follows. For each $x \in U$, there exists an open neighbourhood V of x in U and a function $f \in C^\infty(M)$ such that $h_{|V} = f_{|V}$. Set $(X_{|U}(h))(x) = (X(f))(x)$. By proposition 3 the function $X_{|U}(h)$ is well defined. Clearly, the map $X_{|U} : C^\infty(U) \to C^\infty(U)$ is linear and satisfies the Leibniz' rule. Hence, $X_{|U}$ is a derivaton of $C^\infty(U)$. ■

Proposition 5 ensures that, if U is an open subset of M, every derivation of $C^\infty(M)$ induces a derivation of $C^\infty(U)$. Suppose now that N is a closed submanifold of M defined as the zero level of a family of functions $f_1, ..., f_k \in C^\infty(M)$. In other words,

$$N = \{x \in M \mid f_1(x) = ... = f_k(x) = 0\}.$$

Let X be a derivation of $C^\infty(M)$ such that $X(f_i)(x) = 0$ for $i = 1, ..., k$ and all $x \in M$. Then, X induces a derivation X_N of $C^\infty(N)$ such that

$$X_N(f_{|N}) = (X(f))_{|N}$$

for every $f \in C^\infty(M)$. Proof of this statement requires results which will be obtained later.

Example 6 *Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with its differential structur given by restrictions to S^2 of smooth functions on $\mathbb{R}^3$. The vector fields $X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, $Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$ and $Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}$ on $\mathbb{R}^3$ induce derivations of $C^\infty(S^2)$.*

Let $\varphi : M \to N$ be a diffeomorphism. For each $X \in DerC^\infty(M)$ there exists a derivation $\varphi_* X$ of $C^\infty(M)$, called the *push forward* of X by φ, such that

$$(\varphi_* X)(f) = (\varphi^{-1})^* X(\varphi^* f)$$

for every $f \in C^\infty(N)$. Here $\varphi^* f = f \circ \varphi$ is the pull-back of f by φ. Similarly $(\varphi^{-1})^* X(\varphi^* f) = ((X(\varphi^* f)) \circ \varphi^{-1}$ is the pull-back of $X(\varphi^* f)$ by φ^{-1}.

Let M be an n-dimensional manifold and $\varphi : U \to V$ a chart on M. In other words, U is an open subset of M, V is an open subset of R^n and $\varphi : U \to V$ is a diffeomorphism. Let X be a derivation of $C^\infty(M)$, $X_{|U}$ the restriction of X to U (see Proposition 4) and $\varphi_* X_{|U}$ the push forward of $X_{|U}$ to V. For each $f \in C^\infty(\mathbb{R}^n)$,

$$(\varphi_* X_{|U})(f)(x_1, ..., x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, .., x_n)((\varphi_* X_{|U})(x_i))(x_1, ..., x_n),$$

where the coordinates $x_1, ..., x_n$ are considered as smooth functions on V. We see that $(\varphi_* X_{|U})$ is a vector field on V with the component $(\varphi_* X_{|U})(x_i)$ in the x_i-direction.

2 The tangent bundle

Let M be a manifold. For each $x \in M$, we define a derivation of $C^\infty(M)$ at x to be a linear map $u : C^\infty(M) \to \mathbb{R}$ such that

$$u(f_1 \cdot f_2) = u(f_1)f_2(x) + u(f_2)f_1(x).$$

Let $X : C^\infty(M) \to C^\infty(M)$ be derivation. Then $X(x)$, defined by

$$X(x)(f) = X(f)(x)$$

is a derivation of $C^\infty(M)$ at x. We shall show later that every derivation u of $C^\infty(M)$ at x is of the form $u = X(x)$ for some derivation X of $C^\infty(M)$. Derivations of $C^\infty(M)$ are called *vector fields* on M. Derivations of $C^\infty(M)$ at $x \in M$ are called *vectors* on M at x or *vectors attached* to M at x

For each $x \in M$ we denote by T_xM the space of all vectors at x. In other words, T_xM is the space of all derivations at x. Since derivations at x are linear maps, T_xM is a vector space over $\mathbb{R}$. For each $u, v \in T_xM$ and $a.b \in \mathbb{R}$, the linear combination $au + bv$ is given by

$$(au + bv)f = au(f) + bv(f).$$

The vector space T_xM is called the *tangent space* of M at x. For $u \in T_xM$, we say that

The union of tangent spaces at all points $x \in M$ is called the tangent bundle space of M and denoted TM. In other words,

$$TM = \bigcup_{x \in M} T_xM.$$

We denote by $\tau : TM \to M$ the map associating to each $u \in TM$ the point $\tau(u)$ in M at which u is attached. In other words, for each $x \in M$, we have $\tau^{-1}(x) = T_xM$. That is, the inverse image of $x \in M$ under the map τ is the tangent space of M at x. The map $\tau : TM \to M$ is called the *tangent bundle projection*.

Let M and N be manifolds and $\varphi : M \to N$ be a smooth map. For each $f \in C^\infty(N)$ the pull-back $\varphi^* f$ is in $C^\infty(M)$. For every derivation u of $C^\infty(M)$ at $x \in M$, the map $C^\infty(N) \to \mathbb{R} : f \mapsto u(\varphi^* f)$ is linear. Moreover, for each $f_1, f_2 \in C^\infty(N)$,

$$\varphi^*(f_1 \cdot f_2) = (f_1 \cdot f_2) \circ \varphi = (f_1 \circ \varphi) \cdot (f_2 \circ \varphi) = (\varphi^* f_1) \cdot (\varphi^* f_2).$$

Hence,

$$\begin{aligned} u(\varphi^*(f_1 \cdot f_2)) &= u((\varphi^* f_1) \cdot (\varphi^* f_2)) = u((\varphi^* f_1))(\varphi^* f_2)(x) + u((\varphi^* f_2))(\varphi^* f_1)(x) \\ &= u((\varphi^* f_1))(f_2(\varphi(x))) + u((\varphi^* f_2))(f_1(\varphi(x))). \end{aligned}$$

This implies that $f \mapsto u(\varphi^* f)$ is a derivation of $C^\infty(N)$ at $\varphi(x)$. Thus, a smooth map $\varphi : M \to N$ induces a map $T\varphi : TM \to TN$ such that, for every $u \in TM$ and $f \in C^\infty(N)$,

$$T\varphi(u)(f) = u(\varphi^* f)$$

called the *derived map* of φ or the *tangent map* of φ. If $\tau_M : TM \to M$ and $\tau_N : TN \to N$ are the tangent bundle projections, then $\tau_N \circ (T\varphi) = \varphi \circ \tau_M$. We illustrate this fact by saying that the following diagram

$$\begin{array}{ccccc} & & T\varphi & & \\ & TM & \rightarrow & TN & \\ \tau_M & \downarrow & & \downarrow & \tau_N \\ & M & \rightarrow & N & \\ & & \varphi & & \end{array} \tag{1}$$

commutes.

Theorem 7 *If $M, N,$ and K are manifolds and $\varphi : M \to N$ and $\psi : N \to K$ are smooth maps, then $\psi \circ \varphi : M \to K$ is smooth and $T(\psi \circ \varphi) = (T\psi) \circ (T\varphi)$. Moreover, if $\varphi : M \to N$ is a diffeomorphism then $T\varphi : TM \to TN$ is invertible and $(T\varphi)^{-1} = T\varphi^{-1}$.*

Proof. Let $f \in C^\infty(K)$. Since $\psi : N \to K$ is smooth, it follows that $\psi^* f = f \circ \psi \in C^\infty(N)$. Smoothness of $\varphi : M \to N$ implies that $\varphi^*(\psi^* f) \in C^\infty(M)$. But

$$\varphi^*(\psi^* f) = (\psi^* f) \circ \varphi = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi).$$

Hence, $\psi \circ \varphi$ is smooth.

For each $u \in TM$ and $f \in C^\infty(K)$,

$$\begin{aligned} (T(\psi \circ \varphi)(u))(f) &= u((\psi \circ \varphi)^*(f) = u(f \circ (\psi \circ \varphi)) = u((f \circ \psi) \circ \varphi) = u((\psi^* f) \circ \varphi) \\ &= u(\varphi^*(\psi^* f)) = (T\varphi(u))(\psi^*(f)) = (T\psi(T\varphi(u)))(f) = ((T\psi) \circ (T\varphi))(u))(f). \end{aligned}$$

Therefore, $T(\psi \circ \varphi) = (T\psi) \circ (T\varphi)$.

Moreover, if $id_M : M \to M : x \mapsto x$ is the identity map then, for each $u \in TM$ and $f \in C^\infty(M)$,

$$((Tid_M)(u))(f) = u(id_M^*(f)) = u(f \circ id_M) = u(f)$$

which implies that $Tid_M = id_{TM} : TM \to TM : u \mapsto u$. This implies that if $\varphi : M \to N$ is a diffeomorphism, then $\varphi \circ \varphi^{-1} = id_M$ and $\varphi^{-1} \circ \varphi = id_M$ so that

$$(T\varphi) \circ (T\varphi^{-1}) = (T\varphi^{-1}) \circ (T\varphi) = id_{TM}.$$

Hence,

$$(T\varphi)^{-1} = T\varphi^{-1}.$$

■

Each $f \in C^\infty(M)$ can be pulled back by the tangent bundle projection τ to a function $\tau^* f = f \circ \tau$ on TM. Moreover, $f \in C^\infty(M)$ gives rise to another function on TM called the *differential* of f and denoted $df : TM \to \mathbb{R}$, which is defined by

$$df(u) = u(f) \quad \forall \quad u \in TM.$$

Theorem 8 *The tangent bundle space TM of an n-dimensional manifold M is a manifold of dimension $2n$ with the differential structure $C^\infty(M)$ generated by the family of functions*

$$\mathcal{F}_M = \{\tau^* f, df \mid f \in C^\infty(M)\}.$$

With this differential structure, $\tau : TM \to M$ is smooth and $T\varphi : TM \to TN$ is smooth for every smooth map $\varphi : M \to N$.

Proof. First, we have to show that the familty $\mathcal{F}_M = \{\tau^* f, df \mid f \in C^\infty(M)\}$ generates a differential structure on TM. We define topology on TM by requiring that the family of sets

$$\{(\tau^* f)^{-1}((a,b)), (df)^{-1}((a,b)) \mid a.b \in \mathbb{R} \text{ and } f \in C^\infty(M)\}. \tag{2}$$

is a sub-basis for the topology of TM. Next, we extend the family $\mathcal{F}_M$ to a family

$$\tilde{\mathcal{F}}_M = \{F(df_1, ..., df_k, \tau^* f_{k+1}, ..., \tau^* f_n) \mid n \in \mathbb{N},\ k = 0, ..., n,\ F \in C^\infty(\mathbb{R}^n),\ f_1, ..., f_n \in C^\infty(M)\}.$$

Finally, we define $f : TM \to \mathbb{R}$ to be in $C^\infty(TM)$ if, for each $u \in TM$, there exists an open neighbourhood U of u in TM and a function $f_x \in \tilde{\mathcal{F}}$ such that $f_{|U} = f_{x|U}$. One can easily check that $C^\infty(TM)$ satisfies the assumptions of Definition 2 in Chapter 2.

For each $f \in C^\infty(M)$, $\tau_M^* f \in C^\infty(TM)$ by construction. Hence, $\tau_M : TM \to M$ is smooth.

Let $\varphi : M \to N$ be smooth. For each $h \in C^\infty(N)$ the pull-back $\varphi^* h \in C^\infty(M)$. If $\tau_N : TN \to N$, then

$$\begin{aligned}
(T\varphi)^*(\tau_N^* h) &= (T\varphi)^*(h \circ \tau_N) \\
&= h \circ \tau_N \circ T\varphi \\
&= h \circ \varphi \circ \tau_M \quad \text{by diagram 1} \\
&= \tau_M^*(\varphi^* h) \in \mathcal{F} \subset C^\infty(M).
\end{aligned}$$

Similarly, for every $u \in TM$

$$\begin{aligned}
((T\varphi)^*(dh))(u) &= (dh \circ T\varphi)(u) = (dh)(T\varphi(u)) = (T\varphi(u))(h) \\
&= u(\varphi^* h) = d(\varphi^* h)(u).
\end{aligned}$$

Hence, $(T\varphi)^*(dh) = d(\varphi^* h) \in \mathcal{F}_M$ for every $h \in \mathcal{F}_N = \{\tau^* h, dh \mid h \in C^\infty(N)\}$.

For $a, b \in \mathbb{R}$,

$$\begin{aligned}
(T\varphi)^{-1}(\tau_N^* h)^{-1}((a,b)) &= \{u \in TM \mid (T\varphi)(u) \in (\tau_N^* h)^{-1}((a,b))\} \\
&= \{u \in TM \mid ((\tau_N^* h) \circ (T\varphi))(u) \in (a,b)\} \\
&= \{u \in TM \mid ((h \circ \tau_N \circ T\varphi)(u) \in (a,b)\} \\
&= \{u \in TM \mid (\tau_M^*(\varphi^* h))(u) \in (a,b)\} \\
&= (\tau_M^*(\varphi^* h))^{-1}((a,b)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(T\varphi)^{-1}((dh)^{-1}((a,b))) &= \{u \in TM \mid (T\varphi)(u) \in (dh)^{-1}((a,b))\} \\
&= \{u \in TM \mid (dh)((T\varphi)(u)) \in (a,b)\} \\
&= \{u \in TM \mid (d(\varphi^* h))(u)) \in (a,b)\} \\
&= (d(\varphi^* h))^{-1}((a,b)).
\end{aligned}$$

Hence, the inverse image under $T\varphi$ of each set in the sub-babsis of the the topology of TN is in the sub-basis of the topology of TM. It suffices to conclude that $T\varphi : TM \to TN$ is continuous.

To conclude that $T\varphi$ is smooth, observe that, for every $F \in \mathbb{R}^n$, and $h_1, ..., h_n \in C^\infty(N)$,

$$\begin{aligned}
&(T\varphi)^*(F(dh_1, ..., dh_k, \tau_N^* h_{k+1}, ..., \tau_N^* h_n)) \\
= \; &F((T\varphi)^*(dh_1), ..., (T\varphi)^*(dh_k), (T\varphi)^*(\tau_N^* h_{k+1}), ..., (T\varphi)^*(\tau_N^* h_n)) \\
= \; &F(d(\varphi^* h_1), ..., d(\varphi^* h_k), \tau_M^*(\varphi^* h_{k+1}), ..., \tau_M^*(\varphi^* h_n)) \in C^\infty(M).
\end{aligned}$$

This means that $(T\varphi)^* h \in \tilde{\mathcal{F}}_M$ for each h in

$$\tilde{\mathcal{F}}_N = \{F(dh_1, ..., dh_k, \tau^* h_{k+1}, ..., \tau^* h_n) \mid n \in \mathbb{N},\ k = 0, ..., n,\ F \in C^\infty(\mathbb{R}^n),\ h_1, ..., h_n \in C^\infty(N)\}.$$

Further, if $h \in C^\infty(N)$, for each $y \in N$ there exists a an open neighbourhood U of y in N and $h_y \in \tilde{\mathcal{F}}_N$ such that $h_{|U} = h_{y|U}$. Since $T\varphi$ is continuous it follows that $(T\varphi)^{-1}(U)$ is open in M. Moreover,

$$((T\varphi)^*h)_{|(T\varphi)^{-1}(U)} = h \circ T\varphi_{|(T\varphi)^{-1}(U)} = h_{|U} \circ T\varphi_{|(T\varphi)^{-1}(U)} = ((T\varphi)^*h_y)_{|(T\varphi)^{-1}(U)}.$$

But, we have shown that $h_y \in \tilde{\mathcal{F}}_N$ implies that $(T\varphi)^*h_y \in \tilde{\mathcal{F}}_M$. Hence, $(T\varphi)^*h \in C^\infty(M)$. This concludes the proof that $T\varphi : TM \to TN$ is smooth. ■

Corollary 9 *If $\varphi : M \to N$ is a diffeomorphism then $T\varphi : TM \to TN$ is a diffeomorphism.*

Proof. Since $\varphi : M \to N$ is a diffeomorphism then $\varphi^{-1} : N \to M$ exists and is smooth. Hence, $(T\varphi)^{-1} = T\varphi^{-1}$ is smooth. Therefore, $T\varphi$ is a diffeomorphism. ■

Example 10 *Let U be an open subset of $\mathbb{R}^n$. The tangent bundle space of U is $TU = U \times \mathbb{R}^n$. Each $\mathbf{u} = (x_1, ..., x_n, u_1, ..., u_n)$ gives the derivation*

$$\mathbf{u}(f) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}(x_1, ..., x_n).$$

For an n-dimensional manifold M, let $\varphi : U \to V$ be a chart on M. That is, U is an open subset of M, V is an open subset of $\mathbb{R}^n$ and φ is a diffeoorphism. Then $T\varphi : TU \to TV = V \times \mathbb{R}^n$ is a diffeomorphism. This implies that TU has dimension $2n$.

To visualise TM as a subset of $\mathbb{R}^l$ consider M as a subset of $\mathbb{R}^k$. Let $\iota : M \to \mathbb{R}^k$ be the inclusion map. Then $T\iota : TM \to T\mathbb{R}^k = \mathbb{R}^{2k}$ is an inclusion of TM into $\mathbb{R}^{2k}$.

Remark 11 *A vector field X on M corresponds to a map $\sigma_X : M \to TM : x \mapsto X(x)$.*

3 Integral curves of vector fields

Let X be a vector field on an n-dimensional manifold M. A curve on M is a smooth map $\gamma : (a, b) \to M$, where (a, b) is an interval in $\mathbb{R}$. A curve γ is an integral curve of X if

$$\frac{d}{dt} f(\gamma(t)) = (X(f))(\gamma(t))$$

for each $f \in C^\infty(M)$ and each $t \in (a, b)$.

Let $\varphi : U \to V$ be a chart in M. That is, U is open in M, V is open in $\mathbb{R}^n$, and $\varphi : U \to V$ a diffeomorphism. We denote by $\varphi_* X$ the push-forward of $X_{|U}$ to V. For each $f \in C^\infty(\mathbb{R}^n)$,

$$(\varphi_* X)(f)(x_1, ..., x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, .., x_n)((\varphi_* X)(x_i))(x_1, ..., x_n).$$

Su ppose $\gamma((a, b)) \cap U \neq \emptyset$ and that $(a, b) \cap \gamma^{-1}(U)$ is an interval in (a, b). Then, $\varphi \circ \gamma : (a, b) \cap \gamma^{-1}(U) \to V$ is a curve on V and for every $f \in C^\infty(\mathbb{R}^n)$,

$$\frac{d}{dt} f(\varphi(\gamma(t))) = \frac{d}{dt} \varphi^* f(\gamma(t)) = (X(\varphi^* f)))(\gamma(t)) = (\varphi_* X)(f).$$

Writing $x^i(t) = x^i(\varphi(\gamma(t))$ and $X^i = (\varphi_* X)(x^i)$, for $i = 1, ..., n$, we obtain a differential equation

$$\frac{dx^i(t)}{dt} = X^i(x^1(t), ..., x^n(t)), \text{ for all } i = 1, ...n \text{ and } t \in (a, b) \cap \gamma^{-1}(U). \qquad (3)$$

The existence and uniqueness theorem for ordinary differential equations in $\mathbb{R}^n$ ensures that, for $t_0 \in (a, b) \cap \gamma^{-1}(U)$ and a point $\boldsymbol{x}_0 = (x_0^1, ..., x_0^n) \in V \subseteq \mathbb{R}^n$, there exists a unique maximal solution of equation (3), defined in an interval I_0 contained in $(a, b) \cap \gamma^{-1}(U)$ such that $x^i(t_0) = x_0^i$ for $i = 1, ..., n$. Then, $\gamma : t \mapsto \varphi^{-1}(x^1(t), ..., x^n(t))$ for $t \in I_0 \subseteq (a, b) \cap \gamma^{-1}(U)$ is an integral curve of X such that $\gamma(t_0) = \varphi^{-1}(x_0^1, ..., x_0^n)$. Thus, we have shown the existence of integral curves of a vector field. However, the curve, $\gamma : t \mapsto \varphi^{-1}(x^1(t), ..., x^n(t))$ for $t \in I_0$ need not be maximal. We obtain the maximal integral curve γ of X such that $\gamma(t_0) = \boldsymbol{x}_0$ by patching integral curves obtained in different charts.

Given a vector field X on M, we obtain for each $x \in M$ the maximal integral curve γ_x of X such that $\gamma_x(0) = x$ has domain $(a(x), b(x))$, where $a \geq -\infty$ and $b \leq \infty$. If $a(x) = -\infty$ and $b(x) = \infty$ for all $x \in M$, we say that the vector field X is complete.

Theorem 12 *Every smooth vector field on a compact manifold is complete.*

For a vector field X on M, the set

$$D = \{(x, t) \in M \times \mathbb{R} \mid a(x) < t < b(x)\}$$

is called the *flow box* of X. If the vector field X is complete, then $D = M \times \mathbb{R}$.

For each $(x,t) \in D$, $\gamma_x(t) \in M$ is defined. Moreover, the uniqueness of sulutions of ordinary differential equations implies that, if (x,t) and $(x, t+s)$ are in D, then

$$\gamma_x(t+s) = \gamma_{\gamma_x(t)}(s).$$

In other words, the solution at $(t+s)$ of the initial data problem with the initial condition $\gamma_x(0) = x$ is the same as the solution at s of corresponding to the inital condition at zero given by $\gamma_x(t)$.

Smooth dependence of solutions of ordinary differential equations on initial data implies that, for fixed t, $\gamma_x(t)$ is a sooth function of x.

Notation 13 *Let X be a vector field on M. For each $(x,t) \in D$, let*

$$\exp(tX)(x) = \gamma_x(t).$$

This notation exhibits the role of the vector field X.

Theorem 14 *Let X be a smooth vector field on a manifold M. For each $(x,t) \in D$, there is an open neighbourhood U of x in M such that $\exp(tX)(x')$ is defined for $x' \in U$. Moreover, the map*

$$\exp(tX) : U \to M : x' \mapsto \exp(tX)(x')$$

is a diffeomorphism of U onto its image. The inverse of $\exp(tX)$ is $\exp(-tX)$ obtained by replacing t by $-t$. Similarly, if $t, (t+s) \in (a(x), b(x))$, then

$$\exp((t+s)X)(x) = \exp(sX) \circ \exp(tX)(x).$$

We say that $\exp(tX)$ is a *local one-parameter group of local diffeomorphisms* of M defined by the vector field X.

Corollary 15 *If M is compact then $\exp tX$ is a one-parmeter group of diffeomorphism of M for every smooth vector field X on M.*

Proposition 16 *Let X be a vector field on a manifold M and $\varphi : M \to N$ a diffeomorphism of M onto N. Then $\varphi \circ (\exp tX) \circ \varphi^{-1}$ is a one-parameter local group of local diffeomorphisms of N generated by $\varphi_* X$.*

Proof. Clearly,

$$\begin{aligned} \varphi \circ (\exp(t+s)X) \circ \varphi^{-1} &= \varphi \circ (\exp tX) \circ (\exp tX) \circ \varphi^{-1} \\ &= \varphi \circ (\exp tX) \circ \varphi^{-1} \circ \varphi \circ (\exp tX) \circ \varphi^{-1} \end{aligned}$$

whenever $\exp tXt$, $\exp sX$ and $\exp(t+s)X$ are defined. Hence, $\varphi \circ (\exp tX) \circ \varphi^{-1}$ is a one -parameter local group of local diffeomorphisms of N.

For each $f \in C^\infty(N)$,

$$\begin{aligned}
\frac{d}{dt} f(\varphi \circ (\exp tX) \circ \varphi^{-1})_{|t=0} &= \frac{d}{dt}(\varphi^{-1})^* f((\varphi \circ (\exp tX))_{|t=0} \\
&= (\varphi^{-1})^* \left\{ \frac{d}{dt}(f \circ (\varphi \circ (\exp tX))) \right\}_{|t-0} \\
&= (\varphi^{-1})^* \left\{ \frac{d}{dt}(\exp tX)^*(f \circ \varphi) \right\}_{|t=0} \\
&= (\varphi^{-1})^* \left\{ \frac{d}{dt}(\exp tX)^*(\varphi^* f) \right\}_{|t=0} \\
&= (\varphi^{-1})^* \left\{ X((\exp tX)^*(\varphi^* f)) \right\}_{|t=0} \\
&= (\varphi^{-1})^* \left\{ X((\varphi^* f) \circ (\exp tX)) \right\}_{|t=0} \\
&= (\varphi^{-1})^* X(\varphi^* f) = (\varphi_* X)(f).
\end{aligned}$$

■

3.1 Integration of vector fields on S^2.

3.1.1 Stereographic projections

Recall that $S^2 = \{(x, y, z) \in R^3 \mid x^2 + y^2 + z^2 = 1\}$. The North Pole on S^2 is the point $(0, 0, 1)$. The line joining the North Pole to a point $(x, y, z) \in S^2$ is

$$t \mapsto t(0, 0, 1) + (1 - t)(x, y, z) = ((1 - t)x, (1 - t)y, t + (1 - t)z).$$

It intersects the (x, y)-plane when $t + (1 - t)z = 0$, provided $z \neq 1$. Solving for t we get

$$t = \frac{-z}{1 - z}.$$

The point of intersection of this line with the the (x, y)-plane is called the *stereographic projection of* (x, y, z) *from the North Pole*. We denote its coordinates by (ξ_N, η_N). They are given by.

$$\begin{aligned}
\xi_N &= (1 - t)x = \left(1 - \frac{-z}{1 - z}\right) x = \frac{x}{1 - z}, \\
\eta_N &= (1 - t)y = \left(1 - \frac{-z}{1 - z}\right) y = \frac{y}{1 - z}.
\end{aligned}$$

The stereographic projection from the North Pole gives a chart

$$\varphi_N : U_N = S^2 \backslash \{(0, 0, 1)\} \to R^2 : (x, y, z) \mapsto (\xi_N, \eta_N) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right).$$

To get φ_N^{-1} observe that

$$\xi_N^2 + \eta_N^2 = \frac{x^2}{(1-z)^2} + \frac{y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z}.$$

Hence,

$$(1-z)(\xi_N^2 + \eta_N^2) = 1 + z$$

and

$$z = \frac{\xi_N^2 + \eta_N^2 - 1}{\xi_N^2 + \eta_N^2 + 1}.$$

Therefore,

$$1 - z = 1 - \frac{\xi_N^2 + \eta_N^2 - 1}{\xi_N^2 + \eta_N^2 + 1} = \frac{2}{\xi_N^2 + \eta_N^2 + 1},$$

which yields

$$x = (1-z)\xi_N = \frac{2\xi_N}{\xi_N^2 + \eta_N^2 + 1}$$

and

$$y = (1-z)\eta_N = \frac{2\eta_N}{\xi_N^2 + \eta_N^2 + 1}.$$

Thus,

$$\varphi_N^{-1} : (\xi_N, \eta_N) \mapsto (x, y, z) = \left(\frac{2\xi_N}{\xi_N^2 + \eta_N^2 + 1}, \frac{2\eta_N}{\xi_N^2 + \eta_N^2 + 1}, \frac{\xi_N^2 + \eta_N^2 - 1}{\xi_N^2 + \eta_N^2 + 1} \right)$$

Similarly, the South Pole is the point $(0, 0, -1)$. The line connecting the South pole to a point (x, y, z) is

$$t \mapsto t(0, 0, -1) + (1-t)(x, y, z) = ((1-t)x, (1-t)y, -t + (1-t)z).$$

It intersects the (x, y)-plane when $-t + (1-t)z = 0$, provided $z \neq -1$. Solving for t we get

$$t = \frac{z}{1+z}.$$

The point of intersection of this line with the the (x, y)-plane is called the *stereographic projection of* (x, y, z) *from the South Pole*. We denote its coordinates by (ξ_S, η_S). They are given by.

$$\begin{aligned}
\xi_S &= (1-t)x = \left(1 - \frac{z}{1+z}\right) x = \frac{x}{1+z}, \\
\eta_S &= (1-t)y = \left(1 - \frac{z}{1+z}\right) y = \frac{y}{1+z}.
\end{aligned}$$

The stereographic projection from the South Pole gives a chart

$$\varphi_S : U_S = S^2 \backslash \{(0,0,-1)\} \to \mathbb{R}^2 : (x,y,z) \mapsto (\xi_S, \eta_S) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right).$$

To get φ_S^{-1}, observe that

$$\xi_S^2 + \eta_S^2 = \frac{x^2}{(1+z)^2} + \frac{y^2}{(1+z)^2} = \frac{1-z^2}{(1+z)^2} = \frac{1-z}{1+z}.$$

Hence,

$$(1+z)(\xi_S^2 + \eta_S^2) = 1 - z$$

which yields

$$z = \frac{1 - (\xi_S^2 + \eta_S^2)}{1 + \xi_S^2 + \eta_S^2}.$$

Hence,

$$1 + z = 1 + \frac{1 - (\xi_S^2 + \eta_S^2)}{1 + \xi_S^2 + \eta_S^2} = \frac{2}{1 + \xi_S^2 + \eta_S^2}$$

and

$$\begin{aligned} x &= (1+z)\xi_S = \frac{2\xi_S}{1 + \xi_S^2 + \eta_S^2}, \\ y &= (1+z)\eta_S = \frac{2\eta_S}{1 + \xi_S^2 + \eta_S^2}. \end{aligned}$$

Therefore,

$$\varphi_S^{-1} : (\xi_S, \eta_S) \mapsto (x,y,z) = \left(\frac{2\xi_S}{1 + \xi_S^2 + \eta_S^2}, \frac{2\eta_S}{1 + \xi_S^2 + \eta_S^2}, \frac{1 - (\xi_S^2 + \eta_S^2)}{1 + \xi_S^2 + \eta_S^2} \right).$$

On $U_S \cap U_N = S^2 \backslash \{(0,0,1), (0,0,-1)\}$,

$$\begin{aligned} \xi_S &= \frac{x}{1+z} = \frac{x}{1-z} \frac{1-z}{1+z} = \frac{\xi_N}{\xi_N^2 + \eta_N^2}, \\ \eta_S &= \frac{y}{1+z} = \frac{y}{1-z} \frac{1-z}{1+z} = \frac{\eta_N}{\xi_N^2 + \eta_N^2}, \end{aligned}$$

and

$$\begin{aligned} \xi_N &= \frac{\xi_S}{\xi_S^2 + \eta_S^2}, \\ \eta_N &= \frac{\eta_S}{\xi_S^2 + \eta_S^2}. \end{aligned}$$

3.1.2 Vector fields on S^2

Since $S^2 = \{(x,y,z) \in R^3 \mid x^2+y^2+z^2 = 1\}$, a vector field X on $\mathbb{R}^3$ induces a vector field X_S^2 on S^2 provided, $X(x^2+y^2+z^2) = 0$ when $x^2+y^2+z^2 = 1$. This condition is satisfied by vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

We consider here the integral curve γ of $X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$ satisfying the initial condition $\gamma(0) = (0,0,-1)$

We have

$$\begin{aligned} X(\xi_N) &= \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\frac{x}{1-z} = \frac{yx}{(1-z)^2} = \xi_N\eta_N, \\ X(\eta_N) &= \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\frac{y}{1-z} = \frac{y^2}{(1-z)^2} - \frac{z}{1-z}. \end{aligned}$$

But $z = \frac{\xi_N^2+\eta_N^2-1}{\xi_N^2+\eta_N^2+1}$ implies that

$$\frac{z}{1-z} = \frac{\frac{\xi_N^2+\eta_N^2-1}{\xi_N^2+\eta_N^2+1}}{1-\frac{\xi_N^2+\eta_N^2-1}{\xi_N^2+\eta_N^2+1}} = \frac{1}{2}(\xi_N^2+\eta_N^2-1).$$

Therefore,

$$\begin{aligned} X(\eta_N) &= \frac{y^2}{(1-z)^2} - \frac{z}{1-z} = \eta_N^2 - \frac{1}{2}(\xi_N^2+\eta_N^2-1) \\ &= \frac{1}{2}(\eta_N^2 - \xi_N^2 + 1). \end{aligned}$$

Thus,

$$\varphi_{N*}X = \xi_N\eta_N\frac{\partial}{\partial \xi_N} + \frac{1}{2}(\eta_N^2 - \xi_N^2 + 1)\frac{\partial}{\partial \eta_N},$$

and the integral curves of X satisfy the equations

$$\begin{aligned} \frac{d}{dt}\xi_N &= \xi_N\eta_N, \\ \frac{d}{dt}\eta_N &= \frac{1}{2}(\eta_N^2 - \xi_N^2 + 1). \end{aligned}$$

We consider a special solution for which $\xi_N(t) = 0$. This is compatible with the first equation. The second equation reads

$$\frac{d}{dt}\eta_N = \frac{1}{2}(\eta_N^2 + 1)$$

or

$$\frac{1}{1+\eta_N^2}d\eta_N = \frac{1}{2}dt.$$

Using the initial condition $\eta_N(0) = 0$, we get

$$\arctan \eta_N = \frac{1}{2}t$$

or

$$\eta_N = \tan(\frac{t}{2}) \text{ for } -\pi < t < \pi.$$

Hence, we get the curve

$$t \mapsto (\xi_N(t), \eta_N(t)) = (0, \tan(t/2)) \quad for - \pi < t < \pi.$$

Since S^2 is compact, vector fields on S^2 are complete. Hence, we expect our integral curve to be defined for all $t \in \mathbb{R}$. However, our curve diverges as t approaches $\pm\pi$. This is so because the domain U_N of the chart is not compact. In order to continue our curve, we have to pass to the stereographic projection from the South Pole.

In order to compare curves in both stereographic projections we first consider the integral curve $s \mapsto (\xi_N(s), \eta_N(s))$ with the initial condition $\xi_N(0) = 0$ and $\eta_N(0) = 1$. It corresponds to $(x(0), y(0), z(0) = (0, 1, 0) \in U_N \cap U_S$. Then $\xi_N(s) = 0$ for all s and

$$\frac{1}{1+\eta_N^2}d\eta_N = \frac{1}{2}ds$$

gives

$$\arctan(\eta_N(s)) - \arctan(\eta_N(0)) = \frac{1}{2}s.$$

Hence,

$$\arctan(\eta_N(s)) = \arctan 1 + \frac{s}{2} = \frac{\pi}{4} + \frac{s}{2}$$

which gives $\eta_N(s) = \tan\left(\frac{\pi}{4} + \frac{s}{2}\right)$. Thus, our curve is

$$s \mapsto (\xi_N(s), \eta_N(s)) = (0, \tan\left(\frac{\pi}{4} + \frac{s}{2}\right) \text{ for } -\frac{3\pi}{2} < s < \frac{\pi}{2}. \tag{4}$$

Equation

$$(x, y, z) = \left(\frac{2\xi_N}{\xi_N^2 + \eta_N^2 + 1}, \frac{2\eta_N}{\xi_N^2 + \eta_N^2 + 1}, \frac{\xi_N^2 + \eta_N^2 - 1}{\xi_N^2 + \eta_N^2 + 1}\right)$$

gives

$$\begin{aligned}
x(s) &= 0, \\
y(s) &= \frac{2\tan\left(\frac{\pi}{4}+\frac{s}{2}\right)}{1+\tan^2\left(\frac{\pi}{4}+\frac{s}{2}\right)} = \frac{2\tan\left(\frac{\pi}{4}+\frac{s}{2}\right)}{\frac{1}{\cos^2\left(\frac{\pi}{4}+\frac{s}{2}\right)}} = 2\sin\left(\frac{\pi}{4}+\frac{s}{2}\right)\cos\left(\frac{\pi}{4}+\frac{s}{2}\right) \\
&= \sin\left(\frac{\pi}{2}+s\right). \\
z(s) &= \frac{\tan^2\left(\frac{\pi}{4}+\frac{s}{2}\right)-1}{\tan^2\left(\frac{\pi}{4}+\frac{s}{2}\right)+1} = \sin^2\left(\frac{\pi}{4}+\frac{s}{2}\right) - \cos^2\left(\frac{\pi}{4}+\frac{s}{2}\right) = -\cos\left(\frac{\pi}{2}+s\right).
\end{aligned}$$

Hence our curve in the (x, y, z) variables is

$$s \mapsto (x(s), y(s), z(s)) = \left(0, \sin\left(\frac{\pi}{2}+s\right), -\cos\left(\frac{\pi}{2}+s\right)\right).$$

Npte that in these variables there are no restrictions on the values of the parameter s.

Now, we want to solve the same problem in terms of the stereographic projection from the South Pole. We have

$$\begin{aligned}
\xi_S &= \frac{x}{1+z} = \frac{x}{1-z}\frac{1-z}{1+z} = \frac{\xi_N}{\xi_N^2+\eta_N^2}, \\
\eta_S &= \frac{y}{1+z} = \frac{y}{1-z}\frac{1-z}{1+z} = \frac{\eta_N}{\xi_N^2+\eta_N^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
X(\xi_S) &= \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\frac{x}{1+z} = \frac{-xy}{(1+z)^2} = -\xi_S\eta_S, \\
X(\eta_S) &= \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\frac{y}{1+z} = \frac{-y^2}{(1+z)^2} - \frac{z}{1+z}.
\end{aligned}$$

But

$$z = \frac{1-(\xi_S^2+\eta_S^2)}{1+\xi_S^2+\eta_S^2}$$

implies that

$$\frac{z}{1+z} = \frac{\frac{1-(\xi_S^2+\eta_S^2)}{1+\xi_S^2+\eta_S^2}}{1+\frac{1-(\xi_S^2+\eta_S^2)}{1+\xi_S^2+\eta_S^2}} = \frac{1}{2}(1-(\xi_S^2+\eta_S^2)).$$

Therefore,

$$\begin{aligned}
X(\eta_S) &= \frac{-y^2}{(1+z)^2} - \frac{z}{1+z} = -\eta_S^2 - \frac{1}{2}(1-(\xi_S^2+\eta_S^2)) \\
&= -\frac{1}{2}(1+\eta_S^2-\xi_S^2).
\end{aligned}$$

Thus,

$$\varphi_{S*}X = -\xi_S\eta_S\frac{\partial}{\partial\xi_S} - \frac{1}{2}(1+\eta_S^2-\xi_S^2)\frac{\partial}{\partial\eta_S},$$

and the integral curves of X satisfy the equations

$$\begin{aligned}\frac{d}{ds}\xi_S &= -\xi_S\eta_S,\\ \frac{d}{ds}\eta_S &= -\frac{1}{2}(1+\eta_S^2-\xi_S^2),\end{aligned}$$

where we have used s to denote the parameter along the curve.

We consider a special solution for which $\xi_S(s) = 0$. Hence, $\xi_S(s) = 0$. This is compatible with the first equation. The second equation reads

$$\frac{d}{ds}\eta_S = -\frac{1}{2}(1+\eta_S^2)$$

or

$$\frac{1}{1+\eta_S^2}d\eta_S = -\frac{1}{2}ds.$$

The initial condition $(0,1,0)$ is in the domain of the stereographic projection from the South Pole. Hence, we can use it. Integrating our equation from 0 to s and setting $\eta(0) = 1$ we get

$$\arctan\eta_S - \arctan 1 = -\frac{1}{2}s.$$

But $\arctan 1 = \pi/4$. Therefore,

$$\eta_S(s) = \tan\left(\frac{\pi}{4} - \frac{s}{2}\right).$$

Thus, we have got a curve

$$s \mapsto (\xi_S(s), \eta_S(s)) = \left(0, \tan\left(\frac{\pi}{4} - \frac{s}{2}\right)\right) \text{ for } -\frac{\pi}{2} < s < \frac{3\pi}{2}.$$

Now we want to compare this result with the result obtained by the sereographic projection from the North Pole. Composing this with the change of coordinates formula

$$\begin{aligned}\xi_N &= \frac{\xi_S}{\xi_S^2+\eta_S^2},\\ \eta_N &= \frac{\eta_S}{\xi_S^2+\eta_S^2}.\end{aligned}$$

we get

$$\begin{aligned}
\xi_N(s) &= \frac{\xi_S(s)}{\xi_S^2(s)+\eta_S^2(s)} = 0, \\
\eta_N(s) &= \frac{\eta_S(s)}{\xi_S^2(s)+\eta_S^2(s)} = \frac{\tan((\pi-2s)/4)}{\tan^2((\pi-2s)/4)} = \cot((\pi-2s)/4)) \\
&= \frac{\cos\left(\frac{\pi}{4}-\frac{s}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{s}{2}\right)} = \frac{\cos(\pi/4)\cos(s/2)+\sin(\pi/4)\sin(s/2)}{\sin(\pi/4)\cos(s/2)-\cos(\pi/4)\sin(s/2)}.
\end{aligned}$$

Since $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$, it follows that

$$\begin{aligned}
\eta_N(s) &= \frac{\cos(s/2)+\sin(s/2)}{\cos(s/2)-\sin(s/2)} = \frac{\sin(\pi/4)\cos(s/2)+\cos(\pi/4)\sin(s/2)}{\cos(\pi/4)\cos(s/2)-\sin(\pi/4)\sin(s/2)} \\
&= \frac{\sin\left(\frac{\pi}{4}+\frac{s}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{s}{2}\right)} = \tan\left(\frac{\pi}{4}+\frac{s}{2}\right).
\end{aligned}$$

Thus, we have a curve

$$t \mapsto \left(0, \tan\left(\frac{\pi}{4}+\frac{s}{2}\right)\right) \text{ for } -\frac{\pi}{2} < s < \frac{3\pi}{2} \text{ and } s \neq \frac{\pi}{2}.$$

It agrees with equation (4) in $U_N \cap U_S$.

Problem 17 *Find the integral curve on S^2 of the vector field $Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$ with initial condition $(1,0,0)$ by integration in variables (x,y,z), (ξ_N,η_N) and (ξ_S,η_S) and compare the results using the change of variables formulae given in this section.*

4 Families of vector fields

Let X and Y be vector fields on a manifold M and $\exp tX$ and $\exp sY$ the local one-parameter groups of local diffeomorphisms of M generated by X and Y, respectively. Suppose that the vector field X is complete. Then, for each $t \in \mathbb{R}$, $\exp tX : M \to M$ is a diffeomorphism of M to itself. For each $t \in R$, $x \in M$ and $f \in C^\infty(M)$,

$$\frac{d}{dt}((\exp tX)^* f)(x) = \frac{d}{dt}(f \circ \exp tX)(x) = X(f(\exp tX(x))) = (X(f \circ \exp tX))(x)$$

because $t \mapsto \exp tX(x)$ is the unique integral curve of X through x.

We can push forward the vector field Y on M to a vector field $(\exp tX)_* Y$ on M defined by

$$((\exp tX)_* Y)\, f = (\exp(-tX))^* \left(Y((\exp tX)^* f)\right)$$

for each $f \in C^\infty(M)$, where we have used the fact that

$$(\exp(tX))^{-1} = \exp(-tX).$$

In other words,

$$((\exp tX)_* Y)\, f = (Y(f \circ (\exp tX)) \circ \exp(-tX).$$

Differentiating with respect to t we get

$$\begin{aligned}
\frac{d}{dt}\,((\exp tX)_* Y)\, f &= \frac{d}{dt}\,(Y(f \circ (\exp tX)) \circ \exp(-tX) \\
&= (Y(X(f \circ \exp tX))) \circ \exp(-tX) - X((Y(f \circ (\exp tX)) \circ \exp(-tX)) \\
&= \{Y(X(f \circ \exp tX)) - X\,(Y(f \circ (\exp tX))\} \circ \exp(-tX) \\
&= \{[Y,X]((f \circ \exp tX)\} \circ \exp(-tX) \\
&= ((\exp(tX))_*[Y,X])f
\end{aligned}$$

Hence,

$$\frac{d}{dt}\,((\exp tX)_* Y) = (\exp(tX))_*[Y,X].$$

Corollary 18 *If* $[Y,X] = 0$ *then* $(\exp tX)_* Y = Y$.

Proof. When $t = 0$, the diffeomorphism $\exp tX$ of M to itself is the identity on M. In other words, $\exp 0X = id_M$. Moreover, $(id_M)_* Y = Y$. Integrating the above equation from 0 to t, we get Hence,

$$\begin{aligned}
(\exp tX)_* Y - Y &= \int_0^t \frac{d}{dt'}\,((\exp t'X)_* Y)\, dt' \\
&= \int_0^t (\exp(t'X))_*[Y,X] dt' \\
&= 0 \quad \text{because} \quad [Y,X] = 0.
\end{aligned}$$

■

Proposition 19 *If* $[Y,X] = 0$ *then* $(\exp tX) \circ (\exp sY) = (\exp sY) \circ (\exp tX)$.

Proof. By Proposition 16, for fixed t, the one-parameter local group $(\exp tX) \circ (\exp sY) \circ (\exp(-tX))$ of local diffeomorphisms of M is generated by $(\exp tX)_* Y$. However, $[Y,X] = 0$ implies that $(\exp tX)_* Y = Y$. Hence, $(\exp tX) \circ (\exp sY) \circ (\exp(-tX)) = (\exp sY)$ which implies that $(\exp tX) \circ (\exp sY) = (\exp sY) \circ (\exp tX)$.
■

Consider a family $\mathcal{F} = \{X_a\}_{\alpha \in A}$ of vector fields on M indexed by a set A. For $x \in M$, the orbit of this family through x is a set

$$\begin{aligned}
O_x &= \{(\exp(t_n X_{\alpha_n})) \circ (\exp(t_{n-1} X_{\alpha_{n-1}})) \circ ... \circ (\exp(t_1 X_{\alpha_1}))(x) \\
&\mid \; n \in \mathbb{N},\; (t_1, ..., t_n) \in \mathbb{R}^n,\; X_{\alpha_1}, ..., X_{\alpha_n} \in \mathcal{F}\}.
\end{aligned}$$

Example 20 *Find the orbit of the family $\mathcal{F} = \{X, Y\}$ of vector fields on $\mathbb{R}^2$, where*

$$X = \frac{\partial}{\partial x} \quad \textit{and} \quad Y = \frac{\partial}{\partial x} + h(x)\frac{\partial}{\partial y}$$

and

$$\begin{aligned} h(x) &= e^{-\frac{1}{x}} \textit{ for } x > 0, \\ h(x) &= 0 \textit{ for } x \leq 0. \end{aligned}$$

Solution 21 *First find integral curves of X and Y. Integral curves of X satisfy equations*

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 0. \end{aligned}$$

This implies

$$\begin{aligned} x(t) &= x(0) + t, \\ y(t) &= y(0). \end{aligned}$$

Hence, for every $(x, y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$,

$$(\exp(tX))(x, y) = (x + t, y).$$

Similalry, integral curves of Y satisfy equations

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= h(x). \end{aligned}$$

Hence,

$$\begin{aligned} x(t) &= x(0) + t, \\ y(t) &= y(0) + \int_0^t h(x(s))ds = y(0) + \int_0^t h(x(0) + s)ds. \end{aligned}$$

But,

$$\begin{aligned} h(x(0) + s) &= e^{\frac{1}{x(0)+s}} \textit{ for } x(0) + s > 0, \\ h(x(0) + s) &= 0 \textit{ for } x(0) + s \leq 0. \end{aligned}$$

Since $e^{\frac{1}{x(0)+s}} > 0$ for sufficiently large s, it follows that

$$\int_0^t h(x(0) + s)ds > 0 \textit{ for sufficiently large } t.$$

Theorem 22 *Let $\mathcal{F}$ be a family of smooth vector fields on a manifold M. For each $x \in M$, the orbit O_x of F through x is a manifold. Moreover, the inclusion map $O_x \hookrightarrow M$ is smooth.*

Proof. See H. J. Sussmann, "Orbits of families of vector fields and integrability of distributions", *Trans. Amer. Math. Soc.* **180** (1973) 171-188. ■

Let $\mathcal{F}$ be a family of vector fields on M. By *span*$\mathcal{F}$ we denote the family of all linear combinations of vector fields in $\mathcal{F}$. The set

$$D = \{X(x) \in TM \mid X \in span\mathcal{F},\, x \in M\}$$

is called the *generalized distribution* on M spanned by $\mathcal{F}$. For each $x \in M$, we denote by D_x the intersection of D with TM. That is,

$$D_x = D \cap T_xM = \{X(x) \in TM \mid X \in span\mathcal{F}\}.$$

It is a vector subspace of T_xM. The dimension of D_x is also called the *rank* of D at x. If the rank of D is constant on M, we say that D is a *distribution* on M.

Example 23 *Let $\mathcal{F} = \{X, Y\}$ be the family of vector fields on $\mathbb{R}^2$ considered in the preceding example. The generalized distribution D on $\mathbb{R}^2$ spanned by F has rank 1 at each (x, y) such that $x \leq 0$, and it has rank 2 at all other points.*

Let D be a generalized distribution on M.

Definition 24 *A manifold $N \subseteq M$ is an integral manifold of D if $T_xN = D_x$ for every $x \in N$, where we identify points of N with the corresponding points of M.*

Integral manifolds of a distribution D on M passing through the same point $x \in M$ can be ordered by inclusion. An *integral manifold* N of D passing through x is *maximal* if every other integral manifold of D passing through x is contained in N. Every integral manifold of D through x can be extended to a maximal one.

Theorem 25 *Let D be the generalized distribution on M spanned by the family $\mathcal{F}$ of vector fields on M. An orbit O of $\mathcal{F}$ is an integral manifold of D if and only if*

1. *rank of D is constant on O,*
2. *For each $X_1, X_2 \in \mathcal{F}$ and $x \in O$,*

$$[X_1, X_2](x) \in D_x.$$

Proof. See J. Śniatycki, "Generalizations of Frobenius' theorem on manifolds and subcartesian spaces", *Can. Math. Bull.*, **50** (2007) 447-459. ■

A distribution D on M is *integrable* if, for every $x \in M$, there exists a maximal integral manifold of D through x. A distribution D on M is *involutive* if, for every pair of vector fields X and Y with values in D, their bracket $[X, Y]$ has values in D.

Theorem 26 *(Frobenius) Every involutive distribution is integrable.*

Project Prove Frobenius' theorem. See F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer Verlag, New York, 1983.