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CHAPTER 2

DIFFERENTIAL SPACES

1 Topological spaces

Definition 1 A *topology* on a set S is a family $\mathcal{T}$ of subsets of S satisfying the following conditions.

1. The empty set $\emptyset$ and S belong to $\mathcal{T}$.
2. If $\{U_\alpha\}_{\alpha \in A}$ is an arbitrary family of sets in $\mathcal{T}$, then their union is contained in $\mathcal{T}$.
3. If $\{U_1, ..., U_n\}$ is a finite family of sets in $\mathcal{T}$, then their intersection is in $\mathcal{T}$.

A set S endowed with a topology $\mathcal{T}$ is called a *topological space*. Sets in $\mathcal{T}$ are called *open sets*. Complements of open sets are called *closed* sets. In other words, $C \subseteq S$ is closed if its complement

$$S \backslash C = \{x \in S \mid x \notin C\}$$

is open. The notions of a basis and a subbasis of the topology are used for arbitrary topological spaces. That is, a family $\{U_\alpha\}_{\alpha \in A}$ is a basis of the topology T of S if, for every open set U in S and each $x \in U$, there exists $\alpha \in A$ such that $x \in U_\alpha \subseteq U$. Further, a family $\{V_\beta\}_{\beta \in B}$ of open sets in S is called a *subbasis* for the topology $\mathcal{T}$ of S if the family of sets consisting of intersections of fininte numbers of sets in $\{V_\beta\}_{\beta \in B}$ is a basis for the topology $\mathcal{T}$.

An example of a topological space is the Euclidean space $\mathbb{R}^n$ with the standard topology described in the preceding chapter.

Example 1. (**Trivial topology**) For an arbitrary set S, the trivial topology $\mathcal{T}$ on S consists of the empty set and S. That is , $\mathcal{T} = \{\emptyset, S\}$. It is easy to see that this topology satisfies the conditions of Definition 1.

Example 2. (**Discrete topology**) The discrete topology on a set S is the family of all subsets of S.

Neither trivial topology nor discrete topology on a set S provide any informations about S which is not contained in the definition of S.

Let R and S be a topological spaces. A map $\varphi : S \to R$ from S to R is *continuous* if for every open set U in R the set

$$\varphi^{-1}(U) = \{x \in S \mid \varphi(x) \in U\}$$

is open. If $\varphi : S \to R$ from S to R is continuous and $\varphi^{-1} : R \to S$ is continuous, we say that φ is a homeomorphism (i.e. topological isomorphism). In particular, a real-valued function $f : S \to \mathbb{R}$ is continuous if, for every open interval $(a, b) \subseteq \mathbb{R}$ the inverse image $f^{-1}((a, b))$ is open in S. The space of all continuous functions on S is denoted $C^0(S)$.

Example 3. (**Subspace topology**) Let $\mathcal{T}_S$ be a topology on S and let R be a subset of S. We define a topology $\mathcal{T}_R$ on R by $\mathcal{T}_R = \{U \cap R \mid U \in \mathcal{T}_S\}$. With this topology, the inclusion map $\iota : R \hookrightarrow S$ is continuous. R endowed with this topology is a *topological subspace* of S.

In the following we shall mainly deal with topological subspaces of the Euclidean space $\mathbb{R}^n$. In other words, we shall study subsets of $\mathbb{R}^n$ endowed with the subspace topology.

There are two common construction of forming new topological spaces namely *product* and *quotient*.

Example 4. (Product topology) If R and S are topological spaces with topologies $\mathcal{T}_R$ and $\mathcal{T}_S$, respectively, then the *product topology* on $R \times S$ has subbasis $\{U \times V \mid U \in \mathcal{T}_R,\ V \in \mathcal{T}_S\}$.

Example 5. (Quotient topology) Consider a topological space S with topology T_S. An *equivalence relation* on S is a subset R of $S \times S$ which satisfies the following properties:

Reflexivity $\{(x, x) \in R \mid s \in S\}$;

Symmetry If $(x_1, x_2) \in R$ then $(x_2, x_1) \in R$;

Transitivity If $(x_1, x_2) \in R$ and $(x_2, x_3) \in R$ then $(x_1, x_3) \in R$.

For each $x \in S$, the *class* of x with respect to the relation R is

$$[x] = \{x' \in S \mid (x, x') \in R\}.$$

The *quotient* of S by the equivalence relation R is the set

$$S/R = \{[x] \mid x \in S\}.$$

Let $\rho : S \to S/R$ denote the projection map associating to each $x \in S$ its class $[x]$ in S/R. In other words, $\rho(x) = [x]$ for all $x \in S$. The *quotient topology* in S/R is given by

$$\mathcal{T}_{S/R} = \{U \subseteq S/R \mid \rho^{-1}(U) \in \mathcal{T}_S\}.$$

2 Differential structures

Recall that the ring $C^\infty(U)$ of smooth functions on an open subset U of $\mathbb{R}^n$ has the following

Localization Property. A function $f : U \to \mathbb{R}$ is in $C^\infty(U)$ if, for every $x \in U$, there exists a neighbourhood W_x of x in U and a function $F_x \in C^\infty(\mathbb{R}^n)$ such that the restriction $f_{|W_x}$ of f to W_x coincides with the restriction $F_{x|W_x}$ of F_x to W, i.e.

$$f_{|W_x} = F_{x|W_x}.$$

You can ask if we can get an interesting theory if we replace an open set U in $\mathbb{R}^n$ by an arbitrary subset S of $\mathbb{R}^n$. In other words, we consider the ring $C^\infty(S)$ of functions on a subset S of $\mathbb{R}^n$ defined as follows.

Definition 1. The *induced differential structure* on a subset S of $\mathbb{R}^n$ is a family $C^\infty(S)$ of continuous functions on S such that $f \in C^\infty(S)$ if, for every $x \in S$, there exists a neighbourhood W of x in S and a function $F_x \in C^\infty(\mathbb{R}^n)$ such that the restriction $f_{|W}$ of f to W coincides with the restriction $F_{x|W}$ of F_x to W, i.e.

$$f_{|W} = F_{x|W}.$$

Theorem 1. If S is a closed subset of $\mathbb{R}^n$ then $C^\infty(S)$ consists of restrictions to S of functions in $C^\infty(\mathbb{R}^n)$. In other words, if $f \in C^\infty(S)$ then there exists $F \in C^\infty(\mathbb{R}^n)$ such that $f = F_{|S}$.

Proof. Let $f \in C^\infty(S)$. By definition, for each $x \in S$, there is an open neighbourhood W_x of x in S and $F_x \in C^\infty(\mathbb{R}^n)$ such that $f_{|W_x} = F_{x|W_x}$. The topology of S is the subspace topology. Hence, $W_x = S \cap V_x$ where V_x is an open subset of $\mathbb{R}^n$. Since S is closed, its complement $V_0 = \mathbb{R}^n \backslash S$ is open, and the family of sets

$\{V_x\}_{x\in S}\cup\{V_0\}=\{V_\alpha\}_{\alpha\in A}$, where $A=S\cup\{0\}$, is an open cover of $\mathbb{R}^n$. Let $\{e_\alpha\}_{\alpha\in A}$ be a partition of unity on $\mathbb{R}^n$ subordinate to this cover. Each e_α is a non-negative smooth function on $\mathbb{R}^n$ with supp $e_\alpha\subseteq V_\alpha$. This means that, for $\alpha=x\in S$, supp e_x =supp $e_x\subseteq V_x$, and supp $e_0\subseteq V_0=\mathbb{R}^n\backslash S$. Moreover, the family of sets $\{\text{supp } e_\alpha\}$ is locally finite and, for every $y\in S$,

$$\sum_{\alpha\in A} e_\alpha(y)=\sum_{x\in S} e_x(y)+e_0(y)=1.$$

Consider, a function F on $\mathbb{R}^n$ defined by

$$F(y)=\sum_{x\in S} f_x(y)e_x(y)+0e_0(x)=\sum_{x\in S} f_x(y)e_x(y)$$

for each $y\in\mathbb{R}^n$. Since the family of sets $\{\text{supp } e_\alpha\}$ is locally finite, each $y\in R^n$ has a neighbourhood in which F is the sum of a finite number of smooth functions. Hence, $F\in C^\infty(\mathbb{R}^n)$. Moreover, if $y\in S$, then $e_0(y)=0$, because supp $e_0\in R^n\backslash S$, and

$$\sum_{x\in S} e_x(y)=1.$$

Hence,

$$F(y)=\sum_{x\in S} f_x(y)e_x(y)=\sum_{x\in S} f(y)e_x(y)=f(y)\sum_{x\in S} e_x(y)=f(y)$$

for every $y\in S$. This implies that $f=F_{|S}$. ■

A subset $S\subseteq\mathbb{R}^n$ endowed with the differential structure $C^\infty(S)$ is called a *differential subspace* of $\mathbb{R}^n$. The differential structure $C^\infty(S)$ enables us to use calculus to study the geometry of S. Since S is an arbitrary subset of $\mathbb{R}^n$ you can see how general are results obtained in this way.

In this course we shall study not arbitrary subsets of $\mathbb{R}^n$ but restrict our consideration to manifolds. In order to define the notion of manifold we consider first a more general setting. Namely, we define a differential structure on a topological space S.

Definition 2. A *differential structure* on a topological space S is a family $C^\infty(S)$ of real valued-functions on S such that:

1. The family of sets $\{f^{-1}((a,b))\mid f\in C^\infty(S) \text{ and } (a,b)\subseteq\mathbb{R}\}$ is a subbasis for the topology of S.
2. If $F\in C^\infty(\mathbb{R}^n)$ and $f_1,...,f_n\in C^\infty(S)$, then $F(f_1,...,f_n)\in C^\infty(S)$.

3. If $f : S \to \mathbb{R}$ is such that, for every $x \in S$ there exists an open set U in S containing x and a function $f_x \in C^\infty(S)$ such that the restriction $f_{|U}$ of f to U coincides with the restriction $f_{x|U}$ of f_p to U, i.e.

$$f_{|U} = f_{x|U},$$

then $f \in C^\infty(S)$.

Functions $f \in C^\infty(S)$ are called *smooth functions* on S. Condition 1 of Definition 2 implies that smooth functions on S are continuous. In other words, $C^\infty(S) \subseteq C^0(S)$. A topological space S endowed with a differential structure $C^\infty(S)$ is called a *differential space*. An example of a differential space is the Euclidean space $\mathbb{R}^n$ with the standard topology and the *standard differential structure* $C^\infty(\mathbb{R}^n)$ defined in the calculus.

Remark 1. In definition 1 we assume that S is a topological space. This makes it easy to formulate condition 3 which requires the notion of open sets on S. However, one can proceed differently. One can consider a set S and a family $\mathcal{F}$ of functions on S. Introduce a topology on S by defining $U \subseteq S$ to be open if, for every $x \in U$, there exists a positive integer n, functions $f_1, ..., f_n \in \mathcal{F}$ and open intervals $(a_1, b_1), ..., (a_n, b_n)$ in $\mathbb{R}$ such that

$$x \in f_1^{-1}((a_1, b_1) \cap ... \cap f_n^{-1}((a_n, b_n)).$$

Next, define $C^\infty(S)$ in such a way that conditions 2 and 3 are satisfied. In other words, require that, if $f_1, ..., f_n$ are in $\mathcal{F}$ and F is in $C^\infty(\mathbb{R}^n)$, then $F(f_1, ..., f_n) \in C^\infty(S)$. Moreover, require that a function $f : S \to \mathbb{R}$ is in $C^\infty(S)$ if, for every $x \in S$ there exists an open set U in S containing x and a function $f_x \in C^\infty(S)$ such that $f_{|U} = f_{x|U}$.

Below, we give some more examples of differential spaces.

Example 1. S is an arbitrary set endowed with the trivial topology (see example 1 of section 1) and $C^\infty(S)$ is the set of all constant functions on S.

Example 2. S is an arbitrary set endowed with the discrete topology (see example 2 of section 1) and $C^\infty(S)$ is the set of all functions on S.

Example 3. Let R be a differential space with a differential structure $C^\infty(R)$ and let S be an arbitrary subset of R. We define $C^\infty(S)$ as follows. A function $f : S \to \mathbb{R}$ is smooth if, for every $x \in S$, there exists an open neighbourhood V of x in R and $F_x \in C^\infty(R)$ such that $f_{|S \cap V} = F_{x|S \cap V}$. The set S endowed with this differential structure is called a *differential subspace* of R.

Let S and R be differential spaces with differential structures $C^\infty(S)$ and $C^\infty(R)$, respectively. A map $\varphi : R \to S$ is said to be *smooth* or C^∞ if, for every $f \in C^\infty(S)$ the composition $f \circ \varphi \in C^\infty(R)$. A map $\varphi : R \to S$ is a *diffeomorphism* if it is smooth, invertible and its inverse $\varphi^{-1} : S \to R$ is smooth. We say that R and S are *diffeomorphic* if there exists a diffeomorphism $\varphi : R \to S$.

Definition 3. A differential space S is *locally diffeomorphic* to a differential space R if, for each $x \in S$, there exists an open neighbourhood U of x in S, an open set V in R and a diffeomorphism of the differential subspace U of S onto the differential subspace V of R.

Definition 4. A differential space S is a *smooth manifold of dimension* n if it is locally diffeomorphic to $\mathbb{R}^n$ with the standard differential structure.

For the sake of completeness, we describe below differential structures on the product and the quotient of differential spaces.

Example 4. Let S and R be differential spaces and with differential structures $C^\infty(S)$ and $C^\infty(R)$, respectively. Their product $R \times S$ is a differential space with differential structure $C^\infty(R \times S)$ defined as follows. Consider the family $\mathcal{F}_0$ of functions on $R \times S$ given by the product of a smooth function on R and a smooth function on S. In other words,

$$\mathcal{F}_0 = \{f^R f^S \mid f^R \in C^\infty(R),\ f^S \in C^\infty(S)\}.$$

It follows from the definition of the product topology (Example 4 of the preceding section) and the propery 1 of definition 1 that the family of sets

$$\{f^{-1}((a,b)) \mid f \in \mathcal{F}_0 \text{ and } (a,b) \subseteq \mathbb{R}\}$$

is a subbasis for the topology of $R \times S$. Consider the family

$$\mathcal{F} = \{F(f_1, ..., f_n) \mid n \in \mathbb{N},\ F \in C^\infty(\mathbb{R}^n) \text{ and } f_1, ..., f_n \in \mathcal{F}_0\}$$

of functions on $R \times S$. It satisfies properties 1 and 2 of definition 1. We define $C^\infty(R \times S)$ as in Remark 1.

Example 5. Let S be a differential space with differential structure $C^\infty(S)$, $R \subseteq S \times S$ an equivalence relation on S and $\rho : S \to S/R : x \mapsto [x]$ the projection map from S to the quotient S/R (see example 5 of the preceding section). Let

$$C^\infty(S/R) = \{f : (S/R) \to \mathbb{R} \mid f \circ \rho \in C^\infty(S)\}.$$

If the family of sets

$$\{f^{-1}((a,b)) \mid f \in C^\infty(S/R) \text{ and } (a,b) \subseteq \mathbb{R}\} \tag{1}$$

is a subbasis for the topology of S/R then $C^\infty(S/R)$ is a differential structure on S/R with the quotient topology.

Exercises

Problem 2.1. Verify that the differential structure introduced in example 1 satisfies the conditions of Definition 2.

Problem 2.2. Verify that the differential structure introduced in example 2 satisfies the conditions of Definition 2.

Problem 2.3. Verify that the differential structure introduced in example 3 satisfies the conditions of Definition 2.

Problem 2.4. Verify that the differential structure introduced in example 4 satisfies the conditions of Definition 2

Problem 2.5. Verify that the differential structure described in remark 1 satisfies condition 3 of Definition 2.

Problem 2.6. Verify that the differential structure introduced in example 5 satisfies the conditions of Definition 2.

3 Manifolds

In the preceding section we defined a manifold of dimension n to be a differential space which is locally diffeomorphic to the n-dimensional Euclidean space R^n with the standard differential structure $C^\infty(\mathbb{R}^n)$. In this section we shall discuss consequences of this definition and give examples of manifolds.

Let M be a smooth manifold of dimension n. By definition it is locally diffeomorphic to $\mathbb{R}^n$. That is, for each point $x \in M$, there exists a neighbourhood U of x in M and a diffeomorphism φ of U onto an open subset V of $\mathbb{R}^n$. Such a diffeomorphism $\varphi : U \to V$ is called a *chart* on M. If we want to emphasize that $x \in U$, we say that φ is a chart at x. By definition of a manifold, we have a chart at x for each x in M. Thus, M is covered by a family of open sets each of which is diffeomorphic to an open subset of $\mathbb{R}^n$. The collection of all local charts on M is called a complete *atlas* on M.

You can recognize the above terminology from cartography. In cartography, Earth's surface is studied in terms of charts (or maps) which describe its regions on sheets of paper which can be thought of as subsets of $\mathbb{R}^2$ if we ignore the thickness of the sheet. An atlas is a collection of charts covering the whole surface of the Earth. Clearly, a collection of all charts, if it were ever produced, would deserve the name of a complete atlas.

The traditional approach to manifolds was in terms of local charts. One would define geometric objects in terms of local charts. This approach required to make sure that the definition was independent of the choice of a chart.

In the approach adopted here, the primary information about the manifold M is encoded in its differential structure $C^\infty(M)$, that is the ring of functions on M which we consider to be smooth, and not in the in the complete atlas on M. We still can perform computations in terms of local charts, and we shall do it because this way we can use the power of calculus. However, all our definitions are given in terms of $C^\infty(M)$ and not an atlas on M which frees us from checking transformation properties of our geometric objects under the change of charts.

3.1 Circle S^1

The circle S^1 is the set of points (x, y) in $\mathbb{R}^2$ satisfying the equation $x^2 + y^2 = 1$. In symbols,

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

The space $C^\infty(S^1)$ consists of restrictions to S^1 of smooth functions on $\mathbb{R}^2$. Let $F \in C^\infty(\mathbb{R}^2)$. Introducting polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get $F(x, y) = F(r\cos\theta, r\sin\theta)$. Hence,

$$F_{|S^1} = F(\cos\theta, \sin\theta).$$

Since $\cos\theta$ and $\sin\theta$ are periodic functions of θ with period 2π, it follows that every functions $f \in C^\infty(S^1)$ can be presented as the periodic function of θ with period 2π. Conversely, Every smooth function of a parameter θ can be presented as a function of $\cos\theta$ and $\sin\theta$. Hence, we have have obtained

Proposition 1. The differential structure $C^\infty(S^1)$ of the circle S^1 consists of smooth functions of a single variable $\theta \in \mathbb{R}$ which are periodic with period 2π.

Remark 1. We could have defined S^1 in a different way. Consider an equivalence relation ~ in $\mathbb{R}$ given by $t_1 \sim t_2$ if $t_1 - t_2 = 2\pi n$, where n is any integer. other The set $\mathbb{R}/\sim$ of equivalence classes of this relation is usually called a *torus* and is denoted T^1. It can be visualized as the loop obtained by glueing together the ends of a thin strip of paper.

Let $\rho : \mathbb{R} \to T^1$ denote the projection map associating to each $t \in \mathbb{R}$ its equivalence class $[t]$ in T^1. Each periodic function $F \in C^1(\mathbb{R})$ with period 2π defines a fuction f on $\mathbb{R}/\sim$ such that $F = f \circ \rho$. Conversely, every function f on $\mathbb{R}/\sim$ defines a periodic function $F = f \circ \rho$ on R with period 2π. It can be shown that the space of functions

$$C^\infty(T^1) = \{f : (\mathbb{R}/\sim) \to \mathbb{R} \mid f \circ \rho \in C^\infty(\mathbb{R})\}$$

is a differential structure on $\mathbb{R}/\sim$. The space $\mathbb{R}/\sim$ endowed with this differential structure is called the 1-torus and is denoted T^1. Proposition 1 above states that S^1 and T^1 are diffeomorphic.

Project III. Prove that every smooth periodic function on R with period 2π can be presented as a smooth function of $\sin\theta$ and $\cos\theta$.

3.2 Torus T^n

The n-torus T^n is the product of n-copies of the 1-torus T^1 which is diffeomorphic to the circle S^1. Since each point of T^1 has a neighbourhood U diffeomorphic to an open interval I in $\mathbb{R}$, it follows that each point $(t_1, ..., t_n) \in T^n$, where $t_1, ..., t_n$ are in T^1, has a neighbourhood of the form $V = U_1 \times U_2 \times ... \times U_n$. Each U_i, for $i = 1, ..., n$ is diffeomorphic to some open interval $I_i \subseteq \mathbb{R}$. It follows from Example 5 in Section 2 that the differential structure on V consists of functions of the form $F(f_1, ..., f_n)$ where f_i is a smooth function on U_i and $F \in C^\infty(\mathbb{R}^n)$. Hence, V is diffeomorphic to the open set in $\mathbb{R}^n$ given as the product $I_1 \times I_2 \times ... \times I_n$ of n open intervals. Therefore, T^n is a manifold of dimension n.

One could show an alternative description of the differential structure of T^n. The 1-torus T^1 is the quotient of R by the equovalence relation ˜ in $\mathbb{R}$ given by $t_1 \sim t_2$ if $t_1 - t_2 = 2\pi n$, where n is any integer. In the terminology of Example 5 of Section 2,

$$\begin{aligned} R &= \{(t,s) \in \mathbb{R}^2 \mid t - s = 2\pi k \text{ for } k \in \mathbb{Z}\} \\ &= \{(t,s) \in \mathbb{R}^2 \mid t \; - s \in 2\pi\mathbb{Z}\} \end{aligned}$$

. Hence, T^n is the quotient of $\mathbb{R}^n$ by

$$R^n = \{((t_1, ..., t_n), (s_1, ..., s_n)) \mid 9t_1 - s_1, ..., t_n - s_n) \in 2\pi\mathbb{Z}^n\}.$$

Using the construction of Example 5 we can conclude that $C^\infty(T^n)$ is isomorphic to the ring $C^\infty(\mathbb{R}^n)^{2\pi\mathbb{Z}^n}$ of smooth functions on $\mathbb{R}^n$ which are periodic witth period 2π in each variable.

Problem 3.2.1. Verify that the condition (1) in Example 5 is satisfied in this case.

3.3 Sphere S^n

The sphere S^n is the set of points $(x_1, ..., x_{n+1})$ in $\mathbb{R}^{n+1}$ satisfying the equation $x_1^2 + ... + x_{n+1}^2 = 1$. In symbols,

$$S^1 = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + ...x_{n+1}^2 = 1\}.$$

Since S^n is cloesd in $\mathbb{R}^{n+1}$ the differential structure $C^\infty(S^n)$ consists of restrictions to S^n smooth functions on $\mathbb{R}^{n+1}$.

To show that S^n is locally diffeomorphic to an open subset of R^n consider a point $(\tilde{x}_1, ..., \tilde{x}_{n+1}) \in S^n$. Since $\tilde{x}_1^2 + ...\tilde{x}_{n+1}^2 = 1$, it follows that at least one of the coordinates $\tilde{x}_1, ..., \tilde{x}_{n+1}$ is different from zero. For the sake of definiteness, suppose that $\tilde{x}_{n+1} > 0$. Solving the defining equation for S^n for x^{n+1} in a neighbourhood

$$U = S^n \cap \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

of $(\tilde{x}_1, ..., \tilde{x}_{n+1})$ we get

$$x^{n+1} = \sqrt{1 - x_1^2 - ... - x_n^2} \text{ for } x_1^2 + ... + x_n^2 < 1.$$

Let B^n be the unit open ball in R^n,

$$B^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + ... + x_n^2 < 1\},$$

and

$$\alpha : B^n \to U : (x_1, ..., x_n) \mapsto (x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}).$$

We want to show that α is a diffeomorphism. First, we observe that α is constinuous. Next, we need to show that, for each $f \in C^\infty(S^n)$, $\alpha^* f = f \circ \alpha$ is smooth. Since S^n is closed, there exists a function $F \in C^\infty(R^{n+1})$ such that $f = F_{|S^n}$. The restriction of f to U coincides with the restriction of F to U. Hence,

$$f_{|U} = F_{|U} = F(x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}).$$

The right hand side is a smooth function on B^n.

In order to conclude that $\alpha : B^n \to U$ is smooth, we have also to take into account functions $h \in C^\infty(U)$ which are not restrictions to U of smooth function on S^n. However, for every $h \in C^\infty(U)$, an each point $p \in U$, there exists a neighbourhood V of p in U and a function $f \in C^\infty(S)$ such that $h_{|V} = f_{|V}$. Since, $f = F_{|U}$ for some $F \in C^\infty(\mathbb{R}^{n+1})$ it follows that

$$h_{|V} = F_{|V} = F(x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}).$$

But the right hand side is the restriction to$\alpha^{-1}(V) \subseteq B^n$ of a smooth function on B^n. Thus, $\alpha : B^n \to U$ is smooth.

Next, we need to show that α is invertible and α^{-1} is smooth. Let

$$\beta : U \to B^n : (x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}) \mapsto (x_1, ..., x_n).$$

Then

$$\beta \circ \alpha(x_1, ..., x_n) = \beta(x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}) = (x_1, ..., x_n)$$

and

$$\alpha \circ \beta(x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}) = \alpha(x_1, ..., x_n) = (x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}).$$

Hence $\beta = \alpha^{-1}$. Moreover, β is continuous. To show that β is smooth, consider $h \in C^\infty(B^n)$. Suppose first, that there exists a function $H \in C^\infty(\mathbb{R}^n)$ such that $h = H_{|B^n}$. Let $F \in C^\infty(\mathbb{R}^{n+1})$ be given by

$$F(x_1, ..., x_n, x_{n+1}) = H(x_1, ..., x_n).$$

Then, for $(x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}) \in U$,

$$\begin{aligned}
\beta^* h((x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2})) &= h \circ \beta((x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2})) \\
&= h(x_1, ..., x_n) = H(x_1, ..., x_n) \\
&= F(x_1, ..., x_n, x_{n+1}) = F_{|S^n}(x_1, ..., x_n, x_{n+1}) \\
&= (F_{|S^n})_{|U}(x_1, ..., x_n, x_{n+1})..
\end{aligned}$$

Hence, $\beta^* h \in C^\infty(U)$.

Suppose now that $\tilde{h} \in C^\infty(B^n)$ is not the restriction to B^n of a function $H \in C^\infty(\mathbb{R}^n)$. Consider a point $p \in U$. Let $x = \alpha(p) \in B^n$,. We know that there exists a neighbourhood V of x in B^n and a function $H \in C^\infty(\mathbb{R}^n)$ such that

$$\tilde{h}_{|V} = H_{|V}.$$

Hence, for $(x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}) \in V \subseteq U$, we can repeat the argument above and obtain

$$\beta^* h((x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2})) = (F_{|S^n})_{|U}(x_1, ..., x_n, x_{n+1})$$

for $F \in C^\infty(\mathbb{R}^{n+1})$. Since it holds for every point $p \in U$, it follows that $\beta^* h \in C^\infty(U)$. Hence, β is smooth. Since $\beta = \alpha^{-1}$ and α is smooth, it follows that $\beta : U \to B^n$ and $\alpha : B^n \to U$ are diffeomorphisms.

We can cover S^n by open sets

$$U^i_\pm = S^n \cap \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid \pm x_i > 0\}.$$

For each $U^i_\pm$ we can repeat the above argument. Hence, S^n is locally diffeomorphic to $\mathbb{R}^n$. This implies that S^n is a manifold.

Remark 2. We can generalize the above argument as follows. Let $U \subseteq S$ and $V \subseteq R$ be open differential subspaces of differential spaces S and R, respectivaly and $\alpha : U \to V$ a continuous map. In order to show that α is smooth it suffices to establish that, for each $F \in C^\infty(R)$ the function $F_{|V} \circ \alpha \in C^\infty(U)$.

3.4 Projective space P^n.

The projective space P^n is defined as the space of all hyperplanes in $\mathbb{R}^{n+1}$ passing through the origin. Since every hyperplane through the origin in $\mathbb{R}^{n+1}$ has a unique normal line through the origin, and it is uniquely determined by its normal line, we can identify P^n with the space of all lines in $\mathbb{R}^{n+1}$ passing through the origin. Each line through the origin in $\mathbb{R}^{n+1}$ intersects the unit sphere $S^n \subset \mathbb{R}^{n+1}$ in two points $(x_1, ..., x_{n+1})$ and $(-x_1, ..., -x_{n+1})$. Hence, we can identify P^n with the quotient S^n/R of S^n by the relation

$$R = \{((x_1, ..., x_{n+1}), (-x_1, ..., -x_{n+1})) \mid (x_1, ..., x_{n+1}) \in S^n\} \subset S^n \times S^n.$$

Using the construction of Example 5 of Section 2, we conclude that the differential structure of P^n is isomorphic to the ring $C^\infty(S^n)^R$ of smooth finctions on S^n which are invariant under the transformation $(x_1, ..., x_{n+1}) \mapsto (-x_1, ..., -x_{n+1})$. But $C^\infty(S^n)$ consists of restrictions to S^n of smooth functions on $\mathbb{R}^{n+1}$. Hence, we may identify functions in $C^\infty(P^n)$ with restrictions to S^n of functions on R^{n+1} which are invariant under the $(x_1, ..., x_{n+1})$.

Problem 3.3.1. Verify that the condition (1) in Example 5 is satisfied in this case.