CALCULUS ON MANIFOLDS

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August 31, 2009

CHAPTER 1

ANALYTIC BACKGROUND

1 Open Sets

A point in $\mathbb{R}^n$ is an n-tuple $x = (x^1, ..., x^n)$ of real numbers, called the *coordinates* of the point x. For The Euclidean distance between every pair of points $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$ is given by

$$\|x - y\| = \sqrt{(y^1 - x^1)^2 + ...(y^n - x^n)^2}.$$

It is positive definite, that is $\|x - y\| = 0$ implies $x = y$. Moreover, it satisfies the *triangle inequality*

$$\|x - y\| \leq \|x - z\| + \|z - y\| \tag{1}$$

for every x, y and z in $\mathbb{R}^n$. An *open ball* in R^n centred at x with radius $r > 0$ is the set

$$B_{x,r} = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$$

consisting of points such that their distance from x less than r. In the case when $n = 1$, the Euclidean distance is given by the absolute value function, and open balls reduce to open intervals. Namely, $\|x - y\| = \sqrt{(x - y)^2} = |x - y|$ for every $x, y \in \mathbb{R}$, and $B_{x,r} = (x - r, x + r)$ for every $x \in R$ and $r > 0$.

A subset of $\mathbb{R}^n$ is *open* if it is the union of open balls. In other words, $U \subseteq \mathbb{R}^n$ is open if, for every $x \in U$, there exists $r > 0$ such that $B_{x,r} \subseteq U$.

If $\{U_\alpha\}_{\alpha \in A}$ is a family of open subsets of $\mathbb{R}^n$, parametrized by an arbitrary index set A, then the union $\bigcup_{\alpha \in A} U_\alpha$ is open. The triangle inequality for the Euclidean distance implies that if $(U_1, ..., U_k)$ is a finite family of open sets, then their union $\bigcap_{i=1}^n U_i$ is also open.

A family $\{U_\alpha\}_{\alpha \in A}$ of open sets in $\mathbb{R}^n$ is called a *basis of the topology* of $\mathbb{R}^n$ if every open subset of $\mathbb{R}^n$ can be written as the union $\bigcup_{\alpha \in B} U_\alpha$ for $B \subseteq A$. It follows from the definition of open sets given above that the family of all open balls $B_{x,r}$ is a basis of the topology of $\mathbb{R}^n$. Another convenient basis is the family of all open cubes

$$C_{x,r} = \{y \in \mathbb{R}^n \mid \left|x^i - y^i\right| < r \text{ for } i = 1, ..., n\}.$$

A family $\{V_\beta\}_{\beta \in B}$ of open sets in $\mathbb{R}^n$ is called a *subbasis of the topology* of $\mathbb{R}^n$ if every the family consisting of intersections of a fininte number of sets in $\{V_\beta\}_{\beta \in B}$ is a basis for a topology of $\mathbb{R}^n$.

Let (x_m) be a sequence of points in $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is *a limit of the sequence* (x_m) if, for every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$\|x - x_m\| < \varepsilon, \text{ for all } m > N.$$

The above definition can be rephrased interms of open balls. Namely, a point $x \in \mathbb{R}^n$ is a limit of the sequence (x_m) if, for every open ball $B_{x,\varepsilon}$ centred at x with radius $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $x_m \in B_{x,\varepsilon}$ for all $m > N$. In an analogous way we could define the limit in terms of open cubes, or any other basis of the topology of $\mathbb{R}^n$.

In the reformulation of criteria for a point $x \in R^n$ to be a limit of a sequence (x_m) given above, we used open balls $B_{x,\varepsilon}$ centered at x and open cubes $C_{x,\varepsilon}$ centered at x. We could have also used other families of open sets containing x. In the following, it will be convenient to use the term an *open neighbourhood* of x for an open set containong x.

If a sequence (x_m) has a limit, we say that it is a *convergent sequence*. The triangle inequality implies that the limit of a convergent sequence is unique. If x is the limit of a sequence (x_m), we write $x = \lim_{m\to\infty} x_m$ or $x_m \to x$ as $m \to \infty$. If x_m has components $(x_m^1, ..., x_m^n)$ and x has components $(x^1, ..., x^n)$ then $x_m \to x$ as $m \to \infty$ if and only if $x_m^i \to x^i$ as $m \to \infty$ for every $i = 1, ..., n$.

Complements of open sets are called *closed* sets. In other words, $C \subseteq \mathbb{R}^n$ is closed if its complement

$$\mathbb{R}^n \backslash C = \{x \in \mathbb{R}^n \mid x \notin C\}$$

is open. Closed sets can also be charactrized in terms of sequences. Namely, a subset C of R^n is closed if and only if, for every convergent sequence (x_m) of points in C, its limit $\lim_{m\to\infty} x_m$ is in C.

It follows from the definition that, if $\{C_\alpha\}_{\alpha\in A}$ is a family of closed subsets of $\mathbb{R}^n$ parametrized by an arbitrary index set A, then the intersection $\bigcap_{\alpha\in A} C_\alpha$ is closed. Similarly, if $(C_1, ..., C_k)$ is a finite family of open sets, then their union $\bigcup_{i=1}^n C_i$ is also closed.

Exercises

Problem 1.1 Prove the triangle inequality $\|x - y\| \leq \|x - z\| + \|z - y\|$ for every x, y and z in $\mathbb{R}^n$.

Problem 1.2 Let (x_m) be a sequence and x a point in $\mathbb{R}^n$. Show that the following statements are equivalent.

1. For every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $\|x - x_m\| < \varepsilon$, for all $m > N$.
2. For every open ball $B_{x,\varepsilon}$, there exists $N \in \mathbb{R}$ such that $x_m \in B_{x,\varepsilon}$ for all $m > N$.
3. For every open cube $C_{x,\varepsilon}$, there exists $N \in \mathbb{R}$ such that $x_m \in C_{x,\varepsilon}$ for all $m > N$.
4. For every open neighbourhood U of x there exists $N \in \mathbb{R}$ such that $x_m \in U$ for all $m > N$.

Problem 1.3 Prove that the limit of a convergent sequence in $\mathbb{R}^n$ is unique.

Problem 1.4 Show that a subset C of $\mathbb{R}^n$ is closed if and only if the limit of every convergent sequence of points in C is contained in C.

2 Smooth Functions

2.1 Globally defined functions

For a function $f : \mathbb{R}^n \to \mathbb{R}$, the value of f at a point $x = (x^1, ..., x^n) \in \mathbb{R}^n$ will be denoted by $f(x)$ or $f(x^1, ..., x^n)$. For brevity, we shall also use the notation $x \longmapsto f(x)$. A function f on $\mathbb{R}^n$ is *continuous at a point* $x \in \mathbb{R}^n$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in B_{x,\varepsilon}$. . We say that a function f on $\mathbb{R}^n$ *is continuous* if it is continuous at every point of $\mathbb{R}^n$.

For every $f : \mathbb{R}^n \to \mathbb{R}$, a point $x = (x^1, ..., x^n) \in \mathbb{R}^n$ and $i = 1, ..., n$, the partial derivative of f at $(x^1, ..., x^n)$ with respect to x^i is

$$\frac{\partial f}{\partial x^i}(x^1, ..., x^n) = \lim_{h \to 0} \frac{f(x^1, ..., x^{i-1}, x^i + h, x^{i+1}, ..., x^n) - f(x^1, ..., x^n)}{h}.$$

In order to have a consise notation for higher order derivatives of a function f on $\mathbb{R}^n$, for every n-tupple $\alpha = (\alpha_1, ..., \alpha_n)$ of non–negative integers, we set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

and

$$\frac{\partial^\alpha}{\partial x^\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

If $\alpha_1 = ... = \alpha_n = 0$, then we let

$$\frac{\partial^\alpha}{\partial x^\alpha} f = f.$$

Thus, a function $f : \mathbb{R}^n \to R$ is *smooth* if $\frac{\partial^\alpha}{\partial x^\alpha} f$ is continuous for every n-tupple $\alpha = (\alpha_1, ..., \alpha_n)$ of non-negative integers.

We denote the space of smooth functions on $\mathbb{R}^n$ by $C^\infty(\mathbb{R}^n)$.

Proposition 1 If $f_1, ..., f_k$ are in $C^\infty(\mathbb{R}^n)$ and $F : \mathbb{R}^k \to \mathbb{R}$ is smooth, then the composition $F(f_1, ..., f_k) : \mathbb{R}^n \to \mathbb{R}$ is smooth.

Proof. This is the consequence of of the chain rule. □

Since addition and multiplication are smooth operations, it follows from Proposition 1 that $C^\infty(\mathbb{R}^n)$ is closed with respect of the operations of addition and multiplication.

Example 1 A function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \left\{ \begin{array}{c} e^{-\frac{1}{x}} \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{array} \right\}$$

is smooth.

Proof. Continuity of $h(x)$ for $x \neq 0$ is obvious. Continuity at $x = 0$ follows from the fact that $\lim_{x \to 0^+} e^{-1/x} = 0$. For $x > 0$, the derivative of $e^{-1/x}$ of order m is the product of $e^{-1/x}$ and a polynomial in $\frac{1}{x}$. By l'Hôpital's rule $\lim_{x \to 0^+}(e^{-1/x}/x^m) = 0$ for every $m > 0$. Hence, all derivatives of $h(x)$ are continuous. $\square$

Proposition 2 Let $C \subset \mathbb{R}^n$ be closed and $x = (x^1, ..., x^n) \in \mathbb{R}^n \backslash C$. Then there exists $f \in C^\infty(\mathbb{R}^n)$ such that $f(x) = 1$ and $f(y) = 0$ for every $y \in C$.

Proof. Since $U = \mathbb{R}^n \backslash C$ is open, it follows that there exists $\delta > 0$ such that the ball $B_{x,\delta}$ is contained in U. This means that the intersection $B_{x,\delta} \cap V$ is empty. Let $h \in C^\infty(\mathbb{R})$ be the function given in Example 1. By Proposition 1, the function

$$f(y_1, ..., y_n) = e^{1/\delta^2} h\left(\delta^2 - ((y^1 - x^1)^2 + ... + (y^n - x^n)^2)\right) \tag{2}$$

is smooth, and it vanishes for $y = (y_1, ..., y_n) \notin B_{x,\delta}$. In particular, $f(y) = 0$ for all $y \in V$. Moreover, $f(x) = e^{1/\delta^2} h(\delta^2) = e^{1/\delta^2} e^{-1/\delta^2} = 1$. $\square$

The *closure* of a set $S \subseteq \mathbb{R}^n$ is the smallest closed set $\bar{S}$ containing S. For each $f \in C^\infty(\mathbb{R}^n)$, the *support* of f is the closure of the set $\{x \in \mathbb{R}^n \mid f(x) \neq 0\}$. Note that a function may vanish at some points in its support. For example, the support of the function h defined in Example 1 is the closed half-line $[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$, and $h(0) = 0$.

The content of Proposition 2 is usually summarized by a statement that smooth functions on $\mathbb{R}^n$ *separate points from closed sets.* We could strengthen the statement of Proposition 2 by requiring that $f(x) = 1$ and the support of f has empty intersection with V. A function f satisfying these properties can be obtained by replacing δ in equation 2 by $\delta/2$.

2.2 Locally defined functions

Let U be an open subset of $\mathbb{R}^n$. Given a function f defined on U, and $x \in U$, we can determine the continuity of f at x, and the existence of partial derivatives of f at x in terms of well defined operations on f. We say that $f : U \to \mathbb{R}$ is *smooth* if it is continuous and all its partial derivatives are continuous. We denote by $C^\infty(U)$ the space of all smooth functions on U.

If $h \in C^\infty(R^n)$ then the restriction $h_{|U}$ of h to U is in $C^\infty(U)$. The converse need not hold. For example, let $n = 1$, and U be the open interval $(0, 1)$. Then the function $f : (0, 1) \to \mathbb{R}$, given by $f(x) = 1/x$, is in $C^\infty(U)$, but it has no extension to a smooth function on $\mathbb{R}$.

For each $i = 1, ..., n$, we denote by $q^i : U \to \mathbb{R}$ the restriction to U of the i'th coordinate function on $\mathbb{R}^n$. In other words, $q^i((x^1, ..., x^n)) = x^i$ for all $(x^1, ..., x^n) \in U$. Clearly, the coordinate functions q^i are smooth.

Proposition 3 A function $f : U \to \mathbb{R}$ is in $C^\infty(U)$ if, for every $x \in U$, there exists an open subset W of $\mathbb{R}^n$ containing x, and a function $\tilde{f} \in C^\infty(\mathbb{R}^n)$ such that the restrictions of f and h to $W \cap U$ coincide.

Proof. Let $f \in C^\infty(U)$. Since U is open in $\mathbb{R}^n$, for $x \in U$, there exists $\delta > 0$ such that $B_{x,\delta} \subseteq U$. Then, $0 < \frac{\delta}{4} < \frac{\delta}{2}$ and, by Problem 2 , there exists a function $h \in C^\infty(\mathbb{R}^n)$ such that $h_{|B_{x,\delta/4}} = 1$ and $h(x) = 0$ if $x \notin B_{x,\delta/2}$. Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ be given by

$$\tilde{f}(x) = \left\{ \begin{array}{c} h(x)f(x) \text{ for } x \in U \\ 0 \text{ for } x \notin U \end{array} \right\}.$$

Then $\tilde{f} \in C^\infty(\mathbb{R}^n)$ and $\tilde{f}_{|B_{x,\delta/4}} = f_{|B_{x,\delta/4}}$. □

Exercises

Problem 2.1 Show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \left\{ \begin{array}{c} x \sin(\frac{1}{x}) \text{ for } x \neq 0 \\ 0 \text{ for } x = 0 \end{array} \right\},$$

is continuous but not smooth.

Problem 2.2 Let $0 < \varepsilon < \delta$. For every $x \in \mathbb{R}^n$, construct a function $f \in C^\infty(\mathbb{R}^n)$ such that $f(y) = 1$ for every $y \in B_{x,\varepsilon}$, and $f(y) = 0$ if $y \notin B_{x,\delta}$.

3 The Inverse Function Theorem

A smooth map $F : \mathbb{R}^n \to \mathbb{R}^m$ consists of m functions $f^1, ..., f^m$ in $C^\infty(\mathbb{R}^n)$ such that $F(x) = (f^1(x), ..., f^m(x))$ for every $x \in \mathbb{R}^n$. The matrix-valued function $(\partial f^i / \partial x^j)$ on $\mathbb{R}^n$ is called called the *Jacobi matrix* of F. For each $x \in R^n$, the derivative $DF(x)$ of F at x is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ given by value at x of the Jacobi matrix of F. A number $M \in \mathbb{R}$ is an upper bound for $DF(x)$ if $\|DF(x)u\| \leq M \|u\|$ for all $u \in \mathbb{R}^n$. The *operator norm* of the matrix $DF(x)$ is the least upper bound $\|DF(x)\|$ of $DF(x)$. which exists because the set of upper bounds for $DF(x)$ is bounded from below by zero. Hence,

$$\|DF(x)u\| \leq \|DF(x)\| \|u\| \tag{3}$$

for all $u \in \mathbb{R}^n$.

Proposition 4 If $F = (f^1, ..., f^m)$ is a smooth map from an open subset U of $\mathbb{R}^n$ with values in an open subset V of $\mathbb{R}^m$, such that there exists a smooth map $H = (h^1, ..., h^n) : V \rightarrow U$ such that $F \circ H = id_U$ (the identity on U) and $H \circ F = id_V$, then $m = n$.

Proof. Differentiating equations $F \circ H = id_U$ and $H \circ F = id_V$ we get with the help of the Chain Rule

$$\sum_{k=1}^{m} \left(\frac{\partial f^i}{\partial x^k}\right)(x) \left(\frac{\partial h^k}{\partial y^j}\right)(y) = \delta^i_j \text{ and } \sum_{j=1}^{n} \left(\frac{\partial h^k}{\partial y^j}\right)(y) \left(\frac{\partial f^j}{\partial x^i}\right)(x) = \delta^k_i,$$

where $x \in U$ let $y = F(x) \in V$, and

$$\delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{4}$$

is the *Kronecker symbol.* It follows that the Jacobi matrix of H is the inverse of the Jacobi matrix of F. Since an invertible matrix is square, it follows that $m = n$. □

The Inverse Function Theorem Let U be an open subset of $\mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^n$ be smooth. If the derivative DF is invertible at a point $x_0 \in U$, then there exists an open neigbourhood V of x_0 contained in U such that $F(V)$ is open in $\mathbb{R}^n$, the restriction $F_{|V}$ of F to V is a one-to-one map of V onto $F(V)$ with a smooth inverse $(F_{|V})^{-1} : F(V) \rightarrow V$.

An important consequence of the Inverse Function Theorem is the Implicit Function Theorem stated below. Here, we use the canonical isomorphism

$$\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n+m} : ((x^1, ..., x^n), (y^1, ..., y^m)) \longmapsto (x^1, ..., x^n, y^1, ..., y^m).$$

The Implicit Function Theorem Let (x_0, y_0) be a point in an open subset U of $\mathbb{R}^{n+m} \equiv \mathbb{R}^n \times \mathbb{R}^m$ and

$$F : U \rightarrow R^m : (x^1, ..., x^n, y^1, ..., y^m) \longmapsto (f^1(x, y), ..., f^m(x, y))$$

be a smooth map such that

$$F(x_0, y_0) = 0 \text{ and } \det(\partial f^a / \partial x^i)(x_0, y_0) \neq 0.$$

Then, there exists a open neighbourhood V of x_0 in $\mathbb{R}^n$ an open neighbourhood W of y_0 in $\mathbb{R}^m$ such that $V \times W \subseteq U$ and there exists a smooth map $H : V \rightarrow W$ such that, for every $(x, y) \in V \times W$,

$$F(x, y) = 0 \text{ if and only if } y = H(x).$$

4 Differential equations of first order

In this section we state existence, uniqueness and smooth dependence on initial data of a solution of a Cauchy problem

$$\dot{x}(t) = F(x(t)), \text{ and } x(t_0) = x_0, \tag{5}$$

where $t \longmapsto x(t)$ is a curve in $\mathbb{R}^n$, $\dot{x}(t) = \frac{dx(t)}{dt}$ and $F \in C^\infty(\mathbb{R}^n)$.

Existence and uniqueness For each $x_0 \in \mathbb{R}^n$, there exists a maximal unique solution $x = f_{x_0}(t)$ of the Cauchy problem (5) defined in an open interval $(T_-(x_0), T_+(x_0)$ where $-\infty \leq T_-(x_0) < 0 < T_+(x_0) \leq \infty$. If

$$\begin{aligned} \text{If } T_-(x_0) &\neq -\infty, \text{ then } \|x(t)\| \to \infty \text{ as } t \to T_-(x_0) \text{ from above.} \\ \text{If } T_+(x_0) &\neq +\infty, \text{ then } \|x(t)\| \to \infty \text{ as } t \to T_+(x_0) \text{ from below.} \end{aligned}$$

Smoothness For each $t \in (T_-(x_0), T_+(x_0))$, the function $x_0 \to f_{x_0}(t)$ is of class C^∞.