First order homogeneous O.D.E.

Additional material on substitutions for first order ordinary differential equations.

Definition: A first order o.d.e. is called homogeneous if it can be written in the form $\frac{dy}{dx} = G\left(\frac{y}{x}\right)$ for some function G.

Example: The equation $x^2\frac{dy}{dx} = yx + 3y^2, \ \ x > 0$

is homogeneous since we can rewrite it as $\frac{dy}{dx} = \frac{yx+3y^2}{x^2} = \frac{y}{x} + 3\left(\frac{y}{x}\right)^2$.

To solve these equations try the substitution $u = \frac{y}{x}$. Then $xu = y$ and taking $\frac{d}{dx}$ of both sides gives $u + x\frac{du}{dx} = \frac{dy}{dx}$. The original equation then becomes $u + x\frac{du}{dx} = G(u)$ which is always a separable equation.

Once this equation is solved don't forget to express the solution in terms of the original variables x and y.

Try to solve the example above.

$\left(\text{Answer: } \ y = \frac{-x}{3\ln x + c}, \ \ c \in \mathbb{R}\right).$

Sample problems:

1. $y^2\frac{dy}{dx} = y^3 - x^3, \ \ x > 0.$

   (Answer: $y = x\left(-3\ln x + 3c\right)^{\frac{1}{3}}, \ \ c \in \mathbb{R}$.)

2. $x^2\frac{dy}{dx} - yx = y^2$

   $\left(\text{Answer:} y = \frac{-x}{c + \ln x}, \ \ c \in \mathbb{R}.\right)$

3. $(3x^2 - y^2) + (xy - x^3y^{-1})\frac{dy}{dx} = 0.$

   $\left(\text{Answer: } \ \frac{y^2}{x^2} - \ln\left(\frac{y^2}{x^6}\right) = c, \ \ c \in \mathbb{R}.\right)$