Indivisible homogeneous directed graphs and a game for vertex partitions

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Abstract

Let $\mathcal{T}$ be a set of finite tournaments. We will give a necessary and sufficient condition for the $\mathcal{T}$-free homogeneous directed graph $\mathrm{H}_{\mathcal{T}}$ to be *divisible*; that is, that there is a partition of $\mathrm{H}_{\mathcal{T}}$ into two sets neither of which contains an isomorphic copy of $\mathrm{H}_{\mathcal{T}}$.

Key words: Homogeneous structures, vertex partitions, homogeneous directed graphs, infinite boundary.

1 Introduction

Let H be a directed graph. A local isomorphism of H is an isomorphism of a finite induced subgraph of H to a finite induced subgraph of H. The directed graph H is *homogeneous* if every local isomorphism of H has an extension to an automorphism of H; see [1]. Let aut(H) be the group of automorphisms of H and H be homogeneous. If F is a finite subset of $V(\mathrm{H})$, denote by $\mathrm{aut}_F(\mathrm{H})$ the subset of all automorphisms f of H so that $f(x) = x$ for all $x \in F$. The

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relation on $V(\mathrm{H})-F$ given by: The vertex x is related to the vertex y if there is $f \in \mathrm{aut}_F(\mathrm{H})$ with $f(x)=y$, is an equivalence relation on $V(\mathrm{H})-F$. The equivalence classes of this equivalence relation are called the *orbits* of F. An orbit of H is an orbit of F for some finite set F of vertices's. The restriction of H to an orbit of H is again a homogeneous directed graph, see [2]. We identify the orbits with the restrictions of H to the orbits.

For every set $\mathcal{T}$ of finite tournaments there is a unique homogeneous countable directed graph $\mathrm{H}_\mathcal{T}$ so that the finite induced subgraphs of $\mathrm{H}_\mathcal{T}$ are all of the finite directed graphs into which none of the tournaments of $\mathcal{T}$ can be embedded; see [1]. The countable homogeneous directed graphs of type $\mathrm{H}_\mathcal{T}$ make up the bulk of the countable homogeneous directed graphs; see [3].

The directed graph D is *indivisible* if for every partition of the set $V(\mathrm{D})$ of vertices's of D into red and blue vertices's there is a copy D^* of D in D so that all of the vertices's of D^* are red or all of the vertices's of D^* are blue. Otherwise D is said to be *divisible*.

Let $\mathcal{T}$ be a finite set of finite tournaments. The main result of [2] states that $\mathrm{H}_\mathcal{T}$ is indivisible if and only if for any two orbits X and Y of $\mathrm{H}_\mathcal{T}$ either X can be embedded into Y or Y can be embedded into X. The proof of the necessity of the latter condition in [2] does not rely on the assumption that the set $\mathcal{T}$ of constraints is finite; the proof of the sufficiency on the other hand relies heavily on the finiteness of the chain of orbits under embedding. There are examples to show that the chain of orbits under embedding can be any countable order type, even the order type of the rationals; see [5]. In this paper we strengthen the result of [2] and prove, see Theorem 6.2, that

Theorem 1.1 *Let $\mathcal{T}$ be a possibly infinite set of finite tournaments. Then $H_\mathcal{T}$ is indivisible if and only if for any two orbits X and Y of H either X can be embedded into Y or Y can be embedded into X.*

2 Preliminaries

If f is a function of a set S to a set T then $f[S]$ denotes the image of f, that is, the set $\{f(s) \mid s \in S\}$. The set S is *countable* if there is a bijection of S to ω. The set $S-T$ consists of the elements in S which are not in T. We consider every element $n \in \omega$ to be the set of all smaller numbers and write $x \in n$ for $x \in \omega$ and $x < n$. If $l \in \omega$ and S and T are subsets of ω we write $l < S$ to mean that every element in S is larger than l and $S < T$ to mean that every element of T is larger than any of the elements in S. In particular $l < \emptyset$ for all $l \in \omega$.

A directed graph G is a binary anti reflexive and anti symmetric relation. We denote the set of vertices's of G by $V(G)$ and the set of edges of G by $E(\mathrm{G})$. The vertices's a and b are *adjacent* if either $(a,b) \in E(\mathrm{G})$ or $(b,a) \in E(G)$. The directed graph A is an *induced subgraph* of G if $V(\mathrm{A}) \subseteq V(\mathrm{G})$ and $E(\mathrm{A}) = (V(\mathrm{A}) \times V(\mathrm{A})) \cap E(\mathrm{G})$; A is a *proper induced subgraph* of G if $V(A) \neq V(G)$. If $A \subseteq V(\mathrm{G})$ then the subgraph of G *induced by* A is the induced subgraph A of G with $V(\mathrm{A}) = A$. If a is a vertex of the directed graph G then $\mathrm{G} - a$ is the subgraph of G induced by the set $V(G) - \{a\}$.

An *embedding* of the directed graph A into the directed graph G is an injection f of $V(\mathrm{A})$ into $V(\mathrm{G})$ so that $(a,b) \in E(\mathrm{A})$ if and only if $(f(a), f(b)) \in E(\mathrm{G})$ for all vertices's $a, b \in V(\mathrm{A})$. If f is also onto then f is an *isomorphism*. The *image* $f[\mathrm{A}]$ of f is the subgraph of G induced by $f[V(\mathrm{A})]$. A *copy* of the directed graph A in the directed graph G is an induced subgraph of G which is isomorphic to A. The directed graph G is *A-free* if there is no copy of A in G. A *tournament* T is a directed graph so that any two different vertices's of T are adjacent.

The *skeleton* of a directed graph G is the set of all finite induced subgraphs of G and the *age* of G, age(G), is the set of all finite graphs which are isomorphic to an element of the skeleton of G. An isomorphism of the element A in the skeleton of the directed graph G to the element B in the skeleton of G is a *local isomorphism* of G.

The *boundary of G* is the set of all finite directed graphs A which are not in the age of G but every proper induced subgraph of A is an element in the age of G. We denote by bound(G) the boundary of G. The set Bound(G) is the set of all finite directed graphs which are not in the age of G. It follows that Bound(G)$\cup$age(G) is the set of all finite directed graphs and bound(G) is the set of minimal elements of Bound(G) with respect to embedding.

3 The homogeneous directed graphs $\mathbf{H}_{\mathcal{T}}$

Let $\mathcal{T}$ be a set of finite tournaments which can pairwise not be embedded into each other. We construct the graph $\mathrm{H}_{\mathcal{T}}$ as the limit of the graphs $(\mathrm{B}_i; i \in \omega)$ with $V(\mathrm{B}_0) \subseteq V(\mathrm{B}_1) \subseteq V(\mathrm{B}_2) \subseteq \cdots$ and $E(\mathrm{B}_0) \subseteq E(\mathrm{B}_1) \subseteq E(\mathrm{B}_2) \subseteq \cdots$. The graph B_0 is the directed graph having a single vertex. Given B_n and A, B two subsets of $V(\mathrm{B}_n)$ with $A \cap B = \emptyset$ denote by $\mathrm{C}_{A,B}$ the directed graph which consists of all of the vertices's of B_n together with a new vertex $x_{A,B}$. The restriction of $\mathrm{C}_{A,B}$ to $V(\mathrm{B}_n)$ is B_n and there is an edge from $x_{A,B}$ to every vertex of B and an edge from every vertex of A to $x_{A,B}$ and $x_{A,B}$ is not adjacent to any vertex in $V(\mathrm{B}_n) - (A \cup B)$. Let S_n be the set of vertices's $x_{A,B}$, for A and B two disjoint subsets of $V(\mathrm{B}_n)$, so that no element of $\mathcal{T}$ has an embedding into $C_{A,B}$. Then B_{n+1} is the directed graph with $V(\mathrm{B}_{n+1}) = V(\mathrm{B}_n) \cup S_n$ and the restriction of B_{n+1} to $V(\mathrm{B}_n) \cup \{x_{A,B}\}$ is $\mathrm{C}_{A,B}$ and no two of the vertices's in S_n are adjacent.

The directed graph $\mathrm{H}_{\mathcal{T}}$ so constructed is called the *homogeneous directed $\mathcal{T}$-free graph*. It follows from the construction that it has the following *mapping extension property*; see [2]:

If A is an element of the age of $\mathrm{H}_{\mathcal{T}}$ and $a \in V(\mathrm{A})$ and f an embedding of $\mathrm{A} - a$ into $\mathrm{H}_{\mathcal{T}}$ then there is an extension f^* of f to an embedding of A into $\mathrm{H}_{\mathcal{T}}$.

The mapping extension property implies that a finite directed graph A is an element of the age of $\mathrm{H}_{\mathcal{T}}$ if and only if there is no embedding of any element $\mathrm{T} \in \mathcal{T}$ into A and that every countable directed graph into which none of the elements of $\mathcal{T}$ have an embedding can be embedded into $\mathrm{H}_{\mathcal{T}}$. Actually the following stronger version of the mapping extension property follows directly from the construction of $\mathrm{H}_{\mathcal{T}}$:

If A is an element of the age of $\mathrm{H}_{\mathcal{T}}$ and $a \in V(\mathrm{A})$ and f is an embedding of $\mathrm{A} - a$ into $\mathrm{H}_{\mathcal{T}}$ then there are infinitely many different extensions f^* of f to an embedding of A into $\mathrm{H}_{\mathcal{T}}$. In particular, all orbits are infinite.

The mapping extension property implies, via a standard argument, that every local isomorphism has an extension to an automorphism, that is that $\mathrm{H}_{\mathcal{T}}$ is homogeneous. There is up to isomorphism only one countable homogeneous directed graph with boundary $\mathcal{T}$; see [1].

According to [4] there is an infinite set of tournaments which can pairwise not be embedded into each other.

4 Orbits

Let $\mathcal{T}$ be a set of finite tournaments and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous directed graph.

Let J be a finite subset of $V(\mathrm{H}_{\mathcal{T}})$ and $A, B \subseteq J$ with $A \cap B = \emptyset$. Denote by $O^J_{A,B}$ the set of all elements $x \in V(\mathrm{H}_{\mathcal{T}}) - J$ so that $(a, x) \in E(\mathrm{H}_{\mathcal{T}})$ for all $a \in A$ and $(x, b) \in E(\mathrm{H}_{\mathcal{T}})$ for all $b \in B$ and x is not adjacent to any vertex in $J - (A \cup B)$. If $O^J_{A,B}$ is not empty then $O^J_{A,B}$ is an orbit of $\mathrm{H}_{\mathcal{T}}$. Given an orbit X of $\mathrm{H}_{\mathcal{T}}$ there are sets A, B and J so that $O^J_{A,B} = X$. We define: $F(X) = J$,

$F_1(X) = A$, $F_2(X) = B$, $F_*(X) = F_1(X) \cup F_2(X)$ and $F_0(X) = J - F_*(X)$. Note that $F(X) \cap X = \emptyset$.

Let X be an orbit of $\mathrm{H}_{\mathcal{T}}$. We denote by bound(X) the boundary of the restriction of $\mathrm{H}_{\mathcal{T}}$ to X and by Bound(X) the set of finite directed graphs which are not in the age of the restriction of $\mathrm{H}_{\mathcal{T}}$ to X. Then bound(X) is the set of minimal elements of Bound(X) with respect to embedding. We write bound$(X) \subseteq$ bound(Y) if Bound$(X) \subseteq$ Bound(Y). Note that bound$(X) \subseteq$ bound(Y) if and only if for every $B \in$ bound(X) there is a $B' \in$ bound(Y) which has an embedding into B. We assume that if $\mathrm{T} \in$ bound(X) then $V(\mathrm{T}) \cap V(\mathrm{H}_{\mathcal{T}}) = \emptyset$.

We denote by age(X) the age of the restriction of $\mathrm{H}_{\mathcal{T}}$ to X. Note that age$(X) \subseteq$ age(Y) if and only if bound$(X) \supseteq$ bound(Y) and if age$(Z) =$ age$(X) \cap$ age(Y) then Bound$(Z) =$ Bound$(X) \cup$ Bound(Y). Similarly bound$(X) =$ bound(Y) if and only if age$(X) =$ age(Y).

We state for future reference Lemma 4.1, see [2], and Lemma 4.2 which is easy to prove.

Lemma 4.1 *Every orbit X of $H_{\mathcal{T}}$ has the mapping extension property. That is if $A \in age(X)$ and $a \in V(A)$ and f an embedding of $A - a$ into X then there is an extension f^* of f to an embedding of A into X.*

Lemma 4.2 *If X and Y are two orbits of $H_{\mathcal{T}}$ with $F_1(X) = F_1(Y)$ and $F_2(X) = F_2(Y)$ then $age(X) = age(Y)$.*

Let

$$\mathbf{B} = \mathbf{B}(\mathrm{H}_{\mathcal{T}}) := \{\text{bound}(X) \mid X \text{ is an orbit of } \mathrm{H}_{\mathcal{T}}\}.$$

In [2] an orbit of the form $O^{A \cup B}_{A,B}$ is denoted by $\mathcal{C}(A, B)$ and $(A, B) \preceq (A', B')$ is defined to mean that there is an embedding of $\mathcal{C}(A', B')$ into $\mathcal{C}(A, B)$. Also

$$\mathcal{F} = \{(A, B) \mid A, B \text{ are finite subsets of } V(\mathrm{H}_{\mathcal{T}}) \text{ and } A \cap B = \emptyset\}.$$

Then, Theorem 6 of [2] says that if $\mathrm{H}_{\mathcal{T}}$ is indivisible then $\preceq$ is a total preorder on $\mathcal{F}$. This together with Lemma 4.2 gives the following theorem:

Theorem 4.1 *Let $\mathcal{T}$ be a set of finite tournaments which can pairwise not be embedded into each other. If the homogeneous directed graph $H_{\mathcal{T}}$ is indivisible then the set $\mathbf{B}(H_{\mathcal{T}})$ of the boundaries of the orbits of $H_{\mathcal{T}}$ is a chain under $\subseteq$.*

Let X and Y be two orbits of $\mathrm{H}_{\mathcal{T}}$. If $I \subseteq F(X)$ we denote by X/I the orbit with $F(X/I) = I$ and $F_1(X/I) = F_1(X) \cap I$ and $F_2(X/I) = F_2(X) \cap I$. The orbits X and Y are *compatible* if $X/(F(X) \cap F(Y)) = Y/(F(X) \cap F(Y))$; that is, if

for all $a \in F(X) \cap F(Y)$, $x \in X$ and $y \in Y$

$$(a,x) \in E(\mathrm{H}_{\mathcal{T}}) \text{ if and only if } (a,y) \in E(\mathrm{H}_{\mathcal{T}}) \text{ and}$$
$$(x,a) \in E(\mathrm{H}_{\mathcal{T}}) \text{ if and only if } (y,a) \in E(\mathrm{H}_{\mathcal{T}}).$$

If X and Y are compatible then the *meet* $X \cap Y$ is $O^J_{A,B}$ with $J = F(X) \cup F(Y)$, $A = F_1(X) \cup F_1(Y)$ and $B = F_2(X) \cup F_2(Y)$, which is either empty or an orbit.

The orbit Y is a *continuation* of the orbit X if $F(X) \subseteq F(Y)$ and if $X = F(Y)/F(X)$. Hence if $I \subseteq F(Y)$ then Y is a continuation of $F(Y)/I$ and if Y is a continuation of X then bound$(X) \subseteq$ bound(Y). The meet of two compatible orbits, if nonempty, is a continuation of both orbits. If Y is a continuation of X then X and Y are compatible and $X \cap Y = Y$. The orbit Y is a *refinement* of the orbit X if Y is a continuation of X and bound$(Y) =$ bound(X). Note that a continuation of a continuation is a continuation and that a refinement of a refinement is a refinement.

Lemma 4.3 *Let R and S be compatible orbits of $H_{\mathcal{T}}$. If no vertex in $F_*(R) - F(S)$ is adjacent to a vertex of $F_*(S) - F(R)$ then age$(R \cap S) =$ age$(R) \cap$ age(S).*

PROOF. Clearly age$(R \cap S) \subseteq$ age$(R) \cap$ age(S).

Conversely, if A is in age$(R) \cap$ age(S) let G be an extension of the restriction of $\mathrm{H}_{\mathcal{T}}$ to $F(R) \cup F(S)$ by a copy of A so that the restriction G_R of G to $F(R) \cup V(\mathrm{A})$ embeds into $\mathrm{H}_{\mathcal{T}}$ by an embedding which is the identity on $F(R)$ and maps A into R and the restriction G_S of G to $F(S) \cup V(\mathrm{A})$ embeds into $\mathrm{H}_{\mathcal{T}}$ by an embedding which is the identity on $F(S)$ and maps A into S. By our hypothesis, any tournament embedding in G embeds in G_R or in G_S, hence in $\mathrm{H}_{\mathcal{T}}$. Thus G is in the age of $\mathrm{H}_{\mathcal{T}}$ and A is in the age of $R \cap S$. □

Lemma 4.4 *Let X be an orbit of $H_{\mathcal{T}}$ and $\mathbf{b} \in \mathbf{B} = \mathbf{B}(H_{\mathcal{T}})$ with bound$(X) \subseteq$ $\mathbf{b}$ and L a finite subset of $V(H_{\mathcal{T}})$. Then there is a continuation Z of X with bound$(Z) = \mathbf{b}$ and $F(Z) \cap L = F(X) \cap L$.*

PROOF. There is an orbit Y with bound$(Y) = \mathbf{b}$. Let A be a directed graph with $V(\mathrm{A}) \cap V(\mathrm{H}_{\mathcal{T}}) = \emptyset$ so that there is an isomorphism f of the restriction of $\mathrm{H}_{\mathcal{T}}$ to $F(Y)$ to A. Let B be the directed graph with $V(\mathrm{B}) = F(X) \cup V(\mathrm{A}) \cup L$ and the restriction of B to $F(X) \cup L$ is the restriction of $\mathrm{H}_{\mathcal{T}}$ to $F(X) \cup L$ and the restriction of B to $V(\mathrm{A})$ is A. No vertex in $V(\mathrm{A})$ is adjacent to a vertex in $F(X) \cup L$. The graph B is in the age of $\mathrm{H}_{\mathcal{T}}$ and hence there is an extension g of the identity map on $F(X) \cup L$ to an embedding of B into $\mathrm{H}_{\mathcal{T}}$.

Let Y' be the orbit with $F(Y') = g \circ f[F(Y)]$ and $F_1(Y') = g \circ f[F_1(Y)]$ and $F_2(Y') = g \circ f[F_2(Y)]$. It follows that bound$(Y') = \mathbf{b}$. The orbits X and Y' are compatible because $F(X) \cap F(Y) = \emptyset$. Let Z be the meet of the orbits X and

Y'. Because no element in $F(X)$ is adjacent to any element in $F(Y')$ it follows from Lemma 4.3 that $\text{bound}(Z) = \text{bound}(X) \cup \text{bound}(Y') = \text{bound}(X) \cup \mathbf{b} = \mathbf{b}$. □

Lemma 4.5 *Let X and Q be two compatible orbits so that Q is a refinement of $X/(F(X) \cap F(Q))$. If every vertex $x \in F(X) - F(Q)$ which is adjacent to an element in $F_*(Q) - F(X)$ is an element in Q, then $X \cap Q$ is a refinement of X.*

PROOF. We have to prove that $\text{age}(X) \subseteq \text{age}(X \cap Q)$. Let $\text{A} \in \text{age}(X)$ with $V(A) \subseteq X$. Then $V(\text{A}) \subseteq X/(F(X) \cap F(Q))$. Let S be the set of all elements $x \in F(X) - F(Q)$ which are adjacent to an element in $F_*(Q) - F(X)$. Then $S \subseteq Q$ and because X and Q are compatible $S \subseteq X/(F(X) \cap F(Q))$.

Let B be the restriction of $\text{H}_\mathcal{T}$ to $S \cup V(\text{A})$. Then $V(\text{B}) \subseteq X/(F(X) \cap F(Q))$.

Then $\text{B} \in age(Q)$ because Q is a refinement of $X/(F(X) \cap F(Q))$. The identity map on S has an extension f to an embedding of B into Q because Q has the mapping extension property. The embedding f maps A into $(X \cap Q)/(F(Q) \cup S)$ due to the definition of B. Hence $(X \cap Q)/(F(Q) \cup S)$ is a refinement of X. It follows from Lemma 4.3 that $X \cap Q$ is a refinement of X. □

Let X be an orbit of $\text{H}_\mathcal{T}$. The sequence $(Q_i; i \in [n+1] \in \omega)$ of orbits with $D_{i+1} := F_*(Q_{i+1}) - (F(Q_i) \cup F(X))$ for all $i \in n$ is a *refinement sequence* of X if for all $i \in n$, $j \in [n+1]$:

i. $Q_0 = V(\text{H}_\mathcal{T})$.
ii. Q_{i+1} is a continuation of Q_i.
iii. Q_j is a refinement of $X/(F(X) \cap F(Q_j))$.
iv. If $x \in F(X) - F(Q_j)$ is adjacent to an element in D_j, then $x \in Q_j$.

Lemma 4.6 *Let $(Q_i; i \in [n+1] \in \omega)$ be a refinement sequence of the orbit X of $H_\mathcal{T}$. Then $X \cap Q_n$ is a refinement of X.*

PROOF. Note that if $n = 1$ then Lemma 4.6 follows directly from Lemma 4.5. We proceed by induction. It follows from iii. that the orbits X and Q_n are compatible. The orbits $X \cap Q_{n-1}$ and Q_n are compatible because Q_n is a continuation of Q_{n-1}. Because

$$\begin{aligned}
&\text{bound}(Q_n) \supseteq \text{bound}\Big((X \cap Q_{n-1})/(F(X \cap Q_{n-1}) \cap F(Q_n))\Big) \supseteq \\
&\supseteq \text{bound}\Big(X/(F(X) \cap F(Q_n))\Big) = \text{bound}(Q_n)
\end{aligned}$$

it follows that the orbit Q_n is a refinement of the orbit

$$(X \cap Q_{n-1})/\Big(F(X \cap Q_{n-1}) \cap F(Q_n)\Big).$$

If

$$x \in F(X \cap Q_{n-1}) - F(Q_n) = (F(X) \cup F(Q_{n-1})) - F(Q_n) = F(X) - F(Q_n)$$

is adjacent to a vertex in

$$F_*(Q_n) - F(X \cap Q_{n-1}) = F_*(Q_n) - (F(X) \cup F(Q_{n-1})) = D_n$$

then $x \in Q_n$ according to condition iv.

We apply Lemma 4.5 to the orbits $X \cap Q_{n-1}$ for X and Q_n for Q. Hence $(X \cap Q_{n-1}) \cap Q_n = X \cap Q_n$ is a refinement of $X \cap Q_{n-1}$. The sequence $(Q_i; i \in n)$ is a refinement sequence of X and the orbit $X \cap Q_{n-1}$ is a refinement of X by induction. Hence $X \cap Q_n$ is a refinement of X. $\square$

The pair $(X; \mathcal{R}_X = (Q_i; i \in [n+1]))$ consisting of an orbit X and a refinement sequence $\mathcal{R}_X$ of X is *branched* with the pair $(Y; \mathcal{R}_Y = (P_i; i \in m+1))$ consisting of an orbit Y and a refinement sequence R_Y of Y if:

a. $n = m$ and $F(X) = F(Y)$.
b. There is $\beta \in [n+1]$ so that $Q_j = P_j$ for all $j \leq \beta$.
c. $\Big(F_*(Q_n) - (F(Q_\beta) \cup F(X))\Big) \cap \Big(F_*(P_n) - (F(Q_\beta) \cup F(X))\Big) = \emptyset$.

The number β is the *branching number* of the branched pair $(X; \mathcal{R}_X = (Q_i; i \in [n+1]))$ and $(Y; \mathcal{R}_Y = (P_i; i \in [n+1]))$. It follows that $(X; \mathcal{R}_X = (Q_i; i \in [n+1]))$ is branched with $(X; \mathcal{R}_X = (Q_i; i \in [n+1]))$ with branching number n.

Let X be an orbit and $a \in V(\mathrm{H}_\mathcal{T})$ a vertex not in $F(X)$. We denote by continue(X, a, k) for $k \in 3$ the continuation of X so that
$F(\text{continue}(X, a, k)) = F(X) \cup \{a\}$ and $a \in F_k(\text{continue}(X, a, k))$.

Let X be an orbit with refinement sequence $(Q_i; i \in [n+1])$ and $a \in X \cap Q_n$ and $k \in 3$. It follows that $(Q_i; i \in [n+1])$ is a refinement sequence of continue(X, a, k). Conditions i., ii. and iii. are trivially satisfied. Because $a \in X \cap Q_n$ we get $a \in Q_n$ and hence $a \in Q_i$ for all $i \in [n+1]$.

Lemma 4.7 *Let the pair $(X; \mathcal{R}_X = (Q_i; i \in [n+1]))$ and $(Y; \mathcal{R}_Y = (P_i; i \in [n+1]))$ be branched with branching number β and $a \in X \cap Q_n$ so that it is not in $F(Y \cap P_n)$ and not adjacent to any element in $F_*(P_n) - (F(Q_\beta) \cup F(Y))$. Then $(P_i; i \in n)$ is a refinement sequence of continue(Y, a, k) for every $k \in 3$.*

PROOF. Conditions i., ii. and iii. are trivially satisfied. We have to argue condition iv. for $x = a$. If a is adjacent to an element in $F_*(P_i) - (F(P_{i-1}) \cup F(Y))$ then $i \leq \beta$ and hence $Q_i = P_i$. Because $a \in X \cap Q_n$ and $X \cap Q_n$ is a continuation of Q_i it follows that $a \in Q_i = P_i$. $\square$

Lemma 4.8 *Let $k, l \in 3$. Let the pair $(X; \mathcal{R}_X = (Q_i; i \in n))$ and $(Y; \mathcal{R}_Y = (P_i; i \in n))$ be branched with branching number β and the vertex $a \notin F(X \cap Q_{n-1}) \cup F(Y \cap P_{n-1})$ so that $\mathcal{R}_X$ is a refinement sequence of continue(X, a, k) and $\mathcal{R}_\mathcal{Y}$ is a refinement sequence of continue(Y, a, l). Then $(Q_i; i \in [n+1])$ with $Q_n =$ continue$(X, a, k) \cap Q_{n-1}$ is a refinement sequence of continue(X, a, k) and $(P_i; i \in [n+1])$ with $P_n =$ continue$(Y, a, l) \cap P_{n-1}$ is a refinement sequence of continue(Y, a, l). Also (continue$(X, a, k); (Q_i; i \in [n+1])$) is branched with (continue$(Y, a, l); (P_i; i \in [n+1])$) with branching number β.*

PROOF. The orbit $Q_n = \text{continue}(X, a, k) \cap Q_{n-1}$ is a refinement of the orbit continue(X, a, k) according to Lemma 4.6, affirming condition iii.

There are no elements in $F(\text{continue}(X, a, k)) - F(Q_n)$ and condition iv. follows. Hence $(Q_i; i \in [n+1])$ with $Q_n = \text{continue}(X, a, k) \cap Q_{n-1}$ is a refinement sequence of continue(X, a, k) and similarly $(P_i; i \in [n+1])$ with $P_n = \text{continue}(Y, a, l) \cap P_{n-1}$ is a refinement sequence of continue(Y, a, l).

Using the same branching number β we obtain conditions a. and b. for (continue$(X, a, k); (Q_i; i \in [n+1])$) being branched with (continue$(Y, a, l); (P_i; i \in [n+1])$). Condition c. follows because $a \in F(\text{continue}(X, a, k))$. $\square$

Lemma 4.9 *Let the pair $(X; \mathcal{R}_X = (Q_i; i \in [n+1]))$ and $(Y; \mathcal{R}_Y = (P_i; i \in [n+1]))$ be branched with branching number β and $F(Y) \subseteq F(P_n)$. Let R be a refinement of P_n so that $(F_*(R) - F(P_n)) \cap F(Q_n) = \emptyset$. Let $P'_i = P_i$ for $i \in n$ and $P'_n = R$. Then $\mathcal{R}'_X = (P'_i; i \in [n+1])$ is a refinement sequence of Y and $(X; \mathcal{R}_X)$ and $(Y; \mathcal{R}'_Y)$ are branched with branching number β.*

PROOF. The only condition which is not trivially satisfied is condition c. Condition c. is satisfied by assumption for all vertices's of P'_n which are in P_n and satisfied for all vertices's in $F_*(R) - F(P_n)$ because $(F_*(R) - F(P_n)) \cap F(Q_n) = \emptyset$. $\square$

5 The game

Let $\mathcal{T}$ be a set of finite tournaments which can pairwise not be embedded into each other and $\mathrm{H}_\mathcal{T}$ the homogeneous $\mathcal{T}$-free directed graph. We assume that

$V(\mathrm{H}_{\mathcal{T}}) = \omega$, used in the proof of Lemma 5.1, and that $(\mathbf{B}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ is a chain.

Let X be an orbit of $\mathrm{H}_{\mathcal{T}}$ and $\mathbf{b} \in \mathbf{B} = \mathbf{B}(\mathrm{H}_{\mathcal{T}})$ so that $\text{bound}(X) \subseteq \mathbf{b}$. A **b**-*restriction* of X is a continuation Y of X with $\text{bound}(Y) = \mathbf{b}$. It follows from Lemma 4.4 that such a **b**-restriction exists for every $\mathbf{b} \in \mathbf{B}$ with $\text{bound}(X) \subseteq \mathbf{b}$.

Let ϕ_{blue} and ϕ_{red} be two unary relations on the set of orbits of $\mathrm{H}_{\mathcal{T}}$. We denote by formula (1) the following statement

For all $\mathbf{b} \in \mathbf{B}$ with $\text{bound}(X) \subseteq \mathbf{b}$
there exists a refinement Y of X with $F(X) < F_*(Y) - F(X)$
so that for all refinements Z of Y with $F(Y) < F_*(Z) - F(Y)$ (1)
there exists a **b**-restriction R of Z with $F(Z) < F_*(R) - F(Z)$
so that $\phi_{\text{blue}}(R)$.

Formula (2) is nearly identical to formula (1) except that $\phi_{\text{blue}}(R)$ is replaced by $\phi_{\text{red}}(R)$.

Theorem 5.1 *Let $\mathcal{T}$ be a set of finite tournaments which can pairwise not be embedded into each other and $H_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous directed graph with $V(H_{\mathcal{T}}) = \omega$. Suppose that the set $\mathbf{B}(H_{\mathcal{T}}) = \mathbf{B}$ is a chain under $\subseteq$. Let* $(\text{Blue}, \text{Red})$ *be a partition of ω into blue and red elements.*

Then there are unary relations ϕ_{blue} and ϕ_{red} on the set of orbits of $H_{\mathcal{T}}$ so that for every orbit X of $H_{\mathcal{T}}$ exactly one of $\phi_{\text{blue}}(X)$ and $\phi_{\text{red}}(X)$. If $\phi_{\text{blue}}(X)$ then X contains infinitely many blue vertices's and formula (1) holds. If $\phi_{\text{red}}(X)$, then X contains infinitely many red vertices's and formula (2) holds.

PROOF. Let the conditions of Theorem 5.1 be given.

In order to prove the Theorem we have for every orbit X of $\mathrm{H}_{\mathcal{T}}$ to decide whether $\phi_{\text{red}}(X)$ or $\phi_{\text{blue}}(X)$ and then prove that the so defined relations ϕ_{red} and ϕ_{blue} have the required properties. Because of the condition that if $\phi_{\text{red}}(X)$ then X contains infinitely many red vertices's we are forced to have $\phi_{\text{blue}}(X)$ if X contains only finitely many red vertices's. Note that if X contains only finitely many red vertices's then formula (1) holds. Similarly if X contains only finitely many blue vertices's then we let $\phi_{\text{red}}(X)$. We use the following game to define the relations ϕ_{red} and ϕ_{blue} for all orbits of $\mathrm{H}_{\mathcal{T}}$.

The game $\Gamma_{\text{red}}(X)$ starts in state $(X, 0)$ with player **I** to move.

0. If the game is in state $(U, 0)$ for some orbit U of $\mathrm{H}_{\mathcal{T}}$ then it is the turn of player **I** to move. Player **I** selects $\mathbf{b} \in \mathbf{B}$ with bound$(U) \subseteq \mathbf{b}$ and the game moves into state $(U, \mathbf{b}, 1)$.

1. If the game is in state $(U, \mathbf{b}, 1)$ then it is the turn of player **II** to move. Player **II** selects a refinement V of U with $F(U) < F_*(V) - F(U)$ and the game moves to state $(V, \mathbf{b}, 2)$.

2. If the game is in state $(V, \mathbf{b}, 2)$ then it is the turn of player **I** to move. Player **I** selects a refinement W of V with $F(V) < F_*(W) - F(V)$. The game moves to state $(W, \mathbf{b}, 3)$.

3. If the game is in state $(W, \mathbf{b}, 3)$ then it is the turn of player **II** to move. Player **II** selects a **b**-restriction R of W with $F(W) < F_*(R) - F(W)$ and the game moves to state $(R, 0)$. Then it is again the turn of player I to move.

The game ends with a win of player **I** if it is in a state of the form $(Y, 0)$ for an orbit Y which contains only finitely many blue elements.

We will write $\phi_{\text{blue}}(X)$ if player **I** does not have a winning strategy in the game $\Gamma_{\text{red}}(X)$. It follows that if $\phi_{\text{blue}}(X)$ then there are infinitely many blue elements in X. Note that if player **I** does not have a winning strategy in the game $\Gamma_{\text{red}}(X)$, that is if $\phi_{\text{blue}}(X)$, then formula (1) holds. We will say player **I** *has a win* at a state of the game if player **I** has a winning strategy when the game is at this state.

The orbits of the form U, V, W, R in the states of the game $\Gamma_{\text{red}}(X)$ are all continuations and therefore subsets of the orbit X. Hence if player **I** has a win in the game $\Gamma_{\text{red}}(X)$ then there is an orbit $Y \subseteq X$ with only finitely many blue vertices's. This implies that if player **I** has a win in the game $\Gamma_{\text{red}}(X)$ then X contains infinitely many red elements. We will write $\phi_{\text{red}}(X)$ if player **I** has a winning strategy in the game $\Gamma_{\text{red}}(X)$. Note that either $\phi_{\text{red}}(X)$ or $\phi_{\text{blue}}(X)$.

In the following Lemma 5.1, which completes the proof of Theorem 5.1, we will make use of the fact that if $X_0, X_1, X_2, X_3, \dots, X_i, \dots, X_n$ is a sequence of orbits of $\mathrm{H}_{\mathcal{T}}$ so that for every $i \in n$ X_{i+1} is a continuation of X_i and

$$F(X_i) < F_*(X_{i+1}) - F(X_i)$$

then

$$F(X_0) < F_*(X_n) - F(X_0).$$

□

Lemma 5.1 *If $\phi_{\text{red}}(X)$ then formula (2) holds.*

PROOF. Let $\phi_{\text{red}}(X)$. Then player **I** has a winning strategy in the game $\Gamma_{\text{red}}(X)$. We use this strategy and start the game in state $(X, 0)$ which is a winning state for Player **I**.

We begin at line one of formula (2) and let $\mathbf{b} \in \mathbf{B}$ with bound$(X) \subseteq \mathbf{b}$ be given. Let $\mathbf{b}'$ be the element of $\mathbf{B}$ chosen by player **I**. The game moves to state $(X, \mathbf{b}', 1)$ with a win for player **I**.

We will construct a refinement Y of X with $F(X) < F_*(Y) - F(X)$ and so that $(Y, \mathbf{b}', 3)$ is a winning state for player **I**. This orbit Y will be used to satisfy the second line of formula (2).

We as player **II** select X, moving the game to state $(X, \mathbf{b}', 2)$. Let Y be the refinement of X chosen by player **I** when given the state $(X, \mathbf{b}', 2)$ moving the game to state $(Y, \mathbf{b}', 3)$ with a win for player **I**. Because player **I** has made a legal move, we get $F(X) < F_*(Y) - F(X)$.

Let Z be a refinement of Y with $F(Y) < F_*(Z) - F(Y)$; accounting for line three of formula (2). In order to validate formula (2) we have to prove that there exists a $\mathbf{b}$-restriction R of Z with $F(Z) < F_*(R) - F(Z)$ so that $(R, 0)$ is a winning position for player **I**. This will validate lines four and five of formula (2).

Because $(Y, \mathbf{b}', 3)$ is a winning state for player **I**, player **I** has a win at position $(R, 0)$ for all $\mathbf{b}'$-restrictions R of Y with $F(Y) < F_*(R) - F(Y)$. Because Z is a refinement of Y with $F(Y) < F_*(Z) - F(Y)$ we obtain that $(R, 0)$ is a winning position for player **I** for all $\mathbf{b}'$-restrictions R of Z with $F(Z) < F_*(R) - F(Z)$. This implies that $(Z, \mathbf{b}', 3)$ is a winning state for player **I**.

If $\mathbf{b} \subseteq \mathbf{b}'$ let R be any $\mathbf{b}$-restriction of Z with $F(Z) < F_*(R) - F(Z)$; as bound$(Z) =$ bound$(X) \subseteq \mathbf{b}$ this is possible. Because player **I** has a win in state $(Z, \mathbf{b}', 3)$ player **I** has a win in all states $(R', 0)$ where R' is any $\mathbf{b}'$-restriction of Z with $F(Z) < F_*(R') - F(Z)$. In particular player **I** has a win in all states $(R', 0)$ where R' is any $\mathbf{b}'$-restriction of R with $F(R) < F_*(R') - F(R)$. Hence, if the game continues in state $(R, 0)$ and player **I** chooses the element $\mathbf{b}'$ in $\mathbf{B}$, the game will move to a winning state $(R', 0)$ for player **I** with bound$(R') = \mathbf{b}'$ and a win for player I independent of the legal moves of player **I** and player **II**. It follows that player **I** has a win if the game is in state $(R, 0)$.

If $\mathbf{b}' \subset \mathbf{b}$ we watch the winning game of player **I** starting at the state $(Z, \mathbf{b}', 3)$. We as player **II** choose a $\mathbf{b}'$-restriction R_0 of Z with $F(Z) < F_*(R_0) - F(Z)$. The game moves into state $(R_0, 0)$ with a win for player **I** and player **I** to move. The game will move through winning states

$(R_0, 0)$, $(R_0, \mathbf{b}_0, 1)$, $(V_0, \mathbf{b}_0, 2)$, $(W_0, \mathbf{b}_0, 3)$,
$(R_1, 0)$, $(R_1, \mathbf{b}_1, 1)$, $(V_1, \mathbf{b}_1, 2)$, $(W_1, \mathbf{b}_1, 3)$,
$(R_2, 0)$, $(R_2, \mathbf{b}_2, 1)$, $(V_2, \mathbf{b}_2, 2)$, $(W_2, \mathbf{b}_2, 3)$,
$(R_3, 0)$, $(R_3, \mathbf{b}_3, 1)$, $(V_3, \mathbf{b}_3, 2)$, $(W_3, \mathbf{b}_3, 3)$,
. . . .
$(R_i, 0)$, $(R_i, \mathbf{b}_i, 1)$, $(V_i, \mathbf{b}_i, 2)$, $(W_i, \mathbf{b}_i, 3)$,

$(R_{i+1}, 0)$, $(R_{i+1}, \mathbf{b}_{i+1}, 1)$, $(V_{i+1}, \mathbf{b}_{i+1}, 2)$, $(W_{i+1}, \mathbf{b}_{i+1}, 3)$,
. . . .

of player **I** where we as player **II** will make some arbitrary legal moves when called upon. Note that for all i bound$(R_{i+1}) = \mathbf{b}_i$.

Because $\mathbf{b}' \subseteq \mathbf{b}_0 \subseteq \mathbf{b}_1 \subseteq \mathbf{b}_2 \ldots$ there is either a number i so that $\mathbf{b}_i \subset \mathbf{b} \subseteq \mathbf{b}_{i+1}$ or, because player **I** has a winning strategy at state $(X, 0)$, the game ends after finitely many rounds with a win of player **I** in some state $(R_n, 0)$ with bound$(R_n) = \mathbf{b}_{n-1} \subset \mathbf{b}$ and only finitely many blue vertices's in R_n.

If $\mathbf{b}_i \subset \mathbf{b} \subseteq \mathbf{b}_{i+1}$ for some $i \in \omega$ then because $(W_{i+1}, \mathbf{b_{i+1}}, 3)$ is a winning position of player **I** every state of the form $(S, 0)$ with S a $\mathbf{b_{i+1}}$-restriction of W_{i+1} with $F(W_{i+1}) < F_*(S) - F(W_{i+1})$ is a winning position of player **I**. This implies as before that if R is a $\mathbf{b}$-restriction of W_{i+1} with $F(W_{i+1}) < F_*(R) - F(W_{i+1})$ then $(R, 0)$ is a winning position of player **I**. Let R be such a $\mathbf{b}$-restriction of W_{i+1}. The orbit W_{i+1} is a continuation of the orbit Z. Hence R is a $\mathbf{b}$-restriction of Z with $(R, 0)$ a winning position of player **I**. Making use of the fact mentioned before Lemma 5.1 it follows that $F(Z) < F_*(R) - F(Z)$.

If the game ends after finitely many rounds with a win of player **I** in a state $(R_n, 0)$ and $\mathbf{b}_{n-1} \subset \mathbf{b}$ then the orbit R_n contains only finitely many blue vertices's. Let R be a $\mathbf{b}$-restriction of R_n with $F(R_n) < F_*(R) - F(R_n)$. The orbit R contains only finitely many blue elements. It follows that R is a $\mathbf{b}$-restriction of Z and $(R, 0)$ is a winning position of player **I**. Again, by the fact mentioned before Lemma 5.1 we get $F(Z) < F_*(R) - F(Z)$.

□

Let (Blue, Red) be a partition of ω into blue and red elements and ϕ_{blue} and ϕ_{red} the unary relations on the set of orbits of $\mathrm{H}_{\mathcal{T}}$ given by Theorem 5.1.

We write $\psi_{\text{blue}}(Y)$ for the orbit Y of $\mathrm{H}_{\mathcal{T}}$ if every refinement V of Y with $F(Y) < F_*(V) - F(Y)$ has for every $l \in \omega$ a refinement R with $l < F_*(R) - F(V)$ and with $\phi_{\text{blue}}(R)$.

Lemma 5.2 *Every orbit X of $H_{\mathcal{T}}$ with $\phi_{\text{blue}}(X)$ has a refinement Y with $F(X) < F_*(Y) - F(X)$ and with $\psi_{\text{blue}}(Y)$.*

PROOF. We use formula (1) for the orbit X in the instance $\mathbf{b} := \text{bound}(X)$. Formula (1) returns a refinement Y of X with $F(X) < F_*(Y) - F(X)$. We will prove that $\psi_{\text{blue}}(Y)$.

Let V be a refinement of Y with $F(Y) < F_*(V) - F(Y)$ and let l be a number. We have to prove that V has a refinement R with $l < F_*(R) - F(V)$ and with $\phi_{\text{blue}}(R)$. Let Z be the refinement of V with $F(Z) = F(V) \cup \{l\}$ and $F_1(Z) = F_1(V)$ and $F_2(Z) = F_2(V)$. Note that Z is a refinement of Y with $F(Y) < F_*(Z) - F(Y)$. Hence we can use Z as an instance in line three of formula (1). Formula (1) returns a $\mathbf{b}$-restriction R of Z with $F(Z) < F_*(R) - F(Z)$ and with $\phi_{\text{blue}}(R)$. The orbit R is a continuation of V and hence a refinement of V because $\text{bound}(R) = \mathbf{b} = \text{bound}(X)) = \text{bound}(Y') = \text{bound}(Y) = \text{bound}(V)$. The condition $l < F_*(R) - F(V)$ follows because $l \in F(Z) < F_*(R) - F(Z) = F_*(R) - F(V)$. □

Lemma 5.3 *If X is an orbit of $H_{\mathcal{T}}$ with $\psi_{\text{blue}}(X)$ then $\phi_{\text{blue}}(X)$.*

PROOF. Let $\psi_{\text{blue}}(X)$. Let $\mathbf{b} \in \mathbf{B}$ with $\text{bound}(X) \subseteq \mathbf{b}$ and $l \in \omega$.

To satisfy line two of formula (1) we let $Y = X$. Then, in line three, we let Z be a refinement of $Y = X$ with $F(X) < F_*(Z) - F(X)$.

Because $\psi_{\text{blue}}(X)$ and Z is a refinement of X with $F(X) < F_*(Z) - F(X)$, there is a refinement W of Z with $F(Z) < F_*(W) - F(Z)$ and $\phi_{\text{blue}}(W)$. Using formula (1) for W in the instance $\mathbf{b}$, we obtain a $\mathbf{b}$-restriction R of W with $F(W) < F_*(R) - F(W)$ and $\phi_{\text{blue}}(R)$. This orbit R is a $\mathbf{b}$-restriction of Z with $F(Z) < F_*(R) - F(Z)$. □

Lemma 5.4 *Let X be an orbit of $H_{\mathcal{T}}$ with $\psi_{\text{blue}}(X)$. Then X contains infinitely many blue elements and $\psi_{\text{blue}}(Y)$ holds for every refinement Y of X with $F(X) < F_*(Y) - F(X)$. For every $l \in \omega$ and every $\mathbf{b} \in \mathbf{B}$ with bound$(X) \subseteq \mathbf{b}$ there is a $\mathbf{b}$-restriction R of X with $l < F_*(R) - F(X)$ and $\psi_{\text{blue}}(R)$.*

PROOF. The relation $\psi_{\text{blue}}(X)$ implies $\phi_{\text{blue}}(X)$ by Lemma 5.3. Hence X contains infinitely many blue elements.

Let Y be a refinement of X with $F(X) < F_*(Y) - F(X)$. If V is a refinement of Y with $F(Y) < F_*(V) - F(Y)$ then V is a refinement of X with $F(X) < F(V) - F_*(X)$. Hence there is for every $l \in \omega$ a refinement R of V with $l < F_*(R) - F(V)$ and with $\phi_{\text{blue}}(R)$.

Let $l \in \omega$ and $\mathbf{b} \in \mathbf{B}$ with $\text{bound}(X) \subseteq \mathbf{b}$. Let X' the refinement of X with $F(X') = F(X) \cup \{l\}$ and $F_1(X') = F_1(X)$ and $F_2(X') = F_2(X)$. Then $\psi_{\text{blue}}(X')$ by the first part of this Lemma and $\phi_{\text{blue}}(X')$ by Lemma 5.3.

Using formula (1) for X' in the instance $\mathbf{b}$ we obtain a $\mathbf{b}$-restriction V of X' with $F(X') < F_*(V) - F(X')$ and $\phi_{\text{blue}}(V)$. By Lemma 5.2 there is a refinement R of V with $F(V) < F_*(R) - F(V)$ and $\psi_{\mathrm{blue}}(R)$. It follows that $l \in F(X') < F_*(R) - F(X') = F_*(R) - F(X)$ and hence $l < F_*(R) - F(X)$. □

6 The construction

Theorem 6.1 *Let $\mathcal{T}$ be a set of finite tournaments and $H_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous directed graph. Suppose that for any two orbits X and Y of $H_{\mathcal{T}}$ either bound$(X) \subseteq$ bound(Y) or bound$(Y) \subseteq$ bound(X). Then $H_{\mathcal{T}}$ is indivisible.*

PROOF. Let ω be the base of $\mathrm{H}_{\mathcal{T}}$ and (Blue,Red) a partition of ω. (ω is the orbit with $F(\omega) = F_*(\omega) = \emptyset$.) Let ϕ_{blue} and ϕ_{red} be the unary relations given by Theorem 5.1. Then, by Theorem 5.1, either $\phi_{\mathrm{blue}}(\omega)$ or $\phi_{\mathrm{red}}(\omega)$. We assume $\phi_{\mathrm{blue}}(\omega)$. (The case $\phi_{\mathrm{red}}(\omega)$ is dual, just replace blue by red throughout.) There exists, by Lemma 5.2, a refinement U of ω with $\psi_{\mathrm{blue}}(U)$. (Note that bound$(U) = \mathcal{T}$.)

For $v \in \omega$ let I_v be the restriction of $\mathrm{H}_{\mathcal{T}}$ to v. The subset J of ω having v elements is an *initial segment of length* v if the order preserving map from v to J is an isomorphism of I_v. The initial segment J' of length u is an *extension* of the initial segment J of length v if $v < u$ and every element of J is smaller than every element of $J' - J$.

Let J be an initial segment of length v. Let $\mathcal{P}$ be the set of pairs (A, B) with $A \cup B \subseteq J$ and $A \cap B = \emptyset$ and $O^J_{A,B}$ not empty. (Remember that $O^J_{A,B}$ is the orbit with $F(O^J_{A,B}) = J$ and $F_1(O^J_{A,B}) = A$ and $F_2(O^J_{A,B}) = B$.) The initial segment J is *well chosen* if:

1. Every element in J is blue.
2. For every pair $(A, B) \in \mathcal{P}$ there is a refinement sequence $(Q^J_{A,B}(i); i \in v+1)$ of the orbit $O^J_{A,B}$.
3. The pair $(O^J_{A,B}; (Q^J_{A,B}(i); i \in v+1))$ is branched with $(O^J_{C,D}; (Q^J_{C,D}(i); i \in v+1))$ for every two elements $(A, B), (C, D) \in \mathcal{P}$.
4. $J = F(O^J_{A,B}) \subseteq F(Q^J_{A,B}(v))$ for every pair $(A, B) \in \mathcal{P}$.
5. $\psi_{\mathrm{blue}}(Q^J_{A,B}(v))$ for every element $(A, B) \in \mathcal{P}$.

Note that $Q^J_{A,B}(v)$ is a refinement of $O^J_{A,B}$ according to condition 4. and condition iii. We denote by $\beta(A, B; C, D)$ the branching number of $(O^J_{A,B}; (Q^J_{A,B}(i); i \in v+1))$ and $(O^J_{C,D}; (Q^J_{C,D}(i); i \in v+1))$.

If $v = 0$ then $\mathcal{P} = \{(\emptyset, \emptyset)\}$ and $J = \emptyset$ is well chosen with $O^J_{\emptyset,\emptyset} = V(\mathrm{H}_{\mathcal{T}}) = \omega$ and the refinement sequence (U). Let J be a well chosen initial segment of length v. We will extend J to a well chosen initial segment of length $v + 1$.

Because $\mathrm{H}_\mathcal{T}$ has the mapping extension property there is an extension f of the order map from I_v to J to an embedding of I_{v+1}. Let M be the set of elements x of J with an edge from x to $f(v)$ and N be the set of elements y of J with an edge from $f(v)$ to y. Then the orbit $O^J_{M,N}$ is not empty because it contains the element $f(v)$ which also implies that every element of $O^J_{M,N}$ and hence every element of $Q^J_{M,N}(v)$ together with J forms an extension of J to an initial segment of length $v+1$. We will find an appropriate $a \in Q^J_{M,N}(v) \subseteq O^J_{M,N}$ and refinement sequences so that $J \cup \{a\}$ is well chosen.

Given $(A,B) \in \mathcal{P}$ let

$$\begin{aligned}
\mathbf{c}(A,B,0) &:= \mathrm{bound}(O^{J\cup\{f(v)\}}_{A,B}) = \mathrm{bound}(\mathrm{continue}(O^J_{A,B}, f(v), 0)),\\
\mathbf{c}(A,B,1) &:= \mathrm{bound}(O^{J\cup\{f(v)\}}_{A\cup\{f(v)\},B}) = \mathrm{bound}(\mathrm{continue}(O^J_{A,B}, f(v), 1)),\\
\mathbf{c}(A,B,2) &:= \mathrm{bound}(O^{J\cup\{f(v)\}}_{A,B\cup\{f(v)\}}) = \mathrm{bound}(\mathrm{continue}(O^J_{A,B}, f(v), 2)).
\end{aligned}$$

For each of the orbits $Q^J_{A,B}(v)$ with $(A,B) \in \mathcal{P}$ relation ψ_{blue} holds. Hence if $\mathbf{b} \in \mathbf{B}$ with $\mathrm{bound}(Q^J_{A,B}(v)) \subseteq \mathbf{b}$ and $l \in \omega$ there is, using Lemma 5.4, a continuation R of $Q^J_{A,B}(v)$ with $\mathrm{bound}(R) = \mathbf{b}$ and $l < F_*(R) - F(Q^J_{A,B}(v))$ and $\psi_{\text{blue}}(R)$. Hence there is for every $k \in 3$ and $l \in \omega$ a continuation $[R,k]^J_{A,B}$ of $Q^J_{A,B}(v)$ so that $\mathrm{bound}([R,k]^J_{A,B}) = \mathbf{c}(A,B,k)$ and $l < F_*([R,k]^J_{A,B}) - F(Q^J_{A,B}(v))$ and $\psi_{\text{blue}}([R,k]^J_{A,B})$.

Because we can choose $[R,k]^J_{A,B}$ so that $F_*([R,k]^J_{A,B}) - F(Q^J_{A,B}(v)) > l$ for any $l \in \omega$ we can also obtain that if $(A,B) \neq (C,D)$ or $k \neq j$ and $(A,B) \in \mathcal{P}$ and $(C,D) \in \mathcal{P}$ then

$$(F_*([R,k]^J_{A,B}) - F(Q^J_{A,B}(v))) \cap (F_*([R,j]^J_{C,D})) = \emptyset.$$

Let

$$\ell > \max\Big(\bigcup_{k\in 3}\ \bigcup_{(A,B)\in\mathcal{P}} F_*([R,k]^J_{A,B})\Big).$$

Let X be the refinement of $Q^J_{M,N}(v)$ so that

$$F(X) = \ell \ \text{ and } \ F_0(X) = (\ell - F(Q^J_{M,N}(v)) \cup F_0(Q^J_{M,N}(v)).$$

The relation $\psi_{\text{blue}}(X)$ follows from $\psi_{\text{blue}}(Q^J_{M,N}(v))$ by Lemma 5.4. The orbit X contains therefore infinitely many blue vertices's. Let a be such a blue vertex.

Let $J_a := J \cup \{a\}$ and for all $k \in 3$ let $O^{J_a,k}_{A,B} := \mathrm{continue}(O^J_{A,B}, a, k)$. Because a and $f(v)$ are in the same orbit $O^J_{M,N}$ of J it follows that

$$\mathrm{bound}(O^{J_a,k}_{A,B}) = \mathbf{c}(A,B,k) = \mathrm{bound}([R,k]^J_{A,B}). \tag{3}$$

The vertex a is an element of $Q^J_{M,N}(v) = O^J_{M,N} \cap Q^J_{M,N}(v)$ because $Q^J_{M,N}(v)$ is a refinement of $O^J_{M,N}$. The vertex a is not adjacent to any element in $F_*(Q^J_{A,B}(v)) - (F(Q^J_{M,N}(\beta(A,B,M,N))) \cup J)$ because a is not adjacent to any element smaller than ℓ which is not in $F(Q^J_{M,N}(v))$ and because of condition c. for branched pairs. Hence we can apply Lemma 4.7 and obtain that $(Q^J_{A,B}(i); i \in v+1)$ is a refinement sequence of $O^{J_a,k}_{A,B}$ for any $k \in 3$ and $(A,B) \in \mathcal{P}$.

Let $Q^{J_a,k}_{A,B}(v+1) := O^{J_a,k}_{A,B} \cap Q^J_{A,B}(v)$. It follows from Lemma 4.6 that $Q^{J_a,k}_{A,B}(v+1)$ is a refinement of $O^{J_a,k}_{A,B}$. It follows, by Lemma 4.8, that if we let $Q^{J_a,k}_{A,B}(i) := Q^J_{A,B}$ for all $i \in v+1$ then $(Q^{J_a,k}_{A,B}(i), i \in v+2)$ is a refinement sequence of $O^{J_a,k}_{A,B}$. Also $(O^{J_a,k}_{A,B}; (Q^{J_a,k}_{A,B}(i); i \in v+2))$ is branched with $(O^{J_a,h}_{C,D}; (Q^{J_a,h}_{C,D}(i); i \in v+2))$ for every two elements $(A,B), (C,D) \in \mathcal{P}$ and every pair of numbers $k, j \in 3$.

We obtain from (3) and Lemma 4.6 that

$$\text{bound}([R,k]^J_{A,B}) = \text{bound}(O^{J_a,k}_{A,B}) = \text{bound}(Q^{J_a,k}_{A,B}(v+1)). \tag{4}$$

Both orbits $[R,k]^J_{A,B}$ and $Q^{J_a,k}_{A,B}(v+1)$ are continuations of the orbit $Q^J_{A,B}(v)$ and $F(Q^{J_a,k}_{A,B}(v+1)) - F([R,k]^J_{A,B}) = \{a\}$. According to the choice of a and $F_*([R,k]^J_{A,B}) - F(Q^J_{A,B}(v))$ the vertex a is not adjacent to any vertex in $F_*([R,k]^J_{A,B}) - F(Q^J_{A,B}(v))$.

Using Lemma 4.3 and (4) it follows that the orbit $R^{J_a,k}_{A,B}(v+1) := [R,k]^J_{A,B} \cap Q^{J_a,k}_{A,B}(v+1)$ is a refinement of the orbit $Q^{J_a,k}_{A,B}(v+1)$ and a refinement of the orbit $[R,k]^J_{A,B}$ and hence $\psi_{\text{blue}}(R^{J_a,k}_{A,B}(v+1))$. (Note that $a > F([R,k]^J_{A,B})$.)

Again, according to the choice of the sets $F_*([R,k]^J_{A,B}) - F(Q^J_{A,B}(v))$ to be on different levels, for different pairs (A,B) or different values $k \in 3$, we can apply Lemma 4.9. For every pair $(A,B) \in \mathcal{P}$ and $k \in 3$ let $R^{J_a,k}_{A,B}(i) = Q^J_{A,B}(i)$ for $i \in v+1$. It follows from Lemma 4.9 that $(R^{J_a,k}_{A,B}(i); i \in v+2)$ is a refinement sequence of the orbit $O^{J_a,k}_{A,B}$ and $(O^{J_a,k}_{A,B}; (R^{J_a,k}_{A,B}(i); i \in v+2))$ is branched with $(O^{J_a,j}_{C,D}; (R^{J_a,j}_{C,D}(i); i \in v+2))$ for all $(A,B), (C,D) \in \mathcal{P}$ and $k, j \in 3$.

It follows that the orbits $O^{J_a,k}_{A,B}$ together with the refinement sequences $(R^{J_a,k}_{A,B}(i); i \in v+2)$ satisfy conditions 1. to 5.

We are now in the following position: Given a well chosen initial segment J of length v there is a unique isomorphism, say f_J, from $\mathrm{H}_\mathcal{T}$ restricted to v to J. Every element of J is blue. The empty set is well chosen. If the initial segment J of length v is well chosen then there is an extension J_a of J to a well chosen initial segment of length $v+1$. That is, $f_J \subset f_{J_a}$. We construct successively the well chosen initial segments $J_0 = \emptyset, J_1, J_2, \dots$ of ever increasing lengths so that J_{i+1} is an extension of J_i. Then $f := \bigcup_{i \in \omega} f_{J_i}$ is an embedding of $\mathrm{H}_\mathcal{T}$ into

$H_{\mathcal{T}}$ with every vertex in the image of f being blue.

□

Theorem 6.2 *Let $\mathcal{T}$ be a possibly infinite set of finite tournaments. Then $H_{\mathcal{T}}$ is indivisible if and only if for any two orbits X and Y of $H_{\mathcal{T}}$ either X can be embedded into Y or Y can be embedded into X.*

PROOF. An orbit of $H_{\mathcal{T}}$ is a homogeneous structure; see [2]. Hence if X and Y are two orbits of $H_{\mathcal{T}}$ and age(X) $\subseteq$ age(Y) then X can be embedded into Y. On the other hand if X can be embedded into Y then the age of X is a subset of the age of Y. Hence X can be embedded into Y or Y can be embedded into X if and only if bound(Y) $\subseteq$ bound(X) or bound(X) $\subseteq$ bound(Y).

Hence the necessary part of Theorem 6.2 follows from Theorem 4.1 and the sufficient part of Theorem 6.2 follows from Theorem 6.1.

□

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