

Coloring subgraphs of the Rado graph

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Abstract

Given a universal binary countable homogeneous structure U and $n \in \omega$, there is a partition of the induced n -element substructures of U into finitely many classes so that for any partition C_0, C_1, \dots, C_{m-1} of such a class Q into finitely many parts there is a number $k \in m$ and a copy U^* of U in U so that all of the induced n -element substructures of U^* which are in Q are also in C_k .

The partition of the induced n -element substructures of U is explicitly given and a somewhat sharper result as the one stated above is proven.

1 Introduction

The Rado Graph $\mathbb{R} = (R; E)$ is the countable universal homogeneous graph. It is a countable graph with the defining property that for every finite set $F \subset R$ of vertices of the Rado graph and partition of F into the classes A and B there is a vertex x of the Rado graph which is adjacent to all vertices in A and not adjacent to any of the vertices in B . The injection $f : R \rightarrow R$ is an *embedding* of the Rado graph if x adjacent to y if and only if $f(x)$ adjacent to $f(y)$ for all vertices $x, y \in R$. The image of an embedding of R is a *copy* of \mathbb{R} .

It is just an exercise in the consequences of the defining property of the Rado graph, to show that the Rado graph is indivisible. That is, that for every partition A, B of the the vertices R of the Rado graph there exists a copy $\mathbb{R}^* = (R^*, E^*)$ of the Rado graph so that $R^* \subseteq A$ or $R^* \subseteq B$.

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There is the obvious partition of the two element subsets of the Rado graph into those which contain an edge and those which do not contain an edge. Hence the generalization of vertex indivisibility would be edge indivisibility. But early on in investigations of partition properties of homogeneous structures it was discovered that the Rado graph is not edge indivisible. Erdős, Hajnal and Posa produced in [1] a partition of the edges of the Rado graph, so that every copy of the Rado graph in the Rado graph must contain both types of edges.

They enumerated $(a_i; i \in \omega)$ the vertices of the Rado graph into an ω sequence and associated with every vertex a_n a 0,1-sequence $\sigma(a_n)$ of length n so that $\sigma(a_n)(i) = 1$ if and only if a_n is adjacent to a_i . Then the *up edges* U of E are the ones for which the lexicographic order of the sequences corresponding to the endpoints agrees with the order on R given by the enumeration. All the other edges are *down edges* D . Then $E = U \cup D$ is a partition of E . It is shown in [1] that every copy of \mathbb{R} in \mathbb{R} must contain up edges and down edges.

It is shown in [2] that this partition is indeed best possible. That is, if A, B is a partition of U , the set of up edges, then there is a copy $\mathbb{R}^* = (R^*, E^*)$ of \mathbb{R} in \mathbb{R} so that $E^* \cap A = \emptyset$ or $E^* \cap B = \emptyset$. If A, B is a partition of D then there is a copy $\mathbb{R}^* = (R^*, E^*)$ of \mathbb{R} in \mathbb{R} so that $E^* \cap A = \emptyset$ or $E^* \cap B = \emptyset$. Because the complement of the Rado graph is the Rado graph there is a similar partition result for the non adjacent pairs of vertices. This then provides a best possible partition of the two element subsets of the Rado graph with respect to indivisibility. Such a partition is a *canonical partition* for which we will provide an exact definition later on.

Naturally then the question arises to describe the different types of three element subsets of the Rado graph to obtain a canonical partition. We will provide some answer to this question in the more general case of n element subsets, in the sense that we will define, for every $n \in \omega$, a partition of the n -element subsets of R into finitely many classes, so that each of the classes is indivisible.

To define the different indivisible n -element types some properties of sets of finite sequences have to be defined. The results deal with the more general case of universal binary countable homogeneous structures. In the next section we will define those, make some general remarks on canonical partitions of relational structures, look at sets of finite sequences and state the result.

2 The result

Let $\mathcal{L} = \{E_i \mid i \in n \in \omega\}$ be a finite list of binary relation symbols. The binary relational structure \mathbb{A} with base set A is of *type* \mathcal{L} if each relation symbol $R \in \mathcal{L}$ has an interpretation as a binary relation $R_{\mathbb{A}}$ on A and every relation of \mathbb{A} corresponds in this way to a relation symbol of \mathcal{L} . Let \mathbb{A} and \mathbb{B} be two relational structures of type \mathcal{L} . The injection $f : A \rightarrow B$ is an *embedding* of \mathbb{A} into \mathbb{B} if $xR_{\mathbb{A}}y$ if and only if $f(x)R_{\mathbb{B}}f(y)$ for all $x, y \in A$ and all $R \in \mathcal{L}$. An *isomorphism* of \mathbb{A} to \mathbb{B} is a bijective embedding. Those definitions of embedding and isomorphism extend in the obvious way to relational structures of arbitrary arity.

The \mathcal{L} structure \mathbb{A} is an *induced substructure* of the \mathcal{L} structure \mathbb{B} if the identity map of A is an embedding of \mathbb{A} into \mathbb{B} . If $C \subseteq B$ then the *substructure of \mathbb{B} induced by C* is the \mathcal{L} structure \mathbb{C} with base set C which is an induced substructure of \mathbb{B} . The structure \mathbb{A}^* is a *copy of \mathbb{A} in \mathbb{B}* if there exists an embedding f of \mathbb{A} into \mathbb{B} so that the substructure of \mathbb{B} induced by $f[A]$ is isomorphic to \mathbb{A}^* . ($f[A] := \{f(a) \mid a \in A\}$.)

We will use the notation $\mathbb{A} = (A; \mathcal{L})$ to indicate that \mathbb{A} is a structure of type \mathcal{L} with base set A . Let $\mathbb{A} = (A; \mathcal{L})$ and $x \in A$. Then $\mathbb{A} - x$ is the substructure of \mathbb{A} induced by the set $A \setminus \{x\}$.

Let $\mathbb{A} = (A; \mathcal{L})$ be a relational structure. Let Q be a set of finite subsets of A . The set Q is *indivisible within \mathbb{A}* if for every partition C_0, C_1, \dots, C_{m-1} of Q into $m \in \omega$ subsets there exists a copy $\mathbb{A}^* = (A^*; \mathcal{L})$ of \mathbb{A} in \mathbb{A} so that all of the subsets of A^* which are in Q are in C_k .

Note that if there is a copy $\mathbb{A}^* = (A^*; \mathcal{L})$ so that $2^{A^*} \cap Q = \emptyset$ then the condition that all elements of Q which are subsets of A^* are also in C_k is vacuously satisfied. It is therefore not difficult to find indivisible sets of subsets of relational structures unless one imposes a further condition that the set of subsets is to be large in some sense. Clearly, the set of all subsets will usually not be indivisible as one can partition them according to size.

The obvious first question then is whether the set of all n -element subsets of a given relational structure is indivisible. It turns out that even for the set of one element subsets this is seldom the case. Hence the question becomes, that given a relational structure \mathbb{A} , is there a partition of the n -element subsets of A into a small number of classes each of which is indivisible.

Let $\mathbb{A} = (A; \mathcal{L})$ be a relational structure. A set $Q \subseteq [A]^n$ of n -element subsets of A is *persistent* if there is no copy $\mathbb{A}^* = (A^*; \mathcal{L})$ of \mathbb{A} in \mathbb{A} so that $[A^*]^n \cap Q = \emptyset$. Note again that if the subset Q of A is not persistent then it is indivisible. A *canonical partition* of $[A]^n$ is a partition of $[A]^n$ into finitely many parts each of which is persistent and indivisible. It has been shown

in [3] that any two canonical partitions of the n -element subsets of some relational structure \mathbb{A} are equivalent up to a permutation of the classes. In particular they have the same number of elements. Note that if the subsets P and Q with $P \cap Q = \emptyset$ are persistent then $P \cup Q$ is not indivisible.

The results of this paper apply not just to the Ramsey graph but to the more general countable universal binary structures. Let \mathbf{F} be a set of relational structures of type \mathcal{L} whose base set is the set $\{0, 1\}$ and with the property that if \mathbb{A} and \mathbb{B} are two isomorphic relational structures in the language \mathcal{L} and set of elements $\{0, 1\}$ then either both are in \mathbf{F} or neither one of the two is in \mathbf{F} . Such a set \mathbf{F} is a *universal constraint set*.

Let \mathbf{F} be a universal constraint set. The relational structure $\mathbb{A} = (A, \mathcal{L})$ satisfies the constraint set \mathbf{F} if for every two elements $x, y \in A$ with $x \neq y$ the substructure of \mathbb{A} induced by the set $\{x, y\}$ is isomorphic to one of the elements in \mathbf{F} and if $R(x, x)$ does not hold in \mathbb{A} for every relation symbol $R \in \mathcal{L}$. The second condition says that if \mathbb{A} satisfies the constraints then it does not have any “loops”.

The countable relational structure $\mathbb{U}_{\mathbf{F}} = (U, \mathcal{L})$ is *universal of type \mathcal{L} under the constraints \mathbf{F}* if it satisfies the constraints \mathbf{F} and has the following *mapping extension property*:

Definition 2.1 (mapping extension property).

For every finite relational structure $\mathbb{A} = (A; \mathcal{L})$ which satisfies the constraints \mathbf{F} and every element x in A and every embedding f of $\mathbb{A} - x$ into $\mathbb{U}_{\mathbf{F}}$ there is an extension of f to an embedding of \mathbb{A} into \mathbb{U} .

Universal structures are special cases of homogeneous structures, see [4] for a more detailed description. Starting with a paper by Komjath and Rödl [5], vertex partitions of homogeneous structures have been quite extensively studied, see [6], [7] [8]. Edge partitions of the Rado graph and the triangle free countable homogeneous graph are quite well understood, see [1], [2], [10]. The partition theory of the order structure of the rationals is completely solved, see [11] and [12].

In order to state the result we have, as in the case of the Rado graph, to represent the elements of relational structures by sequences and have to define some notions for sets of finite sequences.

Let $n, m \in \omega$. We denote, for $m \in \omega + 1$, by ${}^n m$ the set of all sequences $s = \langle s_0, s_1, \dots, s_{n-1} \rangle$ of length n with entries s_i in m . Let $\mathfrak{X}_\omega := \bigcup_{n \in \omega} {}^n \omega$. If $s = \langle s_0, s_1, \dots, s_{n-1} \rangle \in \mathfrak{X}_\omega$ we denote by $|s| = n$ the *length* of s and write either s_i or $s(i)$ to denote the i 's entry of the sequence s . The sequence t is an *initial segment* of the sequence s , written $t \subset s$, if $|t| < |s|$ and $t_i = s_i$ for every $i \in |t|$. We write $t \subseteq s$ if t is an initial segment of s or t is equal to s .

Given two sequences s and t we denote by $s \wedge t$, the *meet* of s and t , the longest sequence which is an initial segment of s and an initial segment of t . If $t \subseteq s$ then $s \wedge t = t$. The meet of two sequences always exists, it might be the empty sequence. Let $S \subseteq \mathfrak{T}_\omega$ be a set of sequences. The set $\text{closure}(S)$ is the set S union the set of all meets of elements in S .

Definition 2.2. Let $s, t \in \mathfrak{T}_\omega$ then $x \prec y$ if and only if x and y are incomparable under \subseteq and $x(|x \wedge y|) < y(|x \wedge y|)$.

Note that \prec is not a total order.

Let $s, t \in \mathfrak{T}_\omega$. Let $S \subseteq \mathfrak{T}_\omega$ and $t \in S$. The sequence s is an *immediate successor* of t in S if $t \subset s$ and there is no element $r \in S$ with $t \subset r \subset s$. The *degree* of t in S is the number of immediate successors of t in S .

The set S of sequences is an *antichain* if $x \subseteq y$ implies $x = y$ for all $x, y \in S$. The set S of sequences is *transversal* if $|x| = |y|$ implies $x = y$ for all $x, y \in S$.

Definition 2.3. The set $F \subseteq \mathfrak{T}_\omega$ of sequences is *diagonal* if it is an antichain and $\text{closure}(F)$ is transversal and the degree of every element of $\text{closure}(F)$ is at most two.

Definition 2.4. Let $R, S \subseteq \mathfrak{T}_\omega$ be two sets of sequences. The function f of R to S is a *similarity* of R to S if for all $x, y, z, u \in R$:

1. f is a bijection.
2. $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.
3. $|x \wedge y| < |z \wedge u|$ if and only if $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
4. If $|z| > |x|$ then $z(|x|) = f(z)(|f(x)|)$.
5. If $x \prec y$ then $f(x) \prec f(y)$.

The sets R and S of sequences are *similar*, $R \sim S$, if there is a similarity of R to S . Note that if R is diagonal and R and S are similar then S is diagonal. We denote by $\text{Sim}_R(S)$ the set of all subsets of R which are similar to S . The function f of R into \mathfrak{T}_ω is a *similarity embedding* if f is a similarity of R to $f[R]$.

Note that Item 1. of Definition 2.4 follows from Item 2. and that the composition of similarities is again a similarity and the inverse of a similarity is again a similarity. Hence \sim is an equivalence relation on \mathfrak{T}_ω .

Let \mathbf{F} be a universal constraint set with $|\mathbf{F}| = k \in \omega$ and λ a bijection of \mathbf{F} to k . We say that λ is a labeling of \mathbf{F} . Let $\mathbb{U}_{\mathbf{F}} = (U; \mathcal{L})$ be a universal countable binary relational structure with constraints \mathbf{F} . Let $(u_i; i \in \omega)$ be an enumeration of the elements of U .

For every pair (u_n, u_m) with $n < m$ let $\mu(u_n, u_m) := \mathbb{F} \in \mathbf{F}$ the structure in \mathbf{F} for which the function mapping 0 to u_n and 1 to u_m is an isomorphism of \mathbb{F} to the substructure of $\mathbb{U}_{\mathbf{F}}$ induced by $\{u_n, u_m\}$.

We associate with every element u_n of U a sequence σ_{u_n} of length n so that for every $i \in n$ the i 's entry $\sigma_{u_n}(i) := \lambda(\mu(u_n, u_m))$. If F is a subset of U then $\sigma(F) := \{\sigma_x \mid x \in F\}$. If F and G are two subsets of U then $F \sim G$ if $\sigma(F) \sim \sigma(G)$ and F is *diagonal* if $\sigma(F)$ is diagonal. The sets F and G are *similar* if $F \sim G$. For $R \subseteq U$ we denote by $\text{Sim}_R(F)$ the set of subsets G of R with $F \sim G$. Note that \sim is an equivalence relation which partitions $[U]^n$ into finitely many similarity classes for every $n \in \omega$.

Let T be the subtree of \mathfrak{T}_ω consisting of all sequences with entries in $k = |\mathbf{F}|$. We define a relational structure of type \mathcal{L} on T to obtain the relational structure $\mathbb{T}_{\mathbf{F}} = (T, \mathcal{L})$ as follows: Let t and s be two sequences in T with $|t| > |s|$. Let $\mathbb{F} \in \mathbf{F}$ be such that $\lambda(\mathbb{F}) = t(|s|)$. Then the function which maps 0 to s and 1 to t is an isomorphism of the structure \mathbb{F} to the substructure of $\mathbb{T}_{\mathbf{F}}$ induced by the set $\{s, t\}$. It follows that σ is an isomorphic embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{T}_{\mathbf{F}}$.

Let $\mathbb{U}_{\mathbf{F}} = (U; \mathcal{L})$ be a universal countable binary relational structure. The notions of similarity between subsets of U and of a subset of U being diagonal depend on the enumeration. We always assume that U is enumerated into an ω sequence and that the notions of similar and diagonal are relative to this fixed enumeration.

The embedding f of $\mathbb{U}_{\mathbf{F}} = (U, \mathcal{L})$ into $\mathbb{U}_{\mathbf{F}}$ is a *diagonalization* of $\mathbb{U}_{\mathbf{F}}$ if $f[U]$ is diagonal and $f[F] \sim F$ for every diagonal subset F of U . Let f be a diagonalization of $\mathbb{U}_{\mathbf{F}}$. The copy of \mathbb{U} induced by $f[U]$ is a *diagonal representation* of \mathbb{U} .

Let f be a diagonalization of $\mathbb{U}_{\mathbf{F}}$ and $n < m \in \omega$. The set $\{u_n, u_m\}$ is a diagonal subset of U . It follows that $\mu(u_n, u_m) = \mu(f(u_n), f(u_m))$ and hence that f is an isomorphic embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{U}_{\mathbf{F}}$. ($\sigma_{u_m}(|\sigma_{u_n}|) = \sigma_{f(u_m)}(|\sigma_{f(u_n)}|)$.)

Hence every diagonal representation of $\mathbb{U}_{\mathbf{F}}$ is isomorphic to $\mathbb{U}_{\mathbf{F}}$ that is every diagonalization of $\mathbb{U}_{\mathbf{F}}$ is an embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{U}_{\mathbf{F}}$. It follows that if a diagonal representation of $\mathbb{U}_{\mathbf{F}}$ has a canonical partition then $\mathbb{U}_{\mathbf{F}}$ has a canonical partition.

We will prove in Section 7:

Theorem 2.1. *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and F a finite diagonal subset of U . Let C_0, C_1, \dots, C_{m-1} be a partition of $\text{Sim}_U(F)$ into equivalence classes.*

Then there exists $k \in m$ and a diagonalization f of \mathbb{U} into \mathbb{U} so that $\text{Sim}_{f[U]}(F) \subseteq C_k$.

Corollary 2.1. *Let $\mathbb{V} = (V; \mathfrak{L})$ be a diagonal representation of the universal countable binary relational structure $\mathbb{U} = (U; \mathfrak{L})$ and F a finite diagonal subset of V . Let C_0, C_1, \dots, C_{m-1} be a partition of $\text{Sim}_V(F)$ into equivalence classes.*

Then there exists $k \in m$ and a diagonalization f of \mathbb{V} into \mathbb{V} so that $\text{Sim}_{f[V]}(F) \subseteq C_k$.

Proof. Let $|F| = n$ and let h be a diagonalization of \mathbb{U} so that $h[U] = V$. For $i \in m$ let $C'_i = \{G \in \text{Sim}_U(F) \mid h[G] \in C_i\}$. The sets C'_i form a partition of $\text{Sim}_U(F)$. According to Theorem 2.1 there is a diagonalization g of \mathbb{U} and $k \in m$ so that $\text{Sim}_{g[U]}(F) \subseteq C'_k$.

The set $g[U]$ is diagonal and hence every subset of $g[U]$ is diagonal. It follows that $G \in \text{Sim}_{h \circ g[U]}(F)$ if and only if there is $G' \in \text{Sim}_{g[U]}(F)$ with $h[G'] = G$. The diagonalization h maps elements of C'_k to elements in C_k . Let $f = h \circ g$ restricted to V . \square

Corollary 2.2. *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and F a finite subset of U .*

Then $\text{Sim}_U(F)$ is indivisible. If F is not diagonal then $\text{Sim}_U(F)$ is not persistent, there is a diagonalization f of \mathbb{U} so that $f[U] \cap \text{Sim}_U(F) = \emptyset$. If F is diagonal then $\text{Sim}_U(F)$ may or may not be persistent.

Let $n \in \omega$. It might be the case that there is no similarity class of an n -element induced substructure of \mathbb{U} which is persistent. The following Lemma enables us to find some persistent subset of $[U]^n$. The set $Q \subseteq [U]^n$ is a *diagonal similarity class* if there is a diagonal set F so that $Q = \text{Sim}_U(F)$. The subset $A \subseteq [U]^n$ is a *diagonal union* if it is the union of diagonal similarity classes.

Lemma 2.1. *Let $\mathbb{V} = (V; \mathfrak{L})$ be a diagonal representation of the universal countable binary relational structure $\mathbb{U} = (U; \mathfrak{L})$. Let $A \subseteq [V]^n$ be a diagonal similarity class of V and $B \subseteq [V]^n$ be a diagonal union with $A \cap B = \emptyset$. If B is not persistent then $A \cup B$ is indivisible.*

Proof. Let C_0, C_1, \dots, C_{m-1} be a partition of $A \cup B$ and $C'_i = C_i \cap A$ be the induced partition of A and $A = \text{Sim}_V(F)$. Let f be an embedding of \mathbb{V} into \mathbb{V} so that $B \cap [f[V]]^n = \emptyset$. According to Corollary 2.1 there is a diagonalization g of \mathbb{V} and $k \in m$ so that $\text{Sim}_{g[V]}(F) \subseteq C'_k$.

The diagonalization g preserves similarity classes of diagonal sets. We obtain $[g \circ f[V]]^n \cap B = \emptyset$ and $A \cap [g \circ f[V]]^n = \text{Sim}_{g \circ f[V]}(F) \subseteq C'_k \subseteq C_k$. \square

Theorem 2.2. *Let $\mathbb{V} = (V; \mathfrak{L})$ be a diagonal representation of the universal countable binary relational structure $\mathbb{U} = (U; \mathfrak{L})$ and $n \in \omega$. Then \mathbb{V} has a canonical partition $Q'_0, Q_1, Q_2, \dots, Q_{q-1}$ of $[V]^n$ into finitely many classes so that each the classes of this partition is a union of similarity classes.*

Proof. Let C_0, C_1, \dots, C_{m-1} be a partition of $[V]^n$ into similarity classes. It follows from Lemma 2.1 that there is a subset S_0 of the indices in m so that the set $Q_0 := \bigcup_{i \in S_0} C_i$ is persistent and indivisible. If the set $\bigcup_{m \setminus S_0} C_i$ is persistent then there is a set $S_1 \subseteq m \setminus S_0$ so that the set $Q_2 = \bigcup_{i \in S_1} C_i$ is persistent and indivisible.

Proceeding we obtain a partition $Q_0, Q_1, Q_2, \dots, Q_{m-1}, Q_m$ of $[V]^n$ so that each of the sets Q_i for $i \in m$ is persistent and indivisible. The set Q_m is empty or not persistent. It follows from Lemma 2.1 that $Q'_0 = Q_0 \cup Q_m, Q_1, Q_2, \dots, Q_{m-1}$ is a canonical partition of $[V]^n$. \square

Corollary 2.3. *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and $n \in \omega$. Then there exists a canonical partition of $[U]^n$ into finitely many classes.*

3 Preliminaries

Let $s, t \in \mathfrak{T}_\omega$ and $t \subset s$. We will write s as $\langle t; s_{|t|}, s_{|t|+1}, \dots, s_{|s|-1} \rangle$. In particular $\langle t; l \rangle$ for $t \in \mathfrak{T}_\omega$ and $l \in \omega$ denotes the sequence s of length $|t| + 1$, initial segment t and $s_{|t|} = s_{|s|-1} = l$. The set S is *closed under initial segments* if for every $s \in S$ and $i \in |s|$ the sequence $t \in S$ with $t \subset s$ and $|t| = i$ is also an element of S .

Let S and T be two meet closed subsets of \mathfrak{T}_ω and $f : S \rightarrow T$ a function of S to T . The function f is *meet preserving* if $f(s \wedge r) = f(s) \wedge f(r)$ for any two elements s and r of S . If f is meet preserving and an injection then f^{-1} is meet preserving and $x \subset y$ if and only if $f(x) \subset f(y)$.

The set $T \subseteq \mathfrak{X}_\omega$ of finite sequences is an ω -tree if it is nonempty, closed under initial segments, has no endpoints and every element of T has finite degree. Note that every ω -tree is closed under meet. The subset D of T is *cofinal in T* if for every $t \in T$ there is an element $d \in D$ with $t \subseteq d$.

The sequence l is *monotone* if $l_i < l_j$ for all $i, j \in |l|$ with $i < j$. Let $s, l \in \mathfrak{X}_\omega \cup {}^\omega\omega$, the sequence l monotone and let $m \in \omega$ be minimal so that $s(l(m))$ is undefined, that is so that $m \geq |l|$ or $|s| \geq l(m)$. The *composition* of the sequences s and l is the sequence $s \circ l$ of length m for which $(s \circ l)(i) = s(l(i))$ for all $i \in m$. (That is, composition of sequences is just function composition.)

It follows that the sequence $s \circ l$ is the subsequence of all of those entries s_i of s for which i is an entry of l . Which is of course the same as the subsequence obtained from s by removing all the entries with indices not in l .

Let S be a set of sequences. The sequence $\overrightarrow{\text{levels}}(S)$ is obtained by ordering the elements of the set $\text{levels}(S) := \{|s| \mid s \in S\}$ in strictly increasing order.

If S is a set of sequences and l monotone then $S \circ l := \{s \circ l \mid s \in S\}$. Note that the subsequence $s \circ \overrightarrow{\text{levels}}(S)$ is obtained from s by removing all of those entries s_i for which there is no element $t \in S$ with $|t| = i$. (If $i \in \omega$ and s a sequence with $i \geq |s|$ then the sequence obtained from s by removing the entry with index i is s .)

Let $S \subseteq \mathfrak{X}_\omega$. The function ν_S of S to \mathfrak{X}_ω given by $\nu_S(x) = x \circ \overrightarrow{\text{levels}}(\text{closure}(S))$ is the *normal function* of S . For $i \in \omega$ let $\nu_{S,i}$ be the function which removes from every element $x \in S$ the entry with index i . It follows that ν_S can be expressed as a product $\nu_S = \nu_{S_{n-1}, i_{n-1}} \circ \nu_{S_{n-2}, i_{n-2}} \circ \cdots \circ \nu_{S_1, i_1} \circ \nu_{S_0, i_0}$ with $S_0 = S$ and i_j some number not in $\text{levels}(\text{closure}(S_{i_j}))$ and $S_{j+1} = \nu_{S_j, i_j}(S_j)$ for all $j \in n$.

Example 3.1. Let $S := \{\langle 3, 1 \rangle, \langle 3, 1, 0, 5 \rangle, \langle 3, 1, 2, 4, 6 \rangle\}$.

Then S is closed and $\overrightarrow{\text{levels}}(\text{closure}(S)) = \overrightarrow{\text{levels}}(S) = \langle 2, 4, 5 \rangle := l$ and hence $\nu_S[S] = \{\langle \rangle, \langle 0 \rangle, \langle 2, 6 \rangle\} = S \circ l$. Then $\nu_{S,3} = \{\langle 3, 1 \rangle, \langle 3, 1, 0 \rangle, \langle 3, 1, 2, 6 \rangle\}$. Let $R := \nu_{S,3}$ and $P := \nu_{R,1} = \{\langle 3 \rangle, \langle 3, 0 \rangle, \langle 3, 2, 6 \rangle\}$. Then we obtain finally $\nu_{P,0} = \{\langle \rangle, \langle 0 \rangle, \langle 2, 6 \rangle\} = \nu_S[S]$.

Note that the function $\nu_S : S \rightarrow \mathfrak{X}_\omega$ is an injection. For assume that $s, t \in S$ with $s \neq t$. If $\{s, t\}$ is an antichain then $s(|s \wedge t|) \neq t(|s \wedge t|)$ and both $s(|s \wedge t|)$ and $t(|s \wedge t|)$ are defined. Hence $\nu_S(s) \neq \nu_S(t)$. If $s \subset t$ then $s \wedge t = s$ and s does not have an entry with index $|s|$ while t does have an entry with index $|s|$. Hence $\nu_S(s) \neq \nu_S(t)$.

Example 3.2. Let $S = \{\langle 3, 1, 0, 5 \rangle, \langle 3, 2, 1, 1 \rangle\}$.

Then $\overrightarrow{\text{levels}}(S) = \langle 4 \rangle := l$ and $\langle 3, 1, 0, 5 \rangle \circ l = \langle \rangle = \langle 3, 2, 1, 1 \rangle \circ l$.

Definition 3.1. Let $R, S \subseteq \mathfrak{T}_\omega$ be two sets of sequences. The function f of R to S is a *strong similarity* of R to S if for all $x, y, z, u \in R$:

1. f is a bijection.
2. $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.
3. $|x \wedge y| < |z \wedge u|$ if and only if $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
4. If $|z| > |x \wedge y|$ then $z(|x \wedge y|) = f(z)(|f(x) \wedge f(y)|)$.

The sets R and S of sequences are *strongly similar* if there is a strong similarity of R to S . We write $R \overset{s}{\sim} S$ to indicate that R and S are strongly similar. If $S \subseteq T$ and f is a strong similarity of R to S then f is a *strong similarity embedding* of R into T and S is a *strong similarity copy* of R in T .

The notion of strongly similar sets of sequences is central to this paper. If F is a subset of the set R of sequences then $\text{Sims}_R(F)$ is the set of all subsets of R which are strongly similar to F .

As an easy consequence of the definition of strong similarity we obtain the following Lemma:

Lemma 3.1. Let $R, S, C, D \subseteq \mathfrak{T}_\omega$ be sets of sequences and f a strong similarity of R to S then:

1. f is a similarity of R to S .
2. If $x, y, z, u \in R$ then $x \wedge y = z \wedge u$ if and only if $f(x) \wedge f(y) = f(z) \wedge f(u)$.
3. The strong similarity f has a unique extension f' to a strong similarity of $\text{closure}(R)$ to $\text{closure}(S)$ via $f'(x \wedge y) = f(x) \wedge f(y)$ and f' is a meet preserving function of R .
4. If R and S are meet closed then f is a meet preserving bijection of R to S .
5. The product of strong similarities is a strong similarity and the inverse of a strong similarity is a strong similarity.
6. If $Q \subseteq R$ then the restriction of f to Q is a strong similarity of Q to $f[Q]$.

7. If $i \notin \text{levels}(\text{closure}(C))$ then $\nu_{C,i}$ is a strong similarity and $\nu_{C,i}[\text{closure}(C)] = \text{closure}(\nu_{C,i}[C])$.
8. The function ν_C is a strong similarity and $\nu_C[\text{closure}(C)] = \text{closure}(\nu_C[C])$.
9. If $C \subseteq \mathfrak{F}_\omega$ is meet closed and $l = \overrightarrow{\text{levels}}(C)$ and $s, r \in C$ then $s = r$ if and only if $\nu_C(s) = \nu_C(r)$ if and only if $s \circ l = r \circ l$.
10. If $R = \nu_R[R]$ and $S = \nu_S[S]$ then $R = S$ and f is the identity map on R .
11. A bijection g of C to D is a strong similarity if and only if

$$g = \nu_D^{-1} \circ \nu_C$$

if and only if

$$r \circ \overrightarrow{\text{levels}}(\text{closure}(C)) = g(r) \circ \overrightarrow{\text{levels}}(\text{closure}(D)) \text{ for all } r \in C.$$

12. The sets C and D of sequences are strongly similar if and only if

$$C \circ \overrightarrow{\text{levels}}(\text{closure}(C)) = D \circ \overrightarrow{\text{levels}}(\text{closure}(D)).$$

13. There is at most one strong similarity of R to S .
14. The set C of sequences is strongly similar to the set D of sequences if and only if $\nu_C[C] = \nu_D[D]$.

Definition 3.2. The set $F \subseteq \mathfrak{F}_\omega$ of sequences is *strongly diagonal* if it is an antichain and $\text{closure}(F)$ is transversal and for all $x, y, z \in F$ with $x \neq y$:

1. $|x \wedge y| < |z|$ and $x \wedge y \not\leq z$ implies $z(|x \wedge y|) = 0$.
2. $x(|x \wedge y|) \in \{0, 1\}$.

It follows that every subset of a strongly diagonal set is strongly diagonal. Note that Item 2. of Definition 3.2 implies that the degree of every element of $\text{closure}(F)$ is at most two and hence that every strongly diagonal set is diagonal.

Lemma 3.2. *Let F be a diagonal subset of \mathfrak{T}_ω . If f is a function of F to \mathfrak{T}_ω so that for all $x, y, z, u \in F$*

$$|x \wedge y| < |z \wedge u| \text{ implies } |f(x) \wedge f(y)| < |f(z) \wedge f(u)| \quad (1)$$

then $f[F]$ is diagonal and for all $x, y, z, u \in F$:

1. $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$ implies $|x \wedge y| < |z \wedge u|$.
2. $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.

Proof. That f is an injection and $\text{closure}(f[F])$ transversal and that the degree of every element in $\text{closure}(f[F])$ is at most two follows easily. The set $f[F]$ is an antichain because if $x \neq y$ are elements of the antichain F then $|x \wedge y| < |x|$ and $|x \wedge y| < |y|$ and hence we obtain from condition (1) that $|f(x) \wedge f(y)| < |f(x)|$ and $|f(x) \wedge f(y)| < |f(y)|$.

Let $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$ and assume for a contradiction that $|x \wedge y| = |z \wedge u|$. Then $x \neq y$ and $z \neq u$ because $\text{closure}(F)$ transversal. If $x = z$ then $|x \wedge y| = |x \wedge u|$ and $y \neq u$ and $|x \wedge y| < |u \wedge y|$ because the degree of $x \wedge y$ in F is at most two. It follows that $|f(x) \wedge f(y)| < |f(u) \wedge f(y)|$ and hence $|f(x) \wedge f(y)| = |f(x) \wedge f(u)|$. If the x, y, z, u are pairwise different we may assume without loss that $|x \wedge y| < |x \wedge z|$ and $|x \wedge y| < |u \wedge y|$, which using (1) implies $|f(x) \wedge f(y)| < |f(x) \wedge f(z)|$ and $|f(x) \wedge f(y)| < |f(u) \wedge f(y)|$, which in turn implies that $|f(x) \wedge f(y)| = |f(z) \wedge f(u)|$.

Let $x \wedge y \subseteq z$. If $x = y = z$ then $f(x) \wedge f(y) \subseteq f(z)$ follows. Otherwise $x \neq y$ and the degree of $x \wedge y$ in $\text{closure}(F)$ is two and $x \wedge y \subset z$ because F is an antichain and the degree of $x \wedge y$ is at most two. The fact that the degree of $x \wedge y$ is two and $x \wedge y \subset z$ imply that $x \wedge y \subset x \wedge z$ or $x \wedge y \subset y \wedge z$. Say, $x \wedge y \subset x \wedge z$, which implies due to condition (1) that $|f(x) \wedge f(y)| < |f(x) \wedge f(z)|$ which in turn implies $f(x) \wedge f(y) \subset f(x) \wedge f(z) \subseteq f(z)$.

If $x \wedge y \subseteq z \wedge u$ then $x \wedge y \subseteq z$ and $x \wedge y \subseteq u$ and hence $f(x) \wedge f(y) \subseteq f(z)$ and $f(x) \wedge f(y) \subseteq f(u)$ and hence $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.

If $f(x) \wedge f(y) \subseteq f(z)$ then $x \wedge y \subseteq z$ because the set $f[F]$ of sequences and the function f^{-1} satisfy the conditions of the Lemma. \square

Lemma 3.3. *Let $F \subseteq \mathfrak{T}_\omega$ be a strongly diagonal set and f a strong similarity embedding of F into \mathfrak{T}_ω . Then $f[F]$ is a strongly diagonal set.*

Proof. The strongly diagonal set F of sequences and the strong similarity f satisfy the conditions of Lemma 3.2. Hence $f[F]$ is diagonal. Item 1. of Definition 3.2 follows from Item 4. of Definition 3.1 together with Item 2. of Lemma 3.2. Item 2. of Definition 3.2 follows from Item 4. of Definition 3.1. \square

Lemma 3.4. *Let $F \subseteq \mathfrak{T}_\omega$ be a diagonal set and f a function of F into \mathfrak{T}_ω . The function f is a strong similarity embedding if and only if for all $x, y, z, u \in F$:*

1. $|x \wedge y| < |z \wedge u|$ implies $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
2. If $|z| > |x \wedge y|$ then $z(|x \wedge y|) = f(z)(|f(x) \wedge f(y)|)$.

Proof. Item 1. implies that f is a bijection and Items 2. and 3. of Definition 3.1 follow from Lemma 3.2 and Item 4. is the present Item 2. \square

Definition 3.3. Let S and T be two subsets of \mathfrak{T}_ω . The injection f of S to T is a *strong diagonalization* of S to T if for all $x, y, z, u \in S$:

1. The set of sequences $f[S]$ is strongly diagonal.
2. $|x \wedge y| < |z \wedge u|$ implies $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
3. If $|x| > |y|$ then $x(|y|) = f(x)(|f(y)|)$.
4. If $x \prec y$ then $f(x) \prec f(y)$.

Lemma 3.5. *Let f be a similarity of the diagonal set F to \mathcal{T}_ω and $x, y, z, u \in F$. Then*

$$x \wedge y \subseteq z \wedge u \text{ if and only if } f(x) \wedge f(y) \subseteq f(z) \wedge f(u).$$

Proof. Follows readily from Lemma 3.2 \square

Definition 3.4. Let $S, T \subseteq \mathfrak{T}_\omega$. The function $f : S \rightarrow T$ is a *strong \mathfrak{d} -morphism* if for every strongly diagonal subset F of S the restriction of f to F is a strong similarity embedding of F into T .

Definition 3.5. Let $S, T \subseteq \mathfrak{X}_\omega$. The function $f : S \rightarrow T$ is a \mathfrak{d} -morphism if for every diagonal subset F of S the restriction of f to F is a similarity embedding of F into T .

Lemma 3.6. *If f is a similarity of the strongly diagonal set F to the strongly diagonal set G then f is a strong similarity.*

Proof. Let f be a similarity of S to T and F a strongly diagonal subset of S . We have to verify Item 4. of Definition 3.1. Let $|z| > |x \wedge y|$. If $x = y$ then Item 4. of Definition 3.1 follows from Item 4. of Definition 2.4.

If $x \neq y$ and $x \wedge y \not\subset z$ then $z(|x \wedge y|) = 0$ because F is strongly diagonal. It follows from Lemma 3.5 that $f(x) \wedge f(y) \not\subset f(z)$ and hence that $f(z)(|f(x) \wedge f(y)|) = 0$ because $f[F] = G$ is strongly diagonal. ($f(x) \wedge f(y) \neq f(z)$ otherwise $f[F]$ would not be an antichain.)

If $x \neq y$ and $x \wedge y \subset z$. Then $\{x(|x \wedge y|), y(|x \wedge y|)\} = \{0, 1\}$ and $y \not\subset x$ and $x \not\subset y$ because F is an antichain. We may assume without loss that $x(|x \wedge y|) = 0$ and $y(|x \wedge y|) = 1$. Hence $x \prec y$ and we obtain from Definition 2.4 that $f(x) \prec f(y)$. Because G is strongly diagonal it follows that $\{f(x)(|f(x) \wedge f(y)|), f(y)(|f(x) \wedge f(y)|)\} = \{0, 1\}$ which together with $f(x) \prec f(y)$ implies $f(x)(|f(x) \wedge f(y)|) = 0$ and $f(y)(|f(x) \wedge f(y)|) = 1$. Finally $x(|x \wedge y|) = f(x)(|f(x) \wedge f(y)|)$ and $y(|x \wedge y|) = f(y)(|f(x) \wedge f(y)|)$.

It follows from $|z| > |x \wedge y|$ and the fact that the degree of $|x \wedge y|$ in F is two that $z(|x \wedge y|) = x(|x \wedge y|)$ or $z(|x \wedge y|) = y(|x \wedge y|)$. Say $z(|x \wedge y|) = x(|x \wedge y|)$. Then it follows from the fact that the degree of $x \wedge y$ is two that $x \wedge y \subset z \wedge x$. Using Lemma 3.5 and Item 3. of Definition 2.4 we obtain $f(x) \wedge f(y) \subset f(z) \wedge f(x)$ which in turn implies that $f(x)(|f(x) \wedge f(y)|) = f(z)(|f(x) \wedge f(y)|)$.

Hence $z(|x \wedge y|) = x(|x \wedge y|) = f(x)(|f(x) \wedge f(y)|) = f(z)(|f(x) \wedge f(y)|)$. \square

Lemma 3.7. *Every strong diagonalization is a \mathfrak{d} -morphism.*

Proof. Let f be a strong diagonalization of S to T and F a diagonal subset of S . Item 2. of Definition 2.4 follows from Item 2. of Lemma 3.2 and Item 3. of Definition 2.4 from Item 1. of Lemma 3.2.

Item 4. of Definition 2.4 follows from Item 3. of Definition 3.3. Item 5. of Definition 2.4 follows from Item 4. of Definition 3.3 \square

Corollary 3.1. *If $F \subseteq \mathfrak{T}_\omega$ is diagonal and f a strong diagonalization of F into \mathfrak{T}_ω then $f[F]$ is strongly diagonal and $F \sim f[F]$.*

Corollary 3.2. *Every strong diagonalization is a strong \mathfrak{d} -morphism.*

Proof. Follows from Lemma 3.6 and Lemma 3.7. □

The following Lemma is easily checked.

Lemma 3.8. *If f is a strong similarity of R to S and g a strong diagonalization of S to T then $g \circ f$ is a strong diagonalization of R to T . If f is a strong diagonalization of R to S and g a strong similarity of S to T then $g \circ f$ is a strong diagonalization of R to T .*

4 The strong diagonalization theorem

Definition 4.1. The ω -tree T is *wide*, if:

- a. $t \in T$ implies $\langle t; 0 \rangle \in T$ and $\langle t; 1 \rangle \in T$.
- b. $i \in n \in \omega$ and $\langle t; n \rangle \in T$ implies $\langle t; i \rangle \in T$
- c. If $s, t \in T$ and $|s| < |t|$ then the degree of $s \in T$ is less than or equal to the degree of t in T .

There are three ways of understanding the proof of the following Theorem 4.1. The first is to study the definition of strong diagonalization carefully and then to see that there is certainly enough room in a wide omega tree T to accommodate a strong diagonalization of T into T . The second one is to read the proof of Theorem 4.1 to the end of the construction of the function f and then to see that there is certainly enough room in a wide omega tree T to proceed with an induction argument. The third one is to read through the gory details.

Theorem 4.1. *Let T be a wide ω -tree and D a cofinal subset of T . Then there exists a strong diagonalization f of T into D .*

Proof. For $n \in \omega$ let $T(n) \subseteq T$ be the set of sequences of length n and let $T(\leq n)$ be the set of sequences in T of length at most n . Note that the root of T is the empty sequence $\langle \rangle$, that $T(0) = \{\langle \rangle\}$ and that $T(1)$ is the set of sequences in T which contain exactly one element.

We will use the notation $f[T(n)] := \{f(s) \mid s \in T(n)\}$ and for $S, R \subseteq T$ and $n \in \omega$ the expression $S < R$ to mean $|s| < |r|$ for all $s \in S$ and $r \in R$ and $S < n < R$ to mean that $|s| < n < |r|$ for all $s \in S$ and $r \in R$. If s is an immediate successor of t then t is the *immediate predecessor* of s . We denote the immediate predecessor of a sequence s by \dot{s} .

We will construct the function f so that it is a strong diagonalization of T into D and a function g of $T \setminus \{\langle \rangle\}$ into T and a number $l(n)$ for every $n \in \omega$ which satisfy the following conditions:

i. $|s| < |f(s)|$ for all $s \in T$.

ii.

$$g[T(1)] < l(0) < f(\langle \rangle) < g[T(2)] < l(1) < f[T(1)] < g[T(3)] < l(2)$$

and

$$g[T(n+1)] < l(n) < f[T(n)] < g[T(n+2)] < l(n+1) < f[T(n+1)] < g[T(n+3)] < l(n+2) < f[T(n+2)].$$

iii. If $s \in T$ and $|s| \geq 1$ then $g(s)$ is a predecessor of $f(\dot{s})$ with $g(s) = f(\dot{s}) \wedge f(s)$ and $f(\dot{s})(|g(s)|) = 0$ and $f(s)(|g(s)|) = 1$. (To satisfy this condition we use Definition 4.1 of wide ω -tree.)

iv. If $|s| = |t|$ and $s \prec t$ then $|f(s)| < |f(t)|$ and $|g(s)| > |g(t)|$ and $f(s) \wedge f(t) = g(s) \wedge g(t)$.

v. For all $n \in \omega$. The difference between $l(n)$ and the largest element in $\{|f(t)| \mid t \in T(n)\}$ is larger than the number of elements in $T(n+2)$.

vi. Let $s \in T(n+1)$ and $n \in \omega$. Then $|f(s)| > l(n)$ and the entry $f(s)(i)$ of the sequence $f(s)$ is 0 for all indices i with $|g(s)| < i \leq l(n)$, unless there exists $t \in T(n)$ with $|f(t)| = i$ in which case $f(s)(i) = f(s)(|f(t)|) = s(|t|)$. (To satisfy this condition we use Definition 4.1 of wide ω -tree.) The first $l(0)$ entries of of the sequence $f(\langle \rangle)$ are 0.

Let $l(0) = |T(1)| + 1$ and $f(\langle \rangle)$ be a sequence in D with $|f(\langle \rangle)| > l(0)$ and so that it's first $l(0)$ entries are equal to 0. Let $g[T(1)]$ be the set of predecessors of $f(\langle \rangle)$ so that $g[T(1)] < l(0)$ and if $s, t \in T(1)$ with $s \prec t$ then $|g(s)| > |g(t)|$. Note that the functions f and g defined so far satisfy Items **i.** to **vi.**

If the function f is given on $T(\leq n)$ and the function g is given on $T(\leq n+1)$ and the function l is given on all numbers $\leq n$ and if they and satisfy Items **i.** to **vi.** we extend the functions f and g and l as follows.

Let $l(n+1)$ be a number which satisfies Item **v.**

Then we determine for every $s \in T(n+1)$ a sequence $f(s) \in D$ so that all of those choices satisfy Items **i.**, **ii.**, **iii.**, **iv.** and **vi.** That is, we choose the sequence $f(s)$ to be a successor of $g(s)$ so that $f(|g(s)|) = 1$, then the next entries so that $f(s)$ satisfies Item **vi.** and long enough to satisfy Item **ii.** We make sure that if $s \prec t$ then $|f(s)| < |f(t)|$. Then it follows that $f(\dot{s}) \wedge f(s) = g(s)$ because $g(s)$ is a predecessor of $f(\dot{s})$ with $f(\dot{s})(|g(s)|) = 0$ and we have chosen $f(s)$ to be a successor of $g(s)$ with $f(s)(|g(s)|) = 1$. That $f(s) \wedge f(t) = g(s) \wedge g(t)$ we see as follows:

Item **iv** implies that if $\dot{s} \neq \dot{t}$ then $f(\dot{s}) \wedge f(\dot{t}) = g(\dot{s}) \wedge g(\dot{t})$ and hence we get from Item **ii** that $|f(\dot{s}) \wedge f(\dot{t})| < l(n-1)$. Hence $g(s) \not\subseteq g(t) \not\subseteq g(s)$. If $\dot{s} = \dot{t}$ let $s \prec t$ and hence $|g(s)| > |g(t)|$. Then $g(s)(|g(t)|) = f(\dot{s})(|g(t)|) = 0$ and $f(s)(|g(t)|) = 0$ and $f(t)(|g(t)|) = 1$. Hence $f(s) \wedge f(t) = g(t) = g(s) \wedge g(t)$.

Finally we determine for every $s \in T(n+2)$ a predecessor $g(s)$ of $f(\dot{s})$ so that those choices satisfy Items **ii.** and **iv.** Note that $f(\dot{s})(|g(s)|) = 0$ because $g(s)$ is a predecessor of $f(\dot{s})$ and we have chosen $f(\dot{s})$ so that all entries with with index in between $g(\dot{s})$ and $l(n+1)$ are 0. Hence Item **iii.** follows.

We procede with this construction and obtain functions f and g of T to D and a function l of ω to ω which satisfy Items **i.** to **vi.** It follows that $\text{closure}(f[T]) = f[T] \cup g[T]$ and $f[T] \cap g[T] = \emptyset$ and from Items **ii.** and **iv.** that f is an injection.

Claim 1: *The functions f and g satisfy:*

$$f(s) \wedge f(t) = \begin{cases} g(t) & \text{if } |s| = |t| \geq 1 \text{ and } \dot{s} = \dot{t} \text{ and } s \prec t, \\ f(\dot{s}) \wedge f(\dot{t}) & \text{if } |s| = |t| \geq 1 \text{ and } \dot{s} \neq \dot{t}, \\ f(\dot{s}) \wedge f(t) & \text{if } |s| \geq 1 \text{ and } \dot{s} \neq t \in T(\leq |s| - 1), \\ g(s) & \text{if } |s| \geq 1 \text{ and } t = \dot{s}. \end{cases}$$

Proof of Claim 1: Let $|s| = |t| \geq 1$ and $\dot{s} = \dot{t}$ and $s \prec t$. Then, from Item **iv.**, $f(s) \wedge f(t) = g(s) \wedge g(t)$ and $|g(s)| > |g(t)|$. Because, Item **iii.**, both $g(s)$ and $g(t)$ are predecessors of $f(\dot{s})$, they are ordered under \subset and hence $g(s) \wedge g(t) = g(t)$.

Let $|s| = |t| \geq 1$ and $\dot{s} \neq \dot{t}$. Then $f(s) \wedge f(t) = g(s) \wedge g(t)$. Because $g(s)$ is a predecessor of $f(\dot{s})$ and $g(t)$ is a predecessor of $f(\dot{t})$ and $g(s) \neq g(t)$ it follows that $g(s) \wedge g(t) = f(\dot{s}) \wedge f(\dot{t})$.

Let $|s| \geq 1$ and $\dot{s} \neq t \in T(\leq |s|-1)$. If $|t| = |s|-1 = \dot{s}$ then $|f(\dot{s}) \wedge f(t)| = |g(\dot{s}) \wedge g(t)| < |g(s)|$, from Item **ii**. This together with $f(\dot{s}) \wedge f(s) = g(s)$ implies that $f(s) \wedge f(t) = f(\dot{s}) \wedge f(t)$. If $|t| < |s| - 1$ then we obtain from Item **ii**. that $|f(\dot{s}) \wedge f(t)| \leq |f(t)| < |g(s)|$.

The last case follows directly from Item **iii**.

End of proof of Claim 1.

Claim 2: If $s, t \in T$ with $s \neq t$ then $f(s) \wedge f(t) \in g[T]$. Also $\text{closure}(f[T]) = f[T] \cup g[T]$ and $f[T] \cap g[T] = \emptyset$.

Begin proof of Claim 2: The claim $f(s) \wedge f(t) \in g[T]$ follows readily from Claim 1 by induction on $\max(|s|, |t|)$. Item **ii**. implies $f[T] \cap g[T] = \emptyset$.

End of proof of Claim 2.

Claim 3: Let $n \in \omega$. If f restricted to $T(\leq n)$ is a strong diagonalization of $T(\leq n)$ into T then f restricted to $T(\leq n+1)$ is a strong diagonalization of $T(\leq n+1)$ into T .

Begin proof of Claim 3: Assume that f restricted to $T(\leq n)$ is a strong diagonalization of $T(\leq n)$ into T .

The set $f[T(\leq n+1)]$ of sequences is strongly diagonal:

It follows from Items **ii**. and **iv**. that $\text{closure}(f[T]) = f[T] \cup g[T]$ is transversal.

If there are sequences $s, t \in T(\leq n+1)$ with $s \neq t$ and $f(t) \subset f(s)$ then $f(t) = f(s) \wedge f(t) \in g[T]$ according to and in contradiction to Claim 2. Hence $f[T]$ is an antichain.

Let $x, y, s \in T(\leq n+1)$ with $x \neq y$ and $|f(x) \wedge f(y)| < |f(s)|$ and $f(x) \wedge f(y) \not\subset f(s)$.

Let $s \in T(n+1)$. If $|f(x) \wedge f(y)| < |g(s)|$ then $f(x) \wedge f(y) \not\subset f(\dot{s})$ and hence $f(s)(|f(x) \wedge f(y)|) = f(\dot{s})(|f(x) \wedge f(y)|) = 0$. If $|f(x) \wedge f(y)| \geq |g(s)|$ then $f(x) \wedge f(y) = g(t)$ for some $t \in T(n+1)$ with $s \neq t$ because $f(x) \wedge f(y) \not\subset f(s)$. We have $f(s)(|g(t)|) = 0$ from Item **vi**. and the fact that $\text{closure}(f[T])$ is transversal.

Let $s \in T(\leq n)$ and not both $x, y \in T(n+1)$ with $\dot{x} = \dot{y}$. Because $|f(x) \wedge f(y)| < |f(s)|$ it can not be the case that one of x and y , say y , is an element of $T(n+1)$ and $x = \dot{y}$. Hence it follows from Claim 1, that there are sequences $x', y' \in T(\leq n)$ so that $f(x) \wedge f(y) = f(x') \wedge f(y')$. Hence $f(s)(|f(x) \wedge f(y)|) = f(s)(|f(x') \wedge f(y')|) = 0$ because $f[T(\leq n)]$ is strongly diagonal.

Let $s \in T(\leq n)$ and $|x| = |y| \in T(n+1)$ and $\dot{x} = \dot{y}$ and $x \prec y$. Then $f(x) \wedge f(y) = g(y)$ according to Claim 1. It follows from Items **ii**. and **vi**.

that $f(s)(|g(y)|) = 0$.

Finally, in order to show that $f[T]$ is strongly diagonal we have to prove that $f(x)(|f(x) \wedge f(y)|) \in \{0, 1\}$ if $f(x) \neq f(y)$. If one of x and y , say y , is an element of $T(n+1)$ and $x = \dot{y}$ then $f(x) \wedge f(y) = g(y)$ and $f(x)(|g(y)|) = 0$ and $f(y)(|g(y)|) = 1$ according to Item **iii**. Let $x, y \in T(n+1)$ with $\dot{x} = \dot{y}$. If $x \prec y$ then $f(x) \wedge f(y) = g(y)$ and $f(x)(|g(y)|) = 0$ from Item **vi**. and the fact that $\text{closure}(f[T])$ is transversal. Accordint to Item **iii**. we have $f(y)(|g(y)|) = 1$.

In all other cases it follows from Claim 1, that there are sequences $x', y' \in T(\leq n)$ with $f(x) \wedge f(y) = f(x') \wedge f(y')$ and $f(x')(|f(x') \wedge f(y')|) = f(x)(|f(x) \wedge f(y)|)$ and $f(y')(|f(x') \wedge f(y')|) = f(y)(|f(x) \wedge f(y)|)$.

For $s \in T(\leq n+1)$ let $\bar{s} := s$ if $s \in T(\leq n)$ and let $\bar{s} = \dot{s}$ if $s \in T(n+1)$. It follows that if $|x \wedge y| < n$ then $f(x) \wedge f(y) = f(\bar{x}) \wedge f(\bar{y})$. For if x and y are sequences in $T(\leq n)$ then $f(\bar{x}) = f(x)$ and $f(\bar{y}) = f(y)$. If $x \in T(n+1)$ and $y \in T(\leq n)$ then $f(\bar{x}) \wedge f(\bar{y}) = f(\dot{x}) \wedge f(y) = f(x) \wedge f(y)$ according to Claim 1 unless $y = \dot{x}$ in which case $|x \wedge y| = |\dot{x}| = n$. If $x, y \in T(n+1)$ then $f(\bar{x}) \wedge f(\bar{y}) = f(\dot{x}) \wedge f(\dot{y}) = f(x) \wedge f(y)$ according to Claim 1, unless $\dot{x} = \dot{y}$ in which case $x \wedge y = x$ if $x = y$ and if $x \neq y$ then $x \wedge y = \dot{x}$. Hence in any case $|x \wedge y| \geq n$. Note that if $|x \wedge y| < n$ then $x \wedge y = \bar{x} \wedge \bar{y}$.

In order to establish Item 2. of the definition of strong diagonalization let $|x \wedge y| < |z \wedge u|$.

If $|x \wedge y| < n$ and $|z \wedge u| < n$ then $|\bar{x} \wedge \bar{y}| < |\bar{z} \wedge \bar{u}|$ and hence $|f(x) \wedge f(y)| = |f(\bar{x}) \wedge f(\bar{y})| < |f(\bar{z} \wedge \bar{u})| = |f(z) \wedge f(u)|$.

If $z = u \in T(n)$ and $|x \wedge y| < n$ then $|\bar{x} \wedge \bar{y}| = |x \wedge y| < |z|$. Hence $|f(x) \wedge f(y)| = |f(\bar{x}) \wedge f(\bar{y})| < |f(z) \wedge f(z)|$. If $z = u \in T(n+1)$ and $|x \wedge y| < n$ and $x \neq y$ then because $f(x) \wedge f(y) \in g[T]$ it follows from Item **ii**. that $|f(x) \wedge f(y)| < |f(z)|$. If $z = u \in T(n+1)$ and $|x \wedge y| < n$ and $x = y$ then $|f(x)| < |f(y)|$ follows from Item **ii**.

If $|z \wedge u| \geq n$ and $z \neq u$ then either $z \in T(n+1)$ and $u = \dot{z}$ or $u \in T(n+1)$ and $z = \dot{u}$ or $z, u \in T(n+1)$ and $\dot{z} = \dot{u}$. In either case there is $s \in \{z, u\}$ with $s \in T(n+1)$ and $|g(s)| \leq |f(z) \wedge f(u)|$. If $x = y$ with $|x| < n$ then $|f(x)| < |g(s)|$ according to Item **ii**. If $|x \wedge y| < n$ and $x \neq y$ then $|f(x) \wedge f(y)| = |f(\bar{x}) \wedge f(\bar{y})|$. It follows from $|x \wedge y| < n$ that $n \geq 1$ and because $f(\bar{x}) \wedge f(\bar{y}) \in g[T]$ we obtain from Item **ii**. that $|f(\bar{x}) \wedge f(\bar{y})| < l(n-1)$. Item **ii**. implies that $|g(s)| > l(n-1)$ because $s \in T(n+1)$.

If $|x \wedge y| \geq n$ and $|z \wedge u| \geq n$ then $|x \wedge y| = n$ and $|z \wedge u| = n+1$. Hence $z = u \in T(n+1)$. If $x = y \in T(n)$ then $|f(x)| < |f(z)|$ from Item **ii**. If $x \neq y$ then $x = \dot{y}$ or $\dot{x} = y$ or $\dot{x} = \dot{y}$. If $x = \dot{y}$ then $f(x) \wedge f(y) = g(y)$ according to Item **iii**. and we get $|g(y)| < |f(z)|$ from Item **ii**. If $\dot{x} = \dot{y}$ and

$x \prec y$ then $f(x) \wedge f(y) = g(y)$ according to Claim 1.

Let $|x| > |y|$, then $|f(x)| > |f(y)|$. We wish to prove that $x(|y|) = f(x)(|f(y)|)$. In the case that $x \in T[\leq n]$ equality $x(|y|) = f(x)(|f(y)|)$ follows from the fact that f restricted to $T(\leq n)$ is a strong diagonalization. Let $x \in T(n+1)$ and $y \in T(n)$. Then $x(|y|) = f(x)(|f(y)|)$ follows from Item **vi.** Let $x \in T(n+1)$ and $y \in T(< n)$. Then $x(|y|) = \dot{x}(|y|)$ and $f(\dot{x}) \wedge f(x) = g(x)$ from Item **iii.** with $|g(x)| > |f(y)|$ from Item **ii.** Hence $f(x)(|f(y)|) = f(\dot{x})(|f(y)|)$. We get $x(|y|) = \dot{x}(|y|) = f(\dot{x})(|f(y)|) = f(x)(|f(y)|)$.

Let $x \prec y$, then x and y are incomparable under \subseteq . If $x, y \in T(\leq n)$ then $f(x) \prec f(y)$ follows from the fact that f restricted to $T(\leq n)$ is a strong diagonalization.

If $x \in T(n+1)$ and $y \in T(\leq n)$ then we get from Claim 1 and the fact that $\dot{x} \neq y$ that $f(x) \wedge f(y) = f(\dot{x}) \wedge f(y)$. This implies, together with Item **ii.** that $|g(x)| > |f(x) \wedge f(y)|$. Hence $x \prec y$ implies $\dot{x} \prec y$ implies $f(\dot{x}) \prec f(y)$ implies $g(x) \prec f(y)$ implies $f(x) \prec f(y)$. The case $x \in T(\leq n)$ and $y \in T(n+1)$ is dual.

If $x, y \in T(n+1)$ and $\dot{x} \neq \dot{y}$ then $x \prec y$ implies $\dot{x} \prec \dot{y}$ implies $f(\dot{x}) \prec f(\dot{y})$ implies $g(x) \prec g(y)$ implies $f(x) \prec f(y)$. That $f(\dot{x}) \prec f(\dot{y})$ implies $g(x) \prec g(y)$ follows if both $|g(x)|$ and $|g(y)|$ are larger than $|f(\dot{x}) \wedge f(\dot{y})|$. This is the case because $f(x) \wedge f(y) \in g[T(\leq n)]$ and $g[T(n)] < l(n-1) < g[T(n+1)]$ according to Item **ii.** and because $g(x), g(y) \in g[T(n+1)]$.

If $\dot{x} = \dot{y}$ then it follows from $x \prec y$ that $|g(x)| > |g(y)|$ according to Item **iv.** The sequences $g(x)$ and $g(y)$ are predecessors of the sequence $f(\dot{x})$ with $f(x) \wedge f(\dot{x}) = g(x)$ and $f(y) \wedge f(\dot{x}) = g(y)$. Hence $f(x) \wedge f(y) = g(y) = f(\dot{y}) \wedge f(y) = f(\dot{x}) \wedge f(y)$. We have $f(x)(|f(x) \wedge f(y)|) = f(\dot{x})(|f(x) \wedge f(y)|) = f(\dot{y})(|f(x) \wedge f(y)|) = f(\dot{y})(|g(y)|) = 0$ and $f(y)(|f(x) \wedge f(y)|) = f(y)(|g(y)|) = 1$.

End of proof of Claim 3.

Because f restricted to $\{\langle \rangle\}$ is a strong diagonalization, Claim 3 implies that f restricted to $T(n)$ for every $n \in \omega$ is a strong diagonalization, which in turn implies that f is a strong diagonalization. □

5 On Milliken's result

Theorem 5.1 and Theorem 5.2 are due to K. Milliken, see [13]. We prove Theorem 5.2 as a consequence of Theorem 5.1 because we need it stated in

our notation and the translation from the notation in Millikens paper is a bit cumbersome. Also, we think that the proof provided here is preferable.

The set T of sequences is *closed by levels* if for every $t \in T$ and $s \subseteq t$ with $|s| \in \text{levels}(T)$ the sequence s is an element of T .

Let T be a meet closed set of sequences which is also closed by levels and let $n \in \omega + 1$. The set $S \subseteq T$ is an element of $\text{Str}^n(T)$ if:

- $|\text{levels}(S)| = n$.
- S is meet closed and closed by levels.
- For all $s \in S$, the degree of s in S is equal to the degree of s in T , unless $\text{levels}(S)$ has a maximum which is equal to $|s|$.

Note that if $S \in \text{Str}^n(T)$ and $R \in \text{Str}^m(S)$ then $R \in \text{Str}^m(T)$.

Theorem 5.1 (Milliken). *Let $m, n \in \omega$ and T be an ω -tree. If*

$$\text{Str}^n(T) = \bigcup_{i \in m} C_i$$

then there is $k \in m$ and

$$S \in \text{Str}^\omega(T)$$

with

$$\text{Str}^n(S) \subseteq C_k.$$

The elements of $\bigcup_{n \in \omega+1} \text{Str}^n(T)$ are the *strong subsets* of T .

Let S be a meet closed finite subset of an ω -tree T with $n = |\text{levels}(S)|$. A *cover* C of S in T is an element of $\text{Str}^n(T)$ so that $S \subseteq C$ and $\text{levels}(S) = \text{levels}(C)$. Note that if C is a cover of S then $\nu_S[S] \subseteq \nu_C[C]$.

Lemma 5.1. *Let $S \stackrel{s}{\sim} R$ be two meet closed subsets of an ω -tree T . If there exists a subset C of T which is a cover of S and a cover of R then $S = R$.*

Proof. We have $\overrightarrow{\text{levels}}(S) = \overrightarrow{\text{levels}}(C) = \overrightarrow{\text{levels}}(R) := l$ and $\nu_S[S] = \nu_R[R] \subseteq \nu_C[C]$ from Item 14. of Lemma 3.1. It suffices to show that if $s \in S$ and $r \in R$ with $s \circ l = r \circ l$ then $s = r$. This is the case according to Lemma 3.1 Item 9. because C is meet closed. \square

Lemma 5.2. *Let T be an ω -tree and S a finite meet closed subset of T . Then there exists a cover C of S in T .*

Proof. Let R be a meet closed subset of T and $\overrightarrow{\text{levels}}(R) = \langle l_0, l_1, \dots, l_{n-1} \rangle$ for some $n \in \omega$ and let $r \in R$ with $|r| = l_k$ for some $k < n - 1$. Let $N \subset \omega$ so that for every $i \in n$ there is $s \in R$ with $\langle r; i \rangle \subseteq s$. Let $j \in \omega \setminus N$ and $a \in T$ so that $|a| = l_{k+1}$ and $\langle r; j \rangle \subseteq a$. Then $R \cup \{a\}$ is meet closed and $\overrightarrow{\text{levels}}(R) = \overrightarrow{\text{levels}}(R \cup \{a\})$.

We say that $R \cup \{a\}$ is constructed by degree completion. The set R is degree complete if there is no $a \in T$ so that $R \cup a$ can be constructed by degree completion. Note that if R is degree complete then it has the property that for all $r \in R$, the degree of r in R is equal to the degree of r in T , unless $|r| = l_{n-1}$.

Let $r \in R$ and $|r| > i \in \omega$ so that the initial segment a of r with $|a| = l_i$ is not an element of R . Then $R \cup \{a\}$ is meet closed and $\overrightarrow{\text{levels}}(R) = \overrightarrow{\text{levels}}(R \cup \{a\})$. We say that $R \cup \{a\}$ is constructed by level completion. Note that if there is no $a \in T$ so that $R \cup a$ can be constructed by level completion then R is closed by levels.

It follows that if R is degree complete and closed by levels then R is a strong subset of T .

Because R has finitely many levels and the degree of every element of T is finite there is a subset C of T with $S \subseteq C$ and which is constructed from S via a sequence of successive level completions and degree completions and which is closed by levels and degree complete. Because $\overrightarrow{\text{levels}}(C) = \overrightarrow{\text{levels}}(S)$ it follows that $C \in \text{Str}^n(T)$. \square

Theorem 5.2. *Let $F \in \mathfrak{X}_\omega$ be finite and meet closed, $m \in \omega$ and T be an ω -tree. If*

$$\text{Sims}_T(F) = \bigcup_{i \in m} C_i$$

then there is $k \in m$ and

$$S \in \text{Str}^\omega(T)$$

with

$$\text{Sims}_S(F) \subseteq C_k.$$

Proof. Let $n := |\overrightarrow{\text{levels}}(F)|$. According to Lemma 5.2 there is a cover $f(G)$ for every $G \in \text{Sims}_T(F)$ of F and because of Lemma 5.1 the function f is an injection. Let $A = \text{Str}^n(T) \setminus f[\text{Sims}_T(F)]$ and $C_0^* := f[C_0] \cup A$ and $C_i^* = f[C_i]$ for $1 < i < m$. Then $\text{Str}^n(T) = \bigcup_{i \in m} C_i^*$.

According to Milliken's Theorem there is $k \in m$ and $S \in \text{Str}^\omega(T)$ so that $\text{Str}^n(S) \subseteq C_k^*$. Because f is an injection it follows that $\text{Sims}_S(F) \subseteq C_k$. \square

6 The partition result for strong diagonalization's

Lemma 6.1. *Let F, G be two strongly diagonal sets. Then $F \overset{s}{\sim} G$ if and only if $\text{closure}(F) \overset{s}{\sim} \text{closure}(G)$ and $F = G$ if and only if $\text{closure}(F) = \text{closure}(G)$.*

Proof. Let $F \overset{s}{\sim} G$. It follows from Lemma 3.1 Item 3. that $\text{closure}(F) \overset{s}{\sim} \text{closure}(G)$. Let $\text{closure}(F) \overset{s}{\sim} \text{closure}(G)$. Then F is the set of endpoints of $\text{closure}(F)$ and G the set of endpoints of $\text{closure}(G)$. Checking Definition 3.1 we see that $F \overset{s}{\sim} G$.

If $F = G$ then clearly $\text{closure}(F) = \text{closure}(G)$. If $\text{closure}(F) = \text{closure}(G)$ then $F = G$ because F is the set of endpoints of $\text{closure}(F) = \text{closure}(G)$ and so is G . \square

Theorem 6.1. *Let $F \in \mathfrak{T}_\omega$ be strongly diagonal, $m \in \omega$ and T be an ω -tree. If*

$$\text{Sims}_T(F) = \bigcup_{i \in m} C_i$$

then there is $k \in m$ and

$$S \in \text{Str}^\omega(T)$$

with

$$\text{Sims}_S(F) \subseteq C_k.$$

Proof. Follows from Theorem 5.2 and Lemma 6.1. \square

Lemma 6.2. *Let f be a strong diagonalization of the wide ω -tree T and G a subset of $f[T]$ so that $f^{-1}[G]$ is a strongly diagonal subset of T . Then $G \overset{s}{\sim} f^{-1}[G]$.*

Proof. It follows from Corollary 3.1 that $f^{-1}[G] \sim f[f^{-1}[G]] = G$ and from Lemma 3.6 that $G \overset{s}{\sim} f^{-1}[G]$. \square

Corollary 6.1. *Let f be a strong diagonalization of the wide ω -tree T and let F, G be two finite subsets of $f[T]$ with $f^{-1}[F]$ and $f^{-1}[G]$ strongly diagonal. Then $f^{-1}[F] \overset{s}{\sim} f^{-1}[G]$ if and only if $F \overset{s}{\sim} G$.*

Lemma 6.3. *Let T be a wide ω -tree and $S \in \text{Str}^\omega(T)$. Then there exists a strong similarity embedding of T into S .*

Proof. The following CLAIM establishes the Lemma.

CLAIM. *If f is a strong similarity embedding of $T(< n)$ for some $1 \leq n \in \omega$, that is if f is a strong similarity embedding of the union of the first n levels of T into S , then there is an extension f' of f to a strong similarity embedding of $T(\leq n)$ into S .*

Proof of CLAIM. Note that $|f(x)| \geq |x|$ for every $x \in T(< n)$ and hence because T is a wide ω -tree the degree of $f(x)$ in T is larger than or equal to the degree of x in T . Let $x \in T(n-1)$ with k_x equal to the degree of x . Then $\{\langle f(x); i \rangle \mid i \in k_x\}$ is a subset of the set of all immediate successors of $f(x)$.

For every $i \in k_x$ there is a unique element $y_{i,x} \in S$ so that $\langle f(x); i \rangle \subseteq y_{i,x}$ and if $z \in S$ with $\langle f(x); i \rangle \subseteq z$ then $y_{i,x} \subseteq z$. It follows from the definition of $\text{Str}^\omega(T)$ and the fact that the strong similarity f maps all elements of L_{n-1} into the same level of S and hence into the same level of T , that $|y_{i,x}| = |y_{j,z}|$ for all $i \in k_x, j \in k_z$ and $x, z \in T(n-1)$.

Let f' be the extension of f to R' with $f'(\langle x; i \rangle) = y_{i,x}$ for all $x \in T(n-1)$ and $i \in k_x$. Then f' is a strong similarity embedding of $T(\leq n)$ into S .

End proof of CLAIM. □

Theorem 6.2. *Let T be a wide ω -tree, let f be a strong diagonalization of T , let A be a finite subset of $f[T]$ and $C_0 \cup C_1 \cup \dots \cup C_{m-1} = \text{Sims}_{f[T]}(A)$ be a partition of $\text{Sims}_{f[T]}(A)$. Then there is $k \in m$ and a strong diagonalization g of T with $g[f[T]] \subseteq f[T]$ so that*

$$\text{Sims}_{g \circ f[T]}(A) \subseteq C_k.$$

Proof. Let M be the set of elements F in $\text{Sims}_{f[T]}(A)$ so that $f^{-1}[F]$ is strongly diagonal. For $i \in m-1$ let $D_i := \{f^{-1}[C] \mid C \in C_i \cap M\}$ and put $D_{m-1} := \{f^{-1}[C] \mid C \in C_{m-1} \cap M\} \cup (\text{Sims}_T(A) \setminus \{f^{-1}[C] \mid C \in M\})$. Then D_0, D_1, \dots, D_{m-1} is a partition of $\text{Sims}_T(A)$.

Using Theorem 6.1 there is $S \in \text{Str}^\omega(T)$ and $k \in m$ with $\text{Sims}_S(A) \subseteq D_k$. According to Lemma 6.3 there exists a strong similarity embedding h of T into S . Let $V = f[S]$. It follows from Lemma 6.2 and the definition of V that if $F \in M$ and $F \subseteq V$ then $F \in C_k$.

Let $g := f \circ h$ then g is a strong diagonalization of T according to Lemma 3.8 and maps $f[T]$ into $f[T]$ and $g \circ f[T] = f \circ h \circ f[T] := W \subseteq V$.

The function $h \circ f$ is a strong diagonalization of T according to Lemma 3.8. Hence if $F \in \text{Sims}_W(A)$ then $f^{-1}[F] \in h \circ f[T]$ is strongly diagonal and it follows that $F \in M$ which in turn implies that $F \in C_k$. \square

Theorem 6.3. *Let D be a cofinal subset of the wide ω -tree T and A a finite strongly diagonal subset of D . Let $C_0 \cup C_1 \cup \dots \cup C_{m-1} = \text{Sims}_D(A)$ be a partition of $\text{Sims}_D(A)$. Then there is $k \in m$ and a strong diagonalization h of T into D so that*

$$\text{Sims}_{h[D]}(A) \subseteq C_k$$

Proof. Let f be a strong diagonalization of T into D and for $i \in m$ let $C'_i = \{F \in C_i \mid F \subseteq f[T]\}$. It follows that $C'_0 \cup C'_1 \cup \dots \cup C'_{m-1} = \text{Sims}_{f[T]}(A)$ is a partition of $\text{Sims}_{f[T]}(A)$. According to Theorem 6.2 there exists a number $k \in m$ and a strong diagonalization g of T with $g[f[T]] \subseteq f[T] \subseteq D$ so that $\text{Sims}_{g \circ f[T]}(A) \subseteq C_k$. Let $h = g \circ f$. \square

Theorem 6.4. *Let D be a cofinal subset of the wide ω -tree T and A a finite diagonal subset of D . Let $C_0 \cup C_1 \cup \dots \cup C_{m-1} = \text{Sim}_D(A)$ be a partition of $\text{Sim}_D(A)$. Then there is $k \in m$ and a strong diagonalization h of T so that*

$$\text{Sim}_{h[D]}(A) \subseteq C_k$$

Proof. Let f be a strong diagonalization of T into D . It follows from Corollary 3.1 that $A' = f[A]$ is strongly diagonal with $A' \sim A$. Then $\text{Sim}_D(A) = \text{Sim}_D(A')$.

For $i \in m$ let $C'_i = \{F \in C_i \mid F \subseteq f[T]\}$. It follows that $C'_0 \cup C'_1 \cup \dots \cup C'_{m-1} = \text{Sim}_{f[T]}(A')$ is a partition of $\text{Sim}_{f[T]}(A')$. It follows from the fact that $f[T]$ is strongly diagonal and from Lemma 3.6 that $\text{Sims}_{f[T]}(A') = \text{Sim}_{f[T]}(A')$. According to Theorem 6.2 there exists a number $k \in m$ and a strong diagonalization g of T with $g[f[T]] \subseteq f[T] \subseteq D$ so that $\text{Sims}_{g \circ f[T]}(A') \subseteq C_k$. Let $h = g \circ f$. \square

7 Proof of Theorem 2.1

Theorem 2.1 *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and F a finite diagonal subset of U . Let C_0, C_1, \dots, C_{m-1} be a partition of $\text{Sim}_U(F)$ into equivalence classes.*

Then there exists $k \in m$ and a diagonalization f of \mathbb{U} into \mathbb{U} so that $\text{Sim}_{f[U]}(F) \subseteq C_k$.

Proof. Let \mathbf{F} be a universal constraint set and $\mathbb{U} = \mathbb{U}_{\mathbf{F}}$ the universal homogeneous structures satisfying the constraints \mathbf{F} . Let $|\mathbf{F}| = k \in \omega$ and λ a bijection of \mathbf{F} to k and let $(u_i; i \in \omega)$ be an enumeration of the elements of U . Let T be the regular ω -tree of degree k and \mathbb{F} the relational structure with base set T so that the function σ described in Section 2 is an isomorphic embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{T}_{\mathbf{F}}$.

Then $\sigma[U]$ is a cofinal subset of T . We see this as follows:

Let $s = \langle s_0, s_1, \dots, s_{n-1} \rangle \in T$. Let x be an element not in U and $\mathbb{A} = (\{u_i \mid i \in n\} \cup \{x\}; \mathfrak{L})$ be a relational structure in language \mathfrak{L} and base set $\{u_i \mid i \in n\} \cup \{x\}$ so that \mathbb{A} restricted to $\{u_i \mid i \in n\}$ is equal to \mathbb{U} restricted to $\{u_i \mid i \in n\}$ and so that $\lambda(\mathbb{F}) = s_i$ where $\mathbb{F} \in \mathbf{F}$ is isomorphic the restriction of \mathbb{A} to $\{u_i, x\}$. Then \mathbb{A} is an element of the age of $\mathbb{U}_{\mathbf{F}}$.

We obtain, from the mapping extension property of $\mathbb{U}_{\mathbf{F}}$, an embedding f of \mathbb{A} into $\mathbb{U}_{\mathbf{F}}$ which is the identity on the set $\{u_i \mid i \in n\}$. Let $f(x) = u_l$. Note that $l \geq n$ because f is an injection. It follows that s is a predecessor of $\sigma(f(x)) = \sigma(u_l) \in \sigma[U]$ and hence that $\sigma[U]$ is cofinal in T .

Theorem 2.1 follows now from Theorem 6.4. □

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