Canonical partitions of universal structures

C.Laflamme[*]
University of Calgary
Department of Mathematics and Statistics
2500 University Dr. NW. Calgary Alberta Canada T2N1N4
laf@math.ucalgary.ca

N. W. Sauer[†]
University of Calgary
Department of Mathematics and Statistics
2500 University Dr. NW. Calgary Alberta Canada T2N1N4
nsauer@math.ucalgary.ca

V.Vuksanovic
University of Calgary
Department of Mathematics and Statistics
2500 University Dr. NW. Calgary Alberta Canada T2N1N4
vojkan@math.ucalgary.ca

Abstract

Let $\mathbb{U} = (U, \mathfrak{L})$ be a universal binary countable homogeneous structure and $n \in \omega$. We determine the equivalence relation $\mathcal{C}(n)(\mathbb{U})$ on $[U]^n$ with the smallest number of equivalence classes r so that each one of the classes is indivisible. As a consequence we obtain

$$\mathbb{U} \to (\mathbb{U})^n_{<\omega/r}$$

and a characterization of the smallest number r so that the arrow relation above holds.

For the case of infinitely many colors we determine the canonical set of equivalence relations, extending the result of Erdős and Rado for the integers to countable universal binary homogeneous structures.

---

[*] Supported by NSERC of Canada Grant # 690404

[†] Supported by NSERC of Canada Grant # 691325

1 Introduction

The Rado Graph $\mathbb{R} = (R; E)$ is the countable universal homogeneous graph. It is a countable graph with the defining property that for every finite set $F \subset R$ of vertices of the Rado graph and partition of F into the classes A and B there is a vertex x of the Rado graph which is adjacent to all vertices in A and not adjacent to any of the vertices in B. The injection $f : R \mapsto R$ is an *embedding* of the Rado graph if x adjacent to y if and only if $f(x)$ adjacent to $f(y)$ for all vertices $x, y \in R$. The image of an embedding of R is a *copy* of $\mathbb{R}$.

Let $n \in \omega$. In the present paper we will investigate the lattice of partitions of $[R]^n$ in relation to the graph structure of $\mathbb{R}$. We will show that there is a finite set of equivalence relations on $[R]^n$, the "canonical equivalence relations", which form a "basis" for the relationship between the graph structure of $\mathbb{R}$ and the full partition lattice on $[R]^n$. Those canonical equivalence relations are determined by finitely many different types of finite substructures of the Rado graph expanded by an ω-enumeration of R.

There is not much more effort needed to prove and state our results in the context of universal binary countable homogeneous structures. The Rado graph and the order structure of the rationals are two prominent examples of such homogeneous structures. In the case of the rationals the canonical partitions have been determined in [1]. In the general case of universal binary countable homogeneous structures the canonical partitions for the lattice of partitions with finitely many classes have been found in [2].

We will use the notation $\mathbb{A} = (A; \mathfrak{L})$ to indicate that $\mathbb{A}$ is a relational structure of type $\mathfrak{L}$ with base set A. That is, the set $\mathfrak{L}$ is the set of relation symbols of $\mathbb{A}$. Unless otherwise explicitly stated, we will always assume that the set $\mathfrak{L}$ of relation symbols is finite and that every relationsymbol in $\mathfrak{L}$ is binary. We provide some limited discussion of the case that $\mathfrak{L}$ is infinite.

Note: Except for the very first example below we will only consider relational structures in a binary language $\mathfrak{L}$ for which $R(x, y)$ implies $x \neq y$ for all relationsymbols $R \in \mathfrak{L}$.

For a limited introduction to countable homogenous structures see Section 7 and for a more in depths introduction see the appendix in [5]. We restrict our attention to *countable universal homogeneous structures over a binary language*. To motivate the definition below, let us consider the following example, in which the relational language $\mathfrak{L}$ contains only one relation symbol E.

Let $\mathbb{F}_0$ be the relational structure in the language $\mathfrak{L}$ which contains exactly one element, say a, and for which $E(a, a)$ holds. Let $\mathbb{F}_1$ be the relational structure in the language $\mathfrak{L}$ which contains exactly two different elements, say a and b, and for which $E(a,b)$ but not $E(b,a)$ holds. Let $\mathbb{F}_2$ be the relational structure in the language $\mathfrak{L}$ which contains exactly two different elements, say a and b, and for which $E(a,b)$ and $E(b,a)$ holds.

Then the set of all finite binary relational structures with relation symbol E is an age, see [5] or Section 7. This age is the age of the universal relational structure with one binary relation symbol. If we restrict the age of all relational structures in the language $\mathfrak{L}$ to all those finite relational structures in the language $\mathfrak{L}$ which do not embed the structur $\mathfrak{L}_0$ then we obtain again an age. This is the age of the universal countable directed graph.

If we restrict the age to all those finite relational structures in the language $\mathfrak{L}$ which do not embed the structures $\mathbb{L}_0$ and $\mathbb{F}_2$ we obtain the age of the universal countable oriented graph. If we restrict the age to all those finite relational structures in the language $\mathfrak{L}$ which do not embed the structures $\mathbb{L}_0$ and $\mathbb{F}_1$ we obtain the age of the Rado graph. (Every edge is bi-directed, that is not directed.)

For the definition of binary countable universal homogeneous structure let $\mathfrak{L}$ be a finite set of binary relation symbols. Let $\mathbf{F}$ be a set of relational structures in the language $\mathfrak{L}$ with set of elements $\{0,1\}$ and with the property that if $\mathbb{A}$ and $\mathbb{B}$ are two isomorphic relational structures in the language $\mathfrak{L}$ and set of elements $\{0,1\}$ then either both are in $\mathbf{F}$ or neither one of the two is in $\mathbf{F}$. Such a set $\mathbf{F}$ is a *universal constraint set.*

The age of the countable universal homogeneous structure $\mathbb{U}_{\mathbf{F}}$ consists of all finite relational structures in the language $\mathfrak{L}$ for which every induced two element substructure is isomorphic to one of the structures in $\mathbf{F}$ and which do not contain loops. That is elements a with $R(a,a)$ for $R \in \mathfrak{L}$. For example, in the case of the universal directed graph, the set $\mathbf{F}$ consists of two elements, the edge from 0 to 1 and the edge from 1 to 0.

Every countable universal homogeneous structure $\mathbb{U}_{\mathbf{F}}$ has a representation as a set of sequences. Let $|\mathbf{F}| := k \in \omega$ and label the elements of $\mathbf{F}$ with the numbers $0,1,\dots,k-1$. Enumerate the elements of $\mathbb{U}_{\mathbf{F}}$ into an ω-sequence $v_0, v_1, v_2, \dots$. We assign to the element v_n for $n \in \omega$ the sequence $\sigma(v_n) := \langle s_0, s_1, \dots, s_{n-1}\rangle$ where s_i is the label of the element in $\mathbf{F}$ for which the function mapping 0 to v_i and 1 to v_n is an isomorphism. Note that the sequence associated with the element v_0 is the empty sequence $\langle\,\rangle$.

Conversely, let T be a set of finite sequences of different lenghts and with entries the numbers from 0 to $k-1$. Let $s,t \in T$ and the length of s equal to l and shorter than the length of $t := \langle t_0, t_1, \dots t_r\rangle$. Let $\mathbb{A} \in \mathbf{F}$ be the relational structure having label t_l and let $E \in \mathfrak{L}$. Then $E(s,t)$ if and only if $E(0,1)$ holds in $\mathbb{A}$. We obtain a relational structure with language $\mathfrak{L}$ and base set T. Note that this assignment of sequences to the elements of $\mathbb{U}_{\mathbf{F}}$ is an isomorphism of $\mathbb{U}_{\mathbf{F}}$ to the set of sequences such obtained and endowed with the relations as described.

Hence if T is the tree of all finite sequences with entries in k and we stipulate that between two sequences of the same length no relation holds, then T is endowed with a relational structure of type $\mathfrak{L}$. It is not difficult to see that every cofinal subset of T in which no two elements have the same length forms an isomorphic copy of $\mathbb{U}_{\mathbf{F}}$ inside of T.

We will use this relationship between universal homogeneous structures and trees and known partition results on trees to establish partition results for universal structures. In particular the results are of the following nature:

Let $v_0, v_1, v_2, \dots$ be an enumeration of $\mathbb{U}_{\mathbf{F}}$. Let $n \in \omega$ and F an n-element subset of the elements of $\mathbb{U}_{\mathbf{F}}$. Let S be the set of sequences corresponding to set of elements of F. Then S is a subset of the tree T of all sequences with entries in k. We will define, see Definition 3.3, a "similarity" between subsets of T. Two subsets of T being similar if their meet closures "look alike". This equivalence relation between meet closed subsets of T relates backwords to an equivalence relation between subsets of $\mathbb{U}_{\mathbf{F}}$, defining a partition of the n-element subsets of $\mathbb{U}_{\mathbf{F}}$ into similarity classes.

Some of the similarity classes have a representative in every "copy" of $\mathbb{U}_{\mathbf{F}}$ while others don't. We will use the notion of "strong similarity", a finer partition than similarity, as a technical device in the proofs.

It is proven in [2] that each of the similarity classes of the n-element substructures is "indivisible" and that canonical partitions exist. The minimality of those partitions is not established in [2]. That is, the minimality part of the proof that the partition is indeed canonical is missing. In the present paper we will use the results in [2] to show that the partitions defined there are canonical. Then we will use techniques from [1] to generalize the result for partitions into finitely many classes to partitions into infinitely many classes.

See [1] and [2] for a more extensive discussion of the existing literature. One interesting result should be mentioned here. Erdős and Rado determined in [3] the canonical partitions for $[\mathbb{N}]^n$.

We will use the notation and the results from [2]. For completeness some of those definitions and results will be restated. The techniques developed in [1], to deal with partitions of η with infinitely many parts, can also be used in the case of countable binary homogeneous structures and, except for a change of notation, are just transfered from [1].

2 Canonical partitions

Let $\mathbb{A} = (A; \mathfrak{L})$ be a relational structure. A *copy* $\mathbb{A}^*$ of $\mathbb{A}$ in $\mathbb{A}$ is an induced substructure of $\mathbb{A}$ which is isomorphic to $\mathbb{A}$. More general, if $\mathbb{B} = (B; \mathfrak{L})$ is a relational structure then a copy $\mathbb{A}^*$ of $\mathbb{A}$ in $\mathbb{B}$ is an induced substructure of $\mathbb{B}$ which is isomorphic to $\mathbb{A}$.

Let Q be a set of finite subsets of $[A]^n$. The set Q is *indivisible* if for every partition $C_0, C_1, \dots, C_{m-1}$ of Q into $m \in \omega$ subsets there exists a copy $\mathbb{A}^* = (A^*; \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ so that all of the subsets of A^* which are in Q are in C_k for some $k \in m$.

A set $Q \subseteq [A]^n$ of n-element subsets of A is *persistent* if for every copy $A^* = (A^*; \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ we have $[A^*]^n \cap Q \neq \emptyset$.

Definition 2.1. Let $\mathbb{A} = (A; \mathfrak{L})$ be a relational structure. A *canonical equivalence relation* of $[A]^n$ is an equivalence relation on $[A]^n$ with finitely many equivalence classes so that each of the equivalence classes is persistent and indivisible. The set of equivalence classes of a canonical equivalence relation form a *canonical partition*.

It follows that if $\mathcal{C}(n)$ is a canonical equivalence relation on $[A]^n$ and $\mathcal{E}$ is any equivalence relation on $[A]^n$ with finitely many equivalence classes, then there is a copy $\mathbb{A}^* = (A^*, \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ so that $\mathcal{C}(n)$ restricted to $[A^*]^n$ is a subset of $\mathcal{E}$ restricted to $[A^*]^n$. Hence in particular, any two canonical equivalence relations on $[A]^n$ have the same number of elements.

Conversely, if $\mathcal{C}$ is an equivalence relation on $[A]^n$ so that for every equivalence relation $\mathcal{E}$ on $[A]^n$ with finitely many equivalence classes, there is a copy $\mathbb{A}^* = (A^*, \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ so that $\mathcal{C}$ restricted to $[A^*]^n$ is a subset of $\mathcal{E}$ restricted to $[A^*]^n$ and if every equivalence class of $\mathcal{C}$ is persistent then $\mathcal{C}$ is a canonical equivalence relation on $[A]^n$. If $\mathcal{C}$ and $\mathcal{D}$ are two canonical equivalence relations on $[A]^n$ then there is a copy $\mathbb{A}^* = (A^*, \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ so that $\mathcal{C}$ restricted to $[A^*]^n$ is equal to $\mathcal{D}$ restricted to $[A^*]^n$.

It is usually not the case that the zero definable n-element substructures, that is the different n-element embedding types of $\mathbb{A} = (A; \mathfrak{L})$, determine a canonical partition. A set of induced n-element substructures isomorphic to a given n-element substructure being usually not indivisible. Some additional structure on A is necessary to define a finer partition. In all cases so far known, this additional structure is an order induced by an ω-enumeration of A.

It turns out that the situation is somewhat more intricate in the case of equivalences with infinitely many equivalence classes.

Let $\mathbb{A} = (A; \mathfrak{L})$ be a relational structure and $\mathcal{C}$ a canonical equivalence relation on $[A]^n$. Let $\mathfrak{C}$ be the set of equivalence relations $\mathcal{D}$ with $\mathcal{C} \subseteq \mathcal{D}$. It follows that $\mathfrak{C}$ is finite and if $\mathcal{E}$ is any equivalence relation on $[A]^n$ with finitely many equivalence classes, then there is a copy $\mathbb{A}^* = (A^*; \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ and an equivalence relation $\mathcal{D} \in \mathfrak{C}$ so that the restriction of $\mathcal{D}$ to $[A^*]^n$ is equal to the restriction of $\mathcal{E}$ to $[A^*]^n$. We will prove that this property of $\mathfrak{C}$ extends to equivalence relations with infinitely many equivalence classes in the case of universal binary countable homogeneous structures.

Definition 2.2. Let $\mathbb{A} = (A; \mathfrak{L})$ be a relational structure and $n \in \omega$. The finite set $\mathfrak{C}(n)$ of equivalence relations on $[A]^n$ is a *canonical set of equivalence relations of the n-element subsets of A* if for every partition $\mathcal{E}$ of $[A]^n$ there is a copy $\mathbb{A}^* = (A^*, \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ and an element $\mathcal{C} \in \mathfrak{C}(n)$ so that the restriction of $\mathcal{E}$ to $[A^*]^n$ is equal to the restriction of $\mathcal{C}$ to $[A^*]^n$.

Note that the elements of a canonical set of equivalence classes do not have to be canonical equivalence classes.

Definition 2.3. Let $\mathbb{A} = (A; \mathfrak{L})$ be a relational structure and $n \in \omega$. The finite set $\mathfrak{C}(n)$ of equivalence relations on $[A]^n$ is a *basis for the equivalence relations of the n-element subsets of A* if:

1. For every equivalence relation $\mathcal{E}$ of $[A]^n$ and copy $\mathbb{A}^* = (A^*, \mathfrak{L})$ there is a copy $\mathbb{A}^{**} = (A^{**}, \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}^*$ and an element $\mathcal{C} \in \mathfrak{C}(n)$ so that the restriction of $\mathcal{E}$ to $[A^{**}]^n$ is equal to the restriction of $\mathcal{C}$ to $[A^{**}]^n$.
2. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two different elements of $\mathfrak{C}(n)$ and $\mathbb{A}^* = (A^*, \mathfrak{L})$ is a copy of $\mathbb{A}$ in $\mathbb{A}$ then the restriction of $\mathcal{E}_1$ to $[A^*]^n$ is different from the restriction of $\mathcal{E}_2$ to $[A^*]^n$.

Note that every basis for the equivalence relations of the n-element subsets of A is a canonical set of equivalence relations of the n-element subsets of A. It follows easily, see the proof of Lemma 6.1, that any two bases of the equivalence relations on $[A]^n$ have the same number of elements.

3 Sequences

We denote by $\mathfrak{T}_\omega$ the set of all finite sequences with entries s_i in ω. The *length* of the sequence $s = \langle s_0, s_1, \ldots, s_{n-1} \rangle$, $|s|$, is n. We write $t \subset s$ if t is an initial segment of s. The *meet*, $s \wedge t$, of the sequences s and t is the longest sequence which is an initial segment of s and of t. If $S \subseteq \mathfrak{T}_\omega$ then closure(S) is the set S union the set of all meets of elements in S.

Definition 3.1. Let $s, t \in \mathfrak{T}_\omega$ then $x \prec y$ if if and only if x and y are imcomparable under $\subseteq$ and $x(|x \wedge y|) < y(|x \wedge y|)$.

Note that $\prec$ is not a total order.

Let $s, t \in \mathfrak{T}_\omega$. The sequence s is an *immediate successor* of t if $t \subset s$ and $|t| + 1 = |s|$. Let S be a set of sequences. The *degree* of t in S is the number of immediate successors s of t for which there is $r \in S$ with $s \subseteq r$. The sequence $t \in S$ is an *endpoint* of S if the degree of t in S is zero.

The set S of sequences is an *antichain* if $x \subseteq y$ implies $x = y$ for all $x, y \in S$. The set S of sequences is *transversal* if $|x| = |y|$ implies $x = y$ for all $x, y \in S$.

Definition 3.2. The set $F \subseteq \mathfrak{T}_\omega$ of sequences is *diagonal* if it is an antichain and closure(F) is transversal and the degree of every element of closure(F) is at most two. If S is a set of sequences then $\Delta_n(S)$ is the set of diagonal n-element subsets of S. If $N \subseteq \omega$ then $\Delta_N(S) = \bigcup_{n \in N} \Delta_n(S)$.

Definition 3.3. Let $R, S \subseteq \mathfrak{T}_\omega$ be two sets of sequences. The function f of R to S is a *similarity* of R to S if for all $x, y, z, u \in R$:

1. f is a bijection.
2. $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.
3. $|x \wedge y| < |z \wedge u|$ if and only if $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
4. If $|z| > |x|$ then $z(|x|) = f(z)(|f(x)|)$.
5. If $x \prec y$ then $f(x) \prec f(y)$.

The sets R and S of sequences are *similar*, $R \sim S$, if there is a similarity of R to S. Note that if R is diagonal and R and S are similar then S is diagonal. We denote by $\mathrm{Sim}_R(S)$ the set of all subsets of R which are similar to S. The function f of R into $\mathfrak{T}_\omega$ is a *similarity embedding* if f is a similarity of R to $f[R]$. It follows from Item 5. of Definition 3.3 and the fact that $\prec$ is a total order that if $R \sim S$ then there is exactly one similarity f, the *similarity of R to S*.

Note that Item 1. of Definition 3.3 follows from Item 2. and that the composition of similarities is again a similarity and the inverse of a similarity is again a similarity. Hence $\sim$ is an equivalence relation on $\mathfrak{T}_\omega$. The function f extends via $f(x \wedge y) = f(x) \wedge f(y)$ uniquely to a bijection f^* of closure(R) to closure(S). This extension f^* of f is a meet and $\prec$ preserving function.

An equivalence class of $\sim$ whose elements are diagonal is a *diagonal $\sim$-equivalence class.*

Definition 3.4. The infinite set $T \subseteq \mathfrak{T}_\omega$ is an ω-tree if T has no endpoints and every element of T has finite degree and T is closed under initial segments.

Definition 3.5. The ω-tree T is a *wide ω-tree* if for all $x, y \in T$:

1. $|x| < |y|$ implies that the degree of x in T is less than or equal to the degree of y in T.
2. If $i \in k \in \omega$ and $\langle t; k \rangle \in T$ then $\langle t; i \rangle \in T$.

Definition 3.6. The wide ω-tree T is a *regular ω-tree of degree k* if the degree of every element of T is k.

In this paper, all ω-trees will be regular ω-trees of some degree$k \in \omega$.

Lemma 3.1. *Let T be a regular ω-tree of degree k and $n \in \omega$. Then the equivalence relation $\sim$ restricted to the n-element subsets of T has finitely many equivalence classes.*

Proof. Let R be an n-element subset of T. We associate with every pair $x \prec y$ of elements of R the triple of symbols (a, b, c) so that:

1. a is the symbol $\subseteq$ if $x \subseteq y$ and a is the symbol $\not\subseteq$ if $x \not\subseteq y$.
2. b is the symbol $<$ if $|x| < |y|$ and the symbol $=$ if $|x| = |y|$.
3. $c = y(|x|)$ if $x, y \in R$ and $|x| < |y|$ and $c = 0$ otherwise.

Let R and S be two n-element subsets of T. We write $R \equiv S$ if there is a bijection f of R to S so that for every pair of elements $x \prec y$ we have $f(x) \prec f(y)$ and the triple of symbols associated with $x \prec y$ in R is equal to the triple of symbols associated with $f(x) \prec f(y)$ in S.

It follows that $R \sim S$ if and only if $R \equiv S$. There are at most 2^n elements in closure(R) and hence at most finitely many pairs of elements in closure(R). Because R is a subset of a regular ω-tree there are only finitely many such triples of symbols. Hence the equivalence relation $\equiv$ has at most finitely many equivalence classes. $\square$

Definition 3.7. Let $S, T \subseteq \mathfrak{T}_\omega$. The function $f : S \mapsto T$ is a $\mathfrak{d}$-*morphism* if for every diagonal subset F of S the restriction of f to F is a similarity embedding of F into T.

Definition 3.8. Given any regular ω-tree T, a map $f : T \mapsto T$ is a *passing number preserving* (pnp)map if:

1. $|x| < |y|$ implies $|f(x)| < |f(y)|$.
2. If $|x| < |y|$ then $y(|x|) = f(y)(|f(x)|)$.

The number $y(|x|)$ is the *passing number of* y *at* x.

Lemma 3.2. *Let* $S, T \subseteq \mathfrak{T}_\omega$ *and* S *an antichain. Every* $\mathfrak{d}$*-morphism of* S *to* T *is a pnp map.*

Proof. Let $x, y \in S$ with $|x| < |y|$. The set $\{x, y\}$ is diagonal. $\square$

4 Persistence

Let $s \in \mathfrak{T}_\omega$ and $N \in \omega$ then $s{\restriction}N$ is the restriction of s to N, that is the initial segment t of s so that $t(i) = s(i)$ for all $i < \min\{N, |s|\}$. We extend the defintion to $S{\restriction}N := \{s{\restriction}N \mid s \in S \text{ and } |s| \geq N\}$.

Theorem 4.1. *Let* T *be a wide* ω*-tree and* $D \subseteq T$ *be diagonal and* $\phi : T \mapsto T$ *a pnp map. Then there is a similarity embedding of* D *into* $\phi[T]$.

Proof. Fix a pnp map $\phi : T \mapsto T$. For the purpose of this proof, we call a set $L \subseteq \phi[T]$ *large* if $\phi^{-1}[L]$ is cofinal above some $t \in T$.

Given $s \in \phi(T)$ let $\widehat{s} = \{t \in \phi(T) : s \subseteq t\}$.

Lemma 4.1. *Let* $n \in \omega$. *If* $L = \bigcup_{i<n} L_i$ *is large, then* L_i *is large for some* i.

Proof. Let $\phi^{-1}(L)$ be cofinal above $t \in T$. If none of the L_i's are large, then successively choose $t \subseteq s_0 \subseteq s_1 \subseteq \cdots \subseteq s_{n-1}$ such that $\phi^{-1}[L_i] \cap \widehat{s_i} = \emptyset$. But then $\phi^{-1}[L] \cap \widehat{s_{n-1}} = \emptyset$ as well, a contradiction. $\square$

Enumerate closure(D) as $\{d_i : i < \omega\}$ in increasing length, i.e. $i < j$ implies $|d_i| < |d_j|$. We may assume without loss of generality that $\emptyset \notin$ closure(D) and we define $d_{-1} = \emptyset$. This will facilitate the construction.

We shall define recursively, sets of sequences $T_k \subseteq \phi[T]$ and maps f_k and ψ_k such that:

1. f_k is a similarity embedding of $\{d_i : i \in k\} \cup D$ to T_k.
2. There exists $N_k \in \omega$ so that $|t| \leq N_k$ for all $t \in T_k$.
3. $\widehat{t}$ is large for all $t \in T_k{\restriction}N_k$.
4. All maximal nodes of T_k are either in $T_k{\restriction}N_k$, or else in the range of f_k.

5. ψ_k is a $\prec$-preserving bijection of $D\restriction(|d_{k-1}|+1)$ to $T_k\restriction N_k$.

6. $T_{k-1} \subseteq T_k$, $f_{k-1} \subseteq f_k$, and $N_{k-1} < N_k$ if $1 \leq k$.

Let $N_0 = |\phi(\emptyset)|$. Since

$$\bigcup_{v \in \phi[T]\restriction N_0} \widehat{v} \quad \text{is a large set,}$$

one of these $\widehat{v}$ is large by Lemma 4.1. Put $T_0 = \{v\}$ with $v \in \phi[T]\restriction N_0$ so that $\widehat{v}$ is large. Since $d_{-1} \notin \text{closure}(D)$ by assumption, we have $D\restriction(|d_{-1}|+1) = D\restriction(1)$ is a singleton set s. Let $\psi_0(s) = v$, and $f_0 = \emptyset$.

Assume that T_k, ϕ_k, and f_k have been defined as above. We proceed to construct T_{k+1}, ϕ_{k+1}, and f_{k+1} depending on two different cases.

Case I: $d_k \notin D$.
Then d_k is the meet of (at least) two different elements, say d'_0, d'_1, of D. Put $t = \psi_k(d_k\restriction(|d_{k-1}|+1))$. By assumption, the set $\widehat{t}$ is large, so $\phi^{-1}[\widehat{t}]$ is cofinal above some $s \in T$. Let $N_{k+1} := |\phi(s)+1|$. Since ϕ is a pnp map, we must have in particular:

$$(\forall s' \supset s)\ \phi(s')(|\phi(s)|) = s'(|s|).$$

Therefore, for each $i \in \{d'_0(|d_k|), d'_1(|d_k|)\}$:

$$\bigcup_{t' \in \widehat{t}\restriction|\phi(s)|} \widehat{\langle t'; i\rangle} \quad \text{is a large set.}$$

By Lemma 4.1 we can find, for each i a $t'_i \in \widehat{t}\restriction|\phi(s)|$, such that $\widehat{\langle t'_i; i\rangle}$ is large.
Define ψ_{k+1} on $(D \cap \widehat{d_k})\restriction(|d_k|+1)$ (which has size exactly two), as the unique $\prec$-preserving map onto $\{t'_0, t'_1\}$.

For $u \in D\restriction(|d_{k-1}|+1) \setminus \{d_k\restriction(|d_{k-1}|+1)\}$, choose $v_u \in \widehat{\psi_k(u)}\restriction N_{k+1}$ such that $\widehat{v_u}$ is large. This is again possible since

$$\bigcup_{v \in \widehat{\psi_k(u)}\restriction N_{k+1}} \widehat{v} \quad \text{is a large set,}$$

which implies that one of the sets $\widehat{v}$ is large by Lemma 4.1. Every $u \in D\restriction(|d_{k-1}|+1) \setminus \{d_k\restriction(|d_{k-1}|+1)\}$ has a unique extension $u' \in D\restriction(|d_k|+1)$. Let $\psi_{k+1}(u') := v_u$ and $f_{k+1} = f_k$ and let

$$T_{k+1} := T_k \cup \{t'_0, t'_1\} \cup \{v_u \mid u \in D\restriction(|d_{k-1}|+1) \setminus \{d_k\restriction(|d_{k-1}|+1)\}.$$

This completes the construction in this case, and conditions 1-6 are easily verified.

Case II: $d_k \in D$.

By assumption, $\widehat{t}$ is large for each $t \in T_k \restriction N_k$, so $\phi^{-1}[\widehat{t}]$ is cofinal above some $s^t \in T$. Choose $M > \max\{|s^t| : t \in T_k \restriction N_k\}$.

Let $t := \psi_k(d_k \restriction (|d_{k-1}| + 1)) \in T_k \restriction N_k$, and choose $s \supset s^t$ of length M. Let $N_{k+1} = |\phi(s)| + 1$, and extend f_k so that $f_{k+1}(d_k) = \phi(s)$.

It is worth noting that d_k is the only node of $D \restriction |d_k|$ above $d_k \restriction (|d_{k-1}|+1)$; this is so, due to our chosen ordering of the d_i's and the fact that closure(D) is transversal.

For every $u \in D \restriction (|d_{k-1}| + 1) \setminus \{d_k \restriction (|d_{k-1}| + 1)\}$ there is a unique $u' \in D \restriction (|d_k| + 1)$ above u. We have to define $\psi_{k+1}(u')$. For this fix u and let $v = \psi_k(u) \in T_k \restriction N_k$. We claim that

$$S := \bigcup_{\substack{v' \in \widehat{v} \restriction N_{k+1} \\ v'(|\phi(s)|) = u'(|d_k|)}} \widehat{v'} \quad \text{is a large set.}$$

The reason for this is that $\widehat{v}$ is large by assumption. In fact $\phi^{-1}[\widehat{v}]$ is cofinal above s^v defined above. Hence cofinal above each s' extending s^v of length M satisfying $s'(|s|) = u'(|d_k|)$. Since ϕ is pnp it follows that $\phi(s')(|\phi(s)|) = s'(|s|)$ for all such s' and hence S is a large set.

Using Lemma 4.1 again allows us to find such a v'_u so that $\widehat{v'_u}$ is large. We define $\psi_{k+1}(u') := v'_u$ for all $u \in D \restriction (|d_{k-1}| + 1) \setminus \{d_k \restriction (|d_{k-1}| + 1)\}$ and

$$T_{k+1} := T_k \cup \{\phi(s)\} \cup \{v'_u \mid u \in D \restriction (|d_{k-1}| + 1) \setminus \{d_k \restriction (|d_{k-1}| + 1)\}.$$

This completes the construction in Case II.

We claim that $f = \bigcup_k f_k$ is the desired similarity embedding of D to $f[D] \subseteq \phi[T]$.

Condition 2 of Definition 3.3 follows from requirement 5 of our construction and the fact that each $d \in D$ will be mapped above $\psi_k(d \restriction (|d_k| + 1))$ for each k such that $|d_k| < |d|$. Condition 3 follows by construction since if $|x \wedge y| < |z \wedge u|$, then $x \wedge y$ appears as one of d_i's before $|z \wedge u|$. Condition 4 is built in Case II. Finally condition 5 follows from condition 3 since D is diagonal.

□

5 Partitions of sets of sequences

Definition 5.1. The set $F \subseteq \mathfrak{T}_\omega$ of sequences is *strongly diagonal* if it is an antichain and closure(F) is transversal and for all $x, y, z \in F$ with $x \neq y$:

1. $|x \wedge y| < |z|$ and $x \wedge y \not\subset z$ implies $z(|x \wedge y|) = 0$.
2. $x(|x \wedge y|) \in \{0, 1\}$.

It follows that every subset of a strongly diagonal set is strongly diagonal. Note that Item 2. of Definition 5.1 implies that the degree of every element of closure(F) is at most two and hence that every strongly diagonal set is diagonal.

Definition 5.2. Let $R, S \subseteq \mathfrak{T}_\omega$ be two sets of sequences. The function f of R to S is a *strong similarity* of R to S if for all $x, y, z, u \in R$:

1. f is a bijection.
2. $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.
3. $|x \wedge y| < |z \wedge u|$ if and only if $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
4. If $|z| > |x \wedge y|$ then $z(|x \wedge y)| = f(z)(|f(x) \wedge f(y)|)$.

If F is a subset of a set R of sequences then $\mathrm{Sims}_R(F)$ is the set of all subsets of R which are strongly similar to F.

Every strong similarity is a similarity and every strong similarity of the set R of sequences has a unique extension to a strong similarity of closure(R). The notion of strong similarity will mainly be applied to sets of sequences which are antichains. That is for sets R of sequences for which the meet of two of it's elements is not an element of R.

Definition 5.3. Let S and T be two subsets of $\mathfrak{T}_\omega$. The injection f of S to T is a *strong diagonalization* of S to T if for all $x, y, z, u \in S$:

1. The set of sequences $f[S]$ is strongly diagonal.
2. $|x \wedge y| < |z \wedge u|$ implies $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
3. If $|x| > |y|$ then $x(|y|) = f(x)(|f(y)|)$.
4. If $x \prec y$ then $f(x) \prec f(y)$.

Note that every strong diagonalization is a pnp map.

The following Lemma 5.1 is Lemma 3.6 of [2] and the following Lemma 5.2 is Lemma 3.7. of [2] and the following Theorem 5.1 is Theorem 4.1 of [2] and the following Theorem 5.2 is Theorem 6.2 of [2].

Lemma 5.1. *If f is a similarity of the strongly diagonal set F to the strongly diagonal set G then f is a strong similarity.*

Lemma 5.2. *Every strong diagonalization is a $\mathfrak{d}$-morphism.*

Theorem 5.1. *Let T be a regular ω-tree and D a cofinal subset of T. Then there exists a strong diagonalization f of T into D.*

Theorem 5.2. *Let T be a regular ω-tree, let f be a strong diagonalization of T, let A be a finite subset of $f[T]$ and $C_0 \cup C_1 \cup \cdots \cup C_{m-1} = \mathrm{Sims}_{f[T]}(A)$ be a partition of $\mathrm{Sims}_{f[T]}(A)$. Then there is $k \in m$ and a strong diagonalization g of T with $g[f[T]] \subseteq f[T]$ so that*

$$\mathrm{Sims}_{g\circ f[T]}(A) \subseteq C_k.$$

Corollary 5.1. *Let T be a regular ω-tree, let h be a strong diagonalization of T, let A be a finite diagonal subset of T and $C_0 \cup C_1 \cup \cdots \cup C_{m-1} = \mathrm{Sim}_{h[T]}(A)$ be a partition of $\mathrm{Sim}_{h[T]}(A)$. Then there is $k \in m$ and a strong diagonalization r of T with $r[h[T]] \subseteq h[T]$ so that*

$$\mathrm{Sim}_{r\circ h[T]}(A) \subseteq C_k.$$

Proof. It follows from Lemma 5.1 that $\mathrm{Sims}_{h[T]}(h[A]) = \mathrm{Sim}_{h[T]}(A)$ and $\mathrm{Sims}_{g\circ h[T]}(h[A]) = \mathrm{Sim}_{g\circ h[T]}(A)$. $\square$

Theorem 5.3. *Let T be a regular ω-tree and S a cofinal subset of T and f a pnp map of S into T with $R = f[S]$. Let A be a finite diagonal subset of R and $C_0 \cup C_1 \cup \cdots \cup C_{m-1} = \mathrm{Sim}_R(A)$ be a partition of $\mathrm{Sim}_R(A)$. Then there is $k \in m$ and a $\mathfrak{d}$-morphism and pnp map g of T with $g[T] \subseteq R$ so that*

$$\mathrm{Sim}_{g[T]}(A) \subseteq C_k.$$

If f is the identity on S then g can be taken to be a strong diagonalization.

Proof. Let $n \in \omega$ be the number of elements of A.

According to Theorem 5.1 there exists a strong diagonalization h of T into S. Then $D := h[T]$ is diagonal and $\phi := f \circ h : T \mapsto T$ is a pnp map. Hence, according to Theorem 4.1, there is a similarity embedding l of D into $\phi[T]$. Then $l[D] \subseteq f[S] = R$ and hence $(C_i \cap [l[D]]^n; i \in m)$ is a partition of $\mathrm{Sim}_{l[D]}(A)$.

If $A' \in \mathrm{Sim}_{h[T]}(A)$ then $l[A'] \in \mathrm{Sim}_{l[D]}(A)$. Let $C'_i := \{A' \in \mathrm{Sim}_{h[T]}(A) \mid l[A'] \in C_i\}$. It follows that $(C'_i; i \in m)$ is a partition of $\mathrm{Sim}_{h[T](A)}$.

According to Corollary 5.1 there exists $k \in m$ and a strong diagonalization r of T with $r[h[T]] \subseteq h[T]$ so that $\mathrm{Sim}_{r \circ h[T]}(A) \subseteq C'_k$. Then $\mathrm{Sim}_{l \circ r \circ h[T]}(A) \subseteq C_k$. It follows from Lemma 5.2 that $l \circ r \circ h$ is a similarity embedding and from Lemma 3.2 that $l \circ r \circ h$ is a pnp map. Let $g := l \circ r \circ h$.

If f is the identity on S then l can be taken to be the identity on $h[S]$. $\square$

Corollary 5.2. *Let T be a regular ω-tree and S a cofinal subset of T. Let A be a finite diagonal subset of S and $C_0 \cup C_1 \cup \cdots \cup C_{m-1} = \mathrm{Sim}_S(A)$ be a partition of $\mathrm{Sim}_S(A)$. Then there is $k \in m$ and a strong diagonalization g of T with $g[T] \subseteq S$ so that*

$$\mathrm{Sim}_{g[T]}(A) \subseteq C_k.$$

Corollary 5.3. *Let T be a regular ω-tree and S a cofinal subset of T and f a pnp map of S into T with $R = f[S]$. Let $N \subseteq \omega$ be finite and $l \in \omega$ and K_i for every $i \in l$ a finite set. For all $i \in l$ let $f_i : \Delta_N(R) \mapsto K_i$ be a function.*

Then there exists a $\mathfrak{d}$-morphism $h : T \mapsto R$ so that for every $i \in l$ and $n \in N$ and $\sim$ equivalence class P of $\Delta_n(h[T])$ the function f_i restricted to P is constant.

Proof. Let $n \in \omega$ and $N = \{n\}$ and $l = 1$ and $A \in \Delta_n(T)$. The function f_0 induces a partition of $\mathrm{Sim}_R(A)$ into finitely many classes. It follows from Theorem 5.3 that there is a $\mathfrak{d}$-morphism h of T with $g[T] \subseteq R$ so that the function f_0 is constant on the $\sim$ equivalence class $\mathrm{Sim}_{h[T]}(A)$.

The Theorem follows by repeated application of the above argument because T is a regular ω-tree and hence the number of $\sim$ equivalence classes of $\Delta_n(R)$ is finite. $\square$

6 Equivalences with infinitely many classes

Definition 6.1. Let $S \subseteq \mathfrak{T}_\omega$ and $n \in \omega$. The countable set $\mathfrak{E}$ of equivalence relations on $[S]^n$ is a *basis for the equivalence relations on* $[S]^n$ if:

1. For every pnp copy R of S in S and every equivalence relation Q on $[R]^n$ there exists a $\mathfrak{d}$-similarity h of T into R and an equivalence relation $\mathrm{E} \in \mathfrak{E}$ so that

$$\mathrm{E} \cap [[h[T]]^n]^2 = \mathrm{Q} \cap [[h[T]]^n]^2.$$

2. If E_1 and E_2 are two different elements of $\mathfrak{E}$ and R is a pnp copy of S in S then
$$\mathrm{E}_1 \cap [[R]^n]^2 \neq \mathrm{E}_2 \cap [[R]^n]^2.$$

Lemma 6.1. *Let $S \subseteq \mathfrak{T}_\omega$ and $n \in \omega$. If $\mathfrak{E}$ and $\mathfrak{F}$ are two bases for the equivalence relations on $[S]^n$ then $|\mathfrak{E}| = |\mathfrak{F}|$.*

Proof. We will prove that there exists a bijection of $\mathfrak{E}$ to $\mathfrak{F}$.

Let $l \in \omega$ and R a pnp copy of S in S and $\mathfrak{E}_l := \{\mathrm{E}_i \in \mathfrak{E} \mid i \in l\}$ and $\mathfrak{F}_l = \{\mathrm{F}_i \in \mathfrak{F} \mid i \in l\}$ sets of equivalence relations so that

$$\mathrm{E}_\mathrm{i} \cap [[R]^n]^2 = \mathrm{F}_\mathrm{i} \cap [[R]^n]^2$$

for all $i \in l$. It follows from Item 2. of Definition 6.1 that $\mathrm{E}_i \neq \mathrm{E}_j$ and $\mathrm{F}_i \neq \mathrm{F}_j$ for all $i, j \in l$ with $i \neq j$. This implies that the function f_l of $\mathfrak{E}_l$ to $\mathfrak{F}_l$ which maps E_i to F_i is a bijection.

If $\mathfrak{E}_l \neq \mathfrak{E}$ or $\mathfrak{F}_l \neq \mathfrak{F}$ then there are, according to Item 1. of Definition 6.1, elements $\mathrm{E}_l \in \mathfrak{E}$ and $\mathrm{F}_l \in \mathfrak{F}$ and there is a copy R_0 of S in R so that

$$\mathrm{E}_\mathrm{l} \cap [[R_0]^n]^2 = \mathrm{F}_\mathrm{l} \cap [[R_0]^n]^2,$$

extending the bijection f_l to a bijection f_{l+1}. □

Definition 6.2. Let $S \subseteq \mathfrak{T}_\omega$. Then

$$\Theta_n(S) := \{(A, B) \in [[S]^n]^2 \mid A \cup B \text{ is diagonal } \}.$$

Definition 6.3. Let $A, B, C, D \in [\mathfrak{T}]^n$. We write $A : B = C : D$ if $(A, B) = (C, D)$ or if $(A, B), (C, D) \in \Theta_n(\mathfrak{T}_\omega)$ and $A \cup B \sim C \cup D$ and $f[A] = C$ and $f[B] = D$, where f is the similarity of $A \cup B$ to $C \cup D$. We write $A : B \simeq C : D$ for $A : B = C : D$ or $A : B = D : C$.

Let $A \cup B$ and $C \cup D$ be diagonal. It follows that then $A : B \simeq C : D$ if and only if $A \cup B \sim C \cup D$ and $\{f[A], f[B]\} = \{C, D\}$ where f is the similarity of $A \cup B$ to $C \cup D$. If one of $A \cup B$ or $C \cup D$ is not diagonal then $A : B \simeq C : D$ if and only if $(A, B) = (C, D)$. The relation $\simeq$ is an equivalence relation on $[[\mathfrak{T}_\omega]^n]^2$. We will also write $(A, B) \simeq (C, D)$ for $A : B \simeq C : D$.

Let $n \in \omega$ an d $\Lambda', \Lambda'' \subseteq [[\mathfrak{T}]^n]2$. The bijection $h : \Lambda' \mapsto \Lambda''$ is an *equivalence* if $(A, B) \simeq h(A, B)$ for all (A, B) in Λ'. If there is an equivalence of Λ' to Λ'' then Λ' and Λ'' are *equivalent.*

Definition 6.4. Let $S \subseteq \mathfrak{T}_\omega$ and $n \in \omega$ and $\Lambda \subseteq \Theta_n(S)$. The set Λ is a *saturated set*, more precisely *n-saturated for S*, if for every element $(C, D) \in \Theta_n(S)$ there is a pair $(A, B) \in \Lambda$ such that $A : B \simeq C : D$.

It follows that if the n-saturated set Λ for S is equivalent to the set $\Lambda' \subseteq [[S]^n]^2$ then Λ' is n-saturated for S.

Lemma 6.2. *Let T be a regular ω-tree and $n \in \omega$. Then there exists a finite n-saturated set Λ for T.*

If Λ is an n-saturated set for T with the minimal number of elements then $(A, B) \not\simeq (C, D)$ for any two different elements $(A, B), (C, D) \in \Lambda$ and if Λ' is another n-saturated set with the minimal number of elements then Λ and Λ' are equivalent.

Proof. Given $(A, B) \in \Theta_n(T)$ we have $n \leq |A \cup B| \leq 2n$. Let E be the equivalence relation on $\mathcal{A} := \bigcup_{n \leq i \leq 2n} \Delta_i(T) \times \Delta_i(T)$ given by $(A, B)\mathrm{E}(C, D)$ if $A \cup B \sim C \cup D$. It follows from Lemma 3.1 that E has finitely many equivalence classes.

The equivalence relation $\simeq$ is a subset of the equivalence relation E. It remains to prove that $\simeq$ partitions every equivalence class of E whose elements are diagonal into finitely many equivalence classes of $\simeq$. Let $(A, B), (C, D) \in \mathcal{A}$ with $(A, B), (C, D) \in \Theta_n(T)$ and with $A \cup B \sim C \cup D$ and f the similarity of $A \cup B$ to $C \cup D$ and $m = |A \cup B|$. Then $A : B \simeq C : D$ just in case $A \cup B \sim C \cup D$ and $\{f[A], f[B]\} = \{C, D\}$ where f is the similarity of $A \cup B$ to $C \cup D$.

It follows that the equivalence class of E containing $A \cup B$ is partitioned into $\frac{1}{2} \cdot \binom{m}{n} \cdot \binom{n}{2n-m}$ equivalence classes of the equivalence relation $\simeq$.

The second part of the assertion follows trivially. $\square$

Corollary 6.1. *Let T be a regular ω-tree and S a subset of T and $n \in \omega$. Then there exists a finite n-saturated set $\Lambda_n(S)$ for S.*

If Λ' and Λ'' are two n-saturated sets for S with the minimal number of elements then they are equivalent.

Definition 6.5. Let T be a regular ω-tree and S a subset of T and $n \in \omega$. We denote by $\Lambda_n(S)$ some fixed n-saturated set for S with the minimal number of elements.

Lemma 6.3. *Let T be a wide ω-tree and S a cofinal subset of T and f a pnp map of S into T and $n \in \omega$. Then $\Lambda_n(f[S])$ is equivalent to $\Lambda_n(T)$ for every $n \in \omega$.*

Proof. We obtain from Theorem 5.1 a strong diagonalization h of T into S which is according to Lemma 5.2 a $\mathfrak{d}$-similarity and of course also pnp map. The function $f \circ h$ is a pnp map of T into $f[S]$. According to Theorem 4.1, there is a similarity embedding ϕ of $h[S]$ into $f \circ h[S]$. The function $\phi \circ h$ is a δ-map of T.

It follows that $\{(\phi \circ h(A), \phi \circ h(B)) \mid (A, B) \in \Lambda_n(T)\}$ is a minimal n-saturated set for $\phi \circ h[T]$ and hence for $f[S]$. $\square$

Corollary 6.2. *Let T be a wide ω-tree and S a cofinal subset of T and f a pnp map of S into T and $n \in \omega$. Then $\Lambda_n(f[S])$ is a minimal n-saturated set for every for T and if $f[S] \subseteq S$ also a minimal n-saturated set for S.*

Definition 6.6. Let $S \subseteq \mathfrak{T}_\omega$ and $n \in \omega$ and $\mathcal{T} \subseteq \Theta_n(\mathfrak{T}_\omega)$. The set $\mathcal{T}$ is *n-transitive for S* if:

1. For every $C \in \Delta_n(S)$ there is a pair $(A, A) \in \mathcal{T}$ with $A \sim C$.
2. For all $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ and all $C, D, E \in \Delta_n(S)$ with $C : D \simeq A_1 : B_1$ and $D : E \simeq A_2 : B_2$ there is a pair $(A_3, B_3) \in \mathcal{T}$ so that $C : E \simeq A_3 : B_3$.

Let $\Lambda \subseteq [\Delta_n(S)]^2$. We denote by trans($\Lambda$) the set of transitive subsets of Λ.

It follows that if $\mathcal{T}$ is equivalent to $\mathcal{T}'$ and $\mathcal{T}$ is n-transitive for S then $\mathcal{T}'$ is n-transitive for S.

Let $S, T \subseteq \mathfrak{T}_\omega$ so that there is an equivalence h of $\Lambda_n(S)$ to $\Lambda_n(T)$. Then $(A, B) \simeq h(A, B)$ for all $(A, B) \in \Lambda_n(S)$. Hence the n-transitive subsets of $\Lambda_n(S)$ for S are n-transitive sets for T and the n-transitive subsets of $\Lambda_n(T)$ for T are n-transitive sets for S.

In particular it follows from Lemma 6.3 that if T is a wide ω-tree T and S a cofinal subset of T and f a pnp map of S into T then the set $\mathcal{T} \subseteq \Lambda_n(T)$ is n-transitive for T if and only if it is n-transitive for $f[S]$.

Let $S \subseteq \mathfrak{T}_\omega$ and $\mathcal{T}$ a transitive set for S. We define the relation $\overset{\mathcal{T}}{\sim}$ on $\Delta_n(S)$ as follows:

$$C \overset{\mathcal{T}}{\sim} D \text{ if and only if } A : B \simeq C : D \text{ for some } (A, B) \in \mathcal{T}.$$

Note that if the transitive set $\mathcal{T}'$ is equivalent to the transitive set $\mathcal{T}$ then $C \overset{\mathcal{T}'}{\sim} D$ if and only if $C \overset{\mathcal{T}}{\sim} D$.

Lemma 6.4. *Let $S \subseteq \mathfrak{T}_\omega$ and $\mathcal{T}$ an n-transitive set for S. Then $\overset{\mathcal{T}}{\sim}$ is an equivalence relation on $\Delta_n(S)$.*

Let T be a wide ω-tree and S a cofinal subset of T and $n \in \omega$ and f a pnp map of S into T. If $\mathcal{T}', \mathcal{T}'' \in \operatorname{trans}(\Lambda_n(T))$ so that

$$\overset{\mathcal{T}'}{\sim} \cap [[f[S]]^n]^2 = \overset{\mathcal{T}''}{\sim} \cap [[f[S]]^n]^2$$

then $\mathcal{T}' = \mathcal{T}''$.

Proof. Reflexivity follows from Item 1. of Definition 6.6 and symmetry follows from the definition of $\simeq$, Definition 6.3.

In order to check transitivity let $A \overset{\mathcal{T}}{\sim} B$ and $B \overset{\mathcal{T}}{\sim} C$. If $A = B$ or $B = C$ then $A \overset{\mathcal{T}}{\sim} C$. So suppose that $A \neq B \neq C$. Then there are (A_1, B_1) and (A_2, B_2) in $\mathcal{T}$ such that $A_1 : B_1 \simeq A : B$ and $A_2 : B_2 \simeq B : C$. By Definition 6.6 there is a pair $(A_3, B_3) \in \mathcal{T}$ such that $A' : C' \simeq A_3 : B_3$. This implies $A \overset{\mathcal{T}}{\sim} C$ and hence that the relation $\overset{\mathcal{T}}{\sim}$ is transitive.

Let $(A, B) \in \mathcal{T}'$. According to Lemma 6.3 there is an equivalence h of $\Lambda_n(S)$ to $\Lambda_n(f[S])$. Let $h(A, B) = (C, D)$ which implies $(A, B) \simeq (C, D)$. Hence $C \overset{\mathcal{T}'}{\sim} D$ which implies, because $\overset{\mathcal{T}'}{\sim}$ and $\overset{\mathcal{T}''}{\sim}$ agree on $f[S]$, that $C \overset{\mathcal{T}''}{\sim} D$. This in turn implies that $(A, B) \in \mathcal{T}''$ because $\Lambda_n(S)$ is minimal and hence there is no pair $(E, F) \in \Lambda_n(S)$ with $(E, F) \simeq (C, d)$. $\square$

Let $S \subseteq \mathfrak{T}_\omega$ and $\Lambda_n(S)$ a minimal saturated set for S. Then

$$\mathfrak{E}_n(S) := \{\overset{\mathcal{T}}{\sim} \mid \mathcal{T} \in \operatorname{trans}(\Lambda_n(S))\}.$$

It follows from the discussion after Definition 6.6 that the set $\mathfrak{E}_n(T)$ does not depend on the particular minimal saturated set $\Lambda_n(S)$. The set $\mathfrak{E}_n(T)$ is the *canonical set of partitions of the n-element subsets of T.* We will prove that the name canonical is appropriate.

Theorem 6.1. *Let T be a regular ω-tree and S a cofinal subset of T and $n \in \omega$. Then $\mathfrak{E}_n(S)$ is a finite basis for the equivalence relations on $[S]^n$.*

Proof. We verify Items 1. and 2. of Definition 6.1.

Let f be a pnp map of S into S with $R = f[S]$ and Q an equivalence relation on $[R]^n$. Let Λ_n be a minimal n-saturated set for R and $N = \{|A \cup B| \mid (A, B) \in \Lambda_n\}$. According to Corollary 6.2 Λ_n is a minimal n-saturated set for T and a minimal n-saturated set for S.

We define for every $(A, B) \in \Lambda_n$ a function $\rho_{(A,B)} : \Delta_N(R) \mapsto \{0, 1\}$ such that

$$\rho_{(A,B)}(F) = \begin{cases} 1 & \text{if } A \cup B \sim F \text{ and } l[A] \, \mathrm{Q} \, l[B] \ , \\ 0 & \text{otherwise,} \end{cases}$$

where l is the similarity of $A \cup B$ to F.

According to Corollary 5.3 there exists a $\mathfrak{d}$-similarity $h : T \mapsto R$ so that for every $(A, B) \in \Lambda_N$ and $m \in N$ and $\sim$ equivalence class P of $\Delta_m(h[T])$ the function $\rho_{(A,B)}$ restricted to P is constant. Let

$$\begin{aligned} \mathcal{T} &= \{(A, B) \in \Lambda_n \mid \rho_{(A,B)}[\mathrm{Sim}_{h[T]}(A \cup B)] = \{1\}\} \\ &= \{(A, B) \in \Lambda_n \mid \forall F \in \mathrm{Sim}_{h[T]}(A \cup B)(\rho_{(A,B)}(F) = 1)\}. \end{aligned}$$

It follows that $(A, B) \in \mathcal{T}$ if and only if for all $(C, D) \in [\Delta_n(S)]^2$ with $A : B \simeq C : D$ we have $h[C] \,\mathrm{Q}\, h[D]$ if and only if there exists $(C, D) \in [\Delta_n(S)]^2$ with $A : B \simeq C : D$ so that $h[C] \,\mathrm{Q}\, h[D]$ if and only if $h[A] \,\mathrm{Q}\, h[B]$.

We check that $\mathcal{T}$ is a transitive subset of Λ_n. Let $C \in \Delta_n(S)$. Then there is $(A, A) \in \Lambda_n$ with $A \sim C$. It follows that $(A, A) \in \mathcal{T}$ because $(h[A] \,\mathrm{Q}\, h[A])$.

Let $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$. If there is no triple C, D, E with $C : D \simeq A_1 : B_1$ and $D : E \simeq A_2 : B_2$ let $(A_3, B_3) = (A_1, B_1)$. Let C, D, E be so that $C : D \simeq A_1 : B_1$ and $D : E \simeq A_2 : B_2$. It follows that $h[C] \,\mathrm{Q}\, h[D]$ and $h[D] \,\mathrm{Q}\, h[E]$ and hence $h[C] \,\mathrm{Q}\, h[E]$. Let $(A_3, B_3) \in \Lambda_n$ so that $A_3 : B_3 \simeq h[C] : h[E] \simeq C : E$. Then $(A_3, B_3) \in \mathcal{T}$.

Let $\mathrm{E} = \overset{\mathcal{T}}{\sim}$. Let $(C, D) \in [\Delta_n]^2$ and $(A, B) \in \Lambda_n$ with $A : B \simeq C : D$. Then $C \,\mathrm{E}\, D$ if and only if $C \overset{\mathcal{T}}{\sim} D$ if and only if $(A, B) \in \mathcal{T}$ if and only if $C \,\mathrm{Q}\, D$.

Let $\overset{\mathcal{T}'}{\sim}$ and $\overset{\mathcal{T}''}{\sim}$ be two different elements of $\mathfrak{E}_n(S)$ and $R = f[S]$ a pnp copy of S in S. Let Λ be a minimal n-saturated set for $f[S]$ and hence for S according to Corollary 6.2. It follows from Lemma 6.4 that $\mathcal{T}' \neq \mathcal{T}''$. Let $(A, B) \in \mathcal{T}' \setminus \mathcal{T}''$. Then $A \overset{\mathcal{T}'}{\sim} B$ but not $A \overset{\mathcal{T}''}{\sim} B$.

$\square$

7 Homogeneous structures

Let $\mathfrak{L}$ be a binary relational language. A set $\mathbf{A}$ of finite relational structures in the language $\mathfrak{L}$ has the *amalgamation property* if for any three elements $\mathbb{A}, \mathbb{B}, \mathbb{C}$ of elements in $\mathbf{A}$ and embeddings $f : \mathbb{C} \mapsto \mathbb{A}$ and $g : \mathbb{C} \mapsto \mathbb{B}$ there exists an element $\mathbb{D} \in \mathbf{A}$ and embeddings $f' : \mathbb{A} \mapsto \mathbb{D}$ and $g' : \mathbb{B} \mapsto \mathbb{D}$ so that $f' \circ f = g' \circ g$.

A set $\mathbf{A}$ of finite relational structures in the language $\mathfrak{L}$ is *updirected* if for every two elements $\mathbb{A}$ and $\mathbb{B}$ in $\mathbf{A}$ there exists an element $\mathbb{D} \in \mathbf{A}$ into which both structures $\mathbb{A}$ and $\mathbb{B}$ have an embedding.

A set **A** of finite relational structures in the language $\mathfrak{L}$ is an *age* if it is closed under induced substructures, isomorphic images and updirected. Let $\mathbb{U}$ be a countable relational structure. The set of finite relational structures which have an embedding into $\mathbb{U}$ is *the age of* $\mathbb{U}$.

Theorems 7.1 and 7.2 are due to Fraïssé, see [5].

Theorem 7.1. *Let* $\mathbb{U}$ *be a countable relational structure. The age of* $\mathbb{U}$ *is a countable age. Conversely, if* **A** *is a countable age then there exists a countable relational structure whose age is equal to* **A**.

Definition 7.1 (mapping extension property).
The countable relational structure $\mathbb{U}$ with age **A** has the *mapping extension property* if for every structure $\mathbb{A} = (A; \mathfrak{L}) \in \mathbf{A}$ and every element x in A and every embedding f of $\mathbb{A} - x$ into $\mathbb{U}$ there is an extension of f to an embedding of $\mathbb{A}$ into $\mathbb{U}$.

Definition 7.2. A countable relational structure $\mathbb{U}$ is *homogeneous* if it has the mapping extension property.

Theorem 7.2. *Let* **A** *be a countable age with amalgamation. Then there exists an, up to isomorphism unique, countable homogeneous structure whose age is equal to* **A**. *The age of every countable homogeneous structure is a countable age with amalgamation.*

Let **F** be a universal constraint set int the language $\mathfrak{L}$. It is not difficult to see that the set **A** of all finite relational structures in the language $\mathfrak{L}$, with the property that for every element $\mathbb{A} = (A; \mathfrak{L}) \in \mathbf{A}$ every two element induced substructure of $\mathbb{A}$ is isomorphic to an element of **F**, is an age. Hence there exists a unique homogeneous structure $\mathbb{U}_{\mathbf{F}}$ with age **A**. This structure $\mathbb{U}_{\mathbf{F}}$ is the *universal binary countable homogeneous structure with language* $\mathfrak{L}$ *and constraint set* **F**.

Let **F** be a universal constraint set int the language $\mathfrak{L}$ and with $|\mathbf{F}| = k$. Let λ be a bijection of **F** to k. Let $\mathbb{U}_{\mathbf{F}} = (U; \mathfrak{L})$ be the universal binary countable homogeneous structure with language $\mathfrak{L}$ and constraint set **F**. Let $v_0, v_1, v_2, v_3, \dots$ be an ω-enumeration of U.

As described in the Introduction we associate with every element v_n of U a sequence $\sigma(v_n)$ of length n so that for every $i \in n$ the i's entry $\sigma(v_n)(i)$ is the label $\lambda(\mathbb{A})$ of the element $\mathbb{A} \in \mathbf{F}$ for which the function mapping 0 to i and 1 to v_n is an embedding of $\mathbb{A}$ into $\mathbb{U}_{\mathbf{F}}$. Note that σ is an injection of U into the regular ω-tree T of degree k and that $\sigma(v_n)(i)$ is the passing number $\sigma(v_n)(|\sigma(v_i)|)$ of $\sigma(v_n)$ at v_i. Note also that $\sigma = \sigma_\lambda$ depends on the labelling λ.

Conversly we define a relational structure $\mathbb{T}_{\mathbf{F}} = (T, \mathfrak{L})$ with base set the regular ω-tree T of degree k. Let $s = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ and $t = \langle t_0, t_1, \ldots, t_{m-1}\rangle$ be two elements of T with $m > n$ and let $R \in \mathfrak{L}$ be a binary relation symbol. Then $R(s,t)$ if $R(0,1)$ in the structure $\mathbb{A} \in \mathbf{A}$ for which $\lambda(\mathbb{A}) = t(|s|)$. The relational structure $\mathbb{T}_{\mathbf{F}} = (T, \mathfrak{L})$ is the *tree with constraints* $\mathbf{F}$.

Note: Let $m > n$ and $\mathbb{F} \in \mathbf{F}$. Then $\sigma(v_m)\big(|v_n|\big) = \lambda(\mathbb{F})$ if and only if the function which maps 0 to v_n and 1 to v_m is an isomorphism of $\mathbb{F}$ to the substructure of $\mathbb{T}_{\mathbf{F}}$ induced by $\{v_n, v_m\}$.

Given a universal countable binary relational structure $\mathbb{U} = (U; \mathfrak{L})$ we will always assume that U is ordered into an ω sequence and that the function σ of U into the regular ω-tree T is as defined above.

Theorem 7.3. *Let* $\mathbf{F}$ *be a universal constraint set in a binary relational language* $\mathfrak{L}$ *with* $|\mathbf{F}| = k$ *and let* λ *be a bijection of* $|\mathbf{F}|$ *into* k. *Let* T *be the regular* ω*-tree of degree* k *and* $\mathbb{U}_{\mathbf{F}} = (U; \mathfrak{L})$ *the universal binary homogeneous structure with constraints* $\mathbf{F}$. *Let* $v_0, v_1, v_2, \ldots$ *be an* ω*-enumeration of* U *and* σ *the association of the elements of* U *with elements of* T *via the given enumeration of* U *and the labelling* λ *of the elements of* $\mathbf{F}$. *Let* $\mathbb{T}_{\mathbf{F}} = (T, \mathfrak{L})$ *be the tree with constraints* $\mathbf{F}$.

The function σ *is an isomorphism of* $\mathbb{U}_{\mathbf{F}}$ *to the substructure of* $\mathbb{T}_{\mathbf{F}}$ *induced by* $\sigma[U]$.

Proof. Let $R \in \mathfrak{L}$ and $n < m$ and $R(v_n, v_m)$. Let $\mathbb{F} \in \mathbf{F}$ for which the function mapping 0 to v_n and 1 to v_m is an isomorphism of $\mathbb{F}$ to the substructure of $\mathbb{U}_{\mathbf{F}}$ induced by $\{v_n, v_m\}$. Then $R(0,1)$ and $\sigma(v_m)\big(|\sigma(v_n)|\big) = \sigma(v_m)\big(n\big) = \lambda(\mathbb{F})$. Hence $R(\sigma(v_n), \sigma(v_m))$.

Conversely, let $R \in \mathfrak{L}$ and $n < m$ and $R(\sigma(v_n), \sigma(v_m))$. Let $\mathbb{F} \in \mathbf{F}$ with $\sigma(v_m)(|\sigma(v_m)|) = \lambda(\mathbb{F})$. Then $R(0,1)$ in $\mathbb{F}$ and the function which maps 0 to v_n and 1 to v_m is an isomorphism of $\mathbb{F}$ to the substructure of $\mathbb{U}_{\mathbf{F}}$ induced by $\{v_n, v_m\}$. Hence $R(v_n, v_m)$. □

Theorem 7.4. *Let* $\mathbf{F}$ *be a universal constraint set in a binary relational language* $\mathfrak{L}$ *with* $|\mathbf{F}| = k$ *and let* λ *be a bijection of* $|\mathbf{F}|$ *into* k. *Let* T *be the regular* ω*-tree of degree* k *and* $\mathbb{U}_{\mathbf{F}} = (U; \mathfrak{L})$ *the universal binary homogeneous structure with constraints* $\mathbf{F}$. *Let* $v_0, v_1, v_2, \ldots$ *be an* ω*-enumeration of* U *and* σ *the association of the elements of* U *with elements of* T *via the given enumeration of* U *and the labelling* λ *of the elements of* $\mathbf{F}$. *Let* $\mathbb{T}_{\mathbf{F}} = (T, \mathfrak{L})$ *be the tree with constraints* $\mathbf{F}$.

Then $\sigma[U]$ *is a transversal cofinal subset of the regular* ω*-tree of degree* k.

Let D *be a transversal and cofinal subset of* T. *Then the substructure of* $\mathbb{T}_{\mathbf{F}}$ *induced by* D *is isomorphic to* $\mathbb{U}_{\mathbf{F}}$.

Proof. The set $\sigma[U]$ is obviously transversal. Let $s = \langle s_0, s_1, \dots s_{n-1}\rangle \in T$. Let x be an element not in U and $\mathbb{A} = (\{v_i \mid i \in n\} \cup \{x\}; \mathfrak{L})$ be a relational structure in language $\mathfrak{L}$ and base set $\{v_i \mid i \in n\} \cup \{x\}$ so that $\mathbb{A}$ restricted to $\{v_i \mid i \in n\}$ is equal to $\mathbb{U}$ restricted to $\{v_i \mid i \in n\}$ and so that $\lambda(\mathbb{F}) = s_i$ where $\mathbb{F} \in \mathbf{F}$ is isomorphic the restriction of $\mathbb{A}$ to $\{v_i, x\}$. Then $\mathbb{A}$ is an element of the age of $\mathbb{U}_{\mathbf{F}}$.

We obtain, from the mapping extension property of $\mathbb{U}_{\mathbf{F}}$, an embedding f of $\mathbb{A}$ into $\mathbb{U}_{\mathbf{F}}$ which is the identity on the set $\{v_i \mid i \in n\}$. Let $f(x) = v_m$. Note that $m \geq n$ because f is an injection. It follows that s is a predecessor of $\sigma(f(x)) = \sigma(v_m) \in \sigma[U]$ and hence that $\sigma[U]$ is cofinal in T.

Let D be a transversal and cofinal subset of T. Let $\mathbb{A}$ be an element in the age of $\mathbb{U}_{\mathbf{F}}$. It follows by induction on the size of $\mathbb{A}$ from the cofinalitly of D that there is an embedding of $\mathbb{A}$ into the restriction of $\mathbb{T}_{\mathbf{F}}$ to D. Hence, because the age of $\mathbb{T}_{\mathbf{F}}$ is a subset of the age of $\mathbb{U}_{\mathbf{F}}$, the age of the restriction of $\mathbb{T}_{\mathbf{F}}$ to D is equal to the age of $\mathbb{U}_{\mathbf{F}}$. The restriction of $\mathbb{T}_{\mathbf{F}}$ to D has the mapping extension property because of the cofinality of D. We obtain from Theorem 7.2 that the restriction of $\mathbb{T}_{\mathbf{F}}$ to D is isomorphic to $\mathbb{U}_{\mathbf{F}}$.

□

Theorem 7.5. *Let $\mathbf{F}$ be a universal constraint set in a binary relational language $\mathfrak{L}$ with $|\mathbf{F}| = k$ and let λ be a bijection of $|\mathbf{F}|$ into k. Let T be the regular ω-tree of degree k and $\mathbb{U}_{\mathbf{F}} = (U; \mathfrak{L})$ the universal binary homogeneous structure with constraints $\mathbf{F}$. Let $v_0, v_1, v_2, \dots$ be an ω-enumeration of U and σ the association of the elements of U with elements of T via the given enumeration of U and the labelling λ of the elements of $\mathbf{F}$. Let $\mathbb{T}_{\mathbf{F}} = (T, \mathfrak{L})$ be the tree with constraints $\mathbf{F}$.*

The function $f : U \mapsto U$ is an isomorphism of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{U}_{\mathbf{F}}$ if and only if the function $\sigma \circ f \circ \sigma^{-1}$ is a pnp map of $\sigma[U]$ to $\sigma[U]$.

Proof. The function f is an injection if and only if $\sigma \circ f \circ \sigma^{-1}$ is an injection because σ is an injection. Every pnp map ins injective and every isomorphism is injective.

Let $\mathbb{F} \in \mathbf{F}$ for which the function mapping 0 to v_n and 1 to v_m is an isomorphism of $\mathbb{F}$ to the substructure of $\mathbb{U}_{\mathbf{F}}$ induced by $\{v_n, v_m\}$. Because σ is an isomorphism according to Theorem 7.3, the function which maps 0 to $\sigma(v_n)$ and 1 to v_m is an isomorphism of $\mathbb{F}$, hence $\sigma(v_m)\big(|\sigma(v_n)|\big) = \lambda(\mathbb{F})$.

Let f be an isomorphism. Then $g := \sigma \circ f \circ \sigma^{-1}$ is an isomorphism because σ is an isomorphism according to Theorem 7.3. Hence, the function which maps 0 $g(\sigma(v_n)$ and 1 to $g(\sigma(v_m)$ is an isomorphism of $\mathbb{F}$ into $\mathbb{T}_{\mathbf{F}}$. Therefore

$$g(\sigma(v_m))\big(|g(\sigma(v_n))|\big) = \lambda(\mathbb{F}) = \sigma(v_m)\big(|\sigma(v_n)|\big),$$

which implies that $g := \sigma \circ f \circ \sigma^{-1}$ is a pnp map.

Let $g := \sigma \circ f \circ \sigma^{-1}$ be a pnp map. Then

$$\sigma \circ f(v_m)\big(|\sigma \circ f(v_n)|\big) = g(\sigma(v_m))\big(|g(\sigma(v_n))|\big) = \lambda(\mathbb{F}),$$

which implies that the function mapping 0 to $\sigma \circ f(v_n)$ and 1 to $\sigma \circ f(v_m)$ is an isomorphism of $\mathbb{F}$. Hence the function which maps 0 to $f(v_n)$ and 1 to $f(v_m)$ is an isomorphism of $\mathbb{F}$. Implying that $R(v_n, v_m)$ if and only if $R(f(v_n), f(v_m))$ for $R \in \mathfrak{L}$ and hence that f is an isomorphism.

□

On account of Theorem 7.4 we identify universal countable binary relational structures with the corresponding cofinal subsets of regular ω-trees. This enables us to carry all of the notions for sets of sequences like diagonal, similar, $\mathfrak{d}$-morphism, $\mathfrak{E}_n(U)$ etc. over to universal countable binary relational structures. Furthermore, isomorphisms of the universal homogeneous structures correspond to pnp maps of the corresponding set of sequences according to Theorem 7.5

Definition 7.3. Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and $n \in \omega$. Then $\mathrm{nd}_n(U)$ is the set of all subsets of U with n elements which are not diagonal. Fix an enumeration $\mathcal{C}'_n = (Q_0, Q_1, \dots, Q_{m-1})$ of the different $\sim$ equivalence classes of n-element diagonal subsets of U and let $\mathcal{C}_n(\mathbb{U}) := (Q_0 \cup \mathrm{nd}_n(U), Q_1, Q_2, \dots, Q_{m-1})$.

It follows from Lemma 3.1 that there are only finitely many $\sim$ equivalence classes on the n-element diagonal subsets of U. For $\mathbb{U} = (U; \mathfrak{L})$ a universal countable binary relational structure and $n \in \omega$ we denote by $r_{\mathbb{U}}(n)$ the number of $\sim$ equivalence classes of the n-element diagonal subsets of U.

Theorem 7.6. *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and $n \in \omega$. Then $\mathcal{C}_n(\mathbb{U})$ is a canonical partition of the n-element subsets of U.*

Proof. It follows from Corollary 5.2 that each of the sets Q_i with $i \in m$ is indivisible. The set $Q_0 \cup \mathrm{nd}_n(U)$ is indivisible because the image of a strong diagonalization is diagonal and every subset of a diagonal set is again diagonal.

Embeddings of $\mathbb{U}$ into $\mathbb{U}$ are passing number preserving maps if $\mathbb{U}$ is represented as a cofinal subset of a regular ω-tree. Hence, it follows from Theorem 4.1 that each of the sets Q_i for $i \in m$ is persistent. □

Corollary 7.1. *Let* $\mathbb{U} = (U; \mathfrak{L})$ *be a universal countable binary relational structure and* $n \in \omega$*. Then*

$$\mathbb{U} \to (\mathbb{U})^n_{<\omega/r_{\mathbb{U}}(n)}.$$

If

$$\mathbb{U} \to (\mathbb{U})^n_{<\omega/s}.$$

then $s \geq r_{\mathbb{U}}(n)$*.*

Theorem 7.7. *Let* $\mathbb{U} = (U; \mathfrak{L})$ *be a universal countable binary relational structure and* $n \in \omega$*. Then* $\mathfrak{E}_n(U)$ *is a finite basis for the equivalence relations on* $[U]^n$*.*

Proof. Follows directly from Theorem 6.1. □

References

[1] V. Vuksanovic, Canonical Equivalence Relations, *Ph.D. thesis, Graduate Department of Mathematics, University of Toronto.* (2002).

[2] N. Sauer, Coloring subgraphs of the Rado graph, *manuscript submitted to Combinatorica.*

[3] P. Erdős and R. Rado, *A combinatorial theorem*, J. London Math. Soc. **25** (1950), 249–255.

[4] D. Devlin, Some partition theorems and ultrafilters on ω, *Ph. D. thesis, Dartmouth College.* (1979).

[5] N.W. Sauer, Appendix in: Theory of Relations, by R. Fraïssé, Revised Edition, in: *Studies in Logic and the Foundations of Mathematics*, **145**, North Holland 2000 ISBN 0-444-50542-3 CV 0.