Divisibility of countable metric spaces

Christian Delhommé
E.R.M.I.T.
Département de Mathématiques et d'Informatique
Université de La Réunion
15, avenue René Cassin, BP 71551
97715 Saint-Denis Messag. Cedex 9, La Réunion, France
`delhomme@univ-reunion.fr`

C. Laflamme*
University of Calgary
Department of Mathematics and Statistics
Calgary, Alberta, Canada T2N 1N4
`laf@math.ucalgary.ca`

Maurice Pouzet
PCS, Université Claude-Bernard Lyon1,
Domaine de Gerland -bât. Recherche [B], 50 avenue Tony-Garnier,
F69365 Lyon cedex 07, France
`pouzet@univ-lyon1.fr`

Norbert Sauer†
University of Calgary
Department of Mathematics and Statistics,
Calgary, Alberta, Canada T2N 1N4
`nsauer@math.ucalgary.ca`

Abstract

Prompted by a recent question of G. Hjorth [10] as to whether a bounded Urysohn space is indivisible, that is to say has the property that any partition into finitely many pieces has one piece which contains an isometric copy of the space, we answer this question and more generally investigate partitions of countable metric spaces.

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We show that a countable metric space which is indivisible must be totally Cantor disconnected, which implies in particular that any Urysohn space, bounded or not, is divisible. On the other hand we also show that one can remove "large" pieces from a bounded Urysohn space with the remainder still inducing a copy of this space, providing a certain "measure" of the indivisibility. Associated with every totally Cantor disconnected space is an ultrametric space, and we go on to characterize the countable ultrametric spaces which are homogeneous and indivisible.

Keywords: Partition theory, metric spaces, homogeneous relational structures, Urysohn space, ultrametric spaces.

1 Introduction and basic notions

A metric space $\mathbb{M} := (M; d)$ is called *divisible* if there is a partition of M into two parts, none of which contains an isometric copy of $\mathbb{M}$. If $\mathbb{M}$ is not divisible then it is called *indivisible.* Note that by repeated partition of M into two pieces we obtain that if $\mathbb{M}$ is indivisible then for every partition of M into finitely many pieces there is one piece which contains an isometric copy of the whole space. Every finite metric space (with at least two elements) is divisible, so the interest lies in infinite metric spaces. The uncountable case is different as the indivisibility property may fail badly. For example, every uncountable separable metric space can be divided into two parts such that no part contains a copy of the space via a one-to-one continuous map. This result, based on the Bernstein property 1908)(see [4] p.422) does not really involves the structure of metric spaces. In this paper we deal essentially with the countable case.

After the extension of the above result to uncountable subchains of the real line (Dushnik, Miller, 1940), the notion of indivisibility was considered for chains and then for relational structures (see for example [2] [7]). The notion we consider also falls under the framework of relational structures. Indeed, a metric space can be interpreted 'in several ways' to be a relational structure whose relations are binary and symmetric, the isometries being the isomorphisms of the relational structure. Because of this connection, we will use some basic notions and results about relational structures, and what we need is listed in Section 1.1.

We will show that every indivisible countable metric space is Cantor disconnected, hence in particular, that the bounded Urysohn metric space $\mathbb{U}_{\mathbb{Q},\leq 1}$, which is Cantor connected, is divisible. On the other hand we will show that the space $\mathbb{U}_{\mathbb{Q},\leq 1}$ is "almost" indivisible, in the sense that we can remove “almost” all of the elements of the space in various ways and the remainder still contains an isometric copy of the space.

Ultrametric spaces are special cases of totally Cantor disconnected spaces. We will characterize the indivisible homogeneous one. It seems to be the case that indivisible totally Cantor disconnected spaces are rare and that there is probably no good characterization of such spaces. We will provide various examples of indivisible countable metric spaces.

1.1 Relational structures, homogeneous structures and their ages

A relational structure is a pair $\mathrm{A} := (A; \mathbf{R})$ where $\mathbf{R} := (R_i)_{i\in I}$ is made of relations on the set A, the relation R_i being an n_i-ary relation identified with a subset of E^{n_i}. The family $\mu := (n_i)_{i\in I}$ is the *signature* of A. To $\mu := (n_i)_{i\in I}$, one may attach a family $\rho := (r_i)_{i\in I}$ of predicate symbols and one may see A as a realization of the languages whose non logical symbols are these predicate symbols. Let F be a subset of A, the induced substructure on A is denoted $\mathrm{A}_{\restriction F}$. Let $\mathrm{A}' := (A'; \mathbf{R}')$ having the same signature as A. A local isomorphism from A to A' is an isomorphism f from an induced substructure of A onto an induced substructure of A'; if the domain of f is A then f is an *embedding* of A to A'. The image of an embedding of A in A' is called a *copy* of A in A'.

A relational structure $\mathrm{A} := (A; \mathbf{R})$ is *divisible* if there is a partition $A = X \cup Y$ none of X and Y containing a copy of A. A relational structure which is not divisible is called *indivisible.* The *age* of a relational structure is the class of all finite relational structures which have an embedding into the structure.

We will use several properties of homogeneous structures (also called ultrahomogenous structures). Most are restatements or consequences of the Theorem of R. Fraïssé (Point 6 below). A more detailed account can be found in the book [2].

1. A countable relational structure $\mathrm{H} := (H, \mathbf{R})$ is *homogeneous* if every local isomorphism defined on a finite subset of H into H has an extension to an automorphism of H.

2. A countable relational structure $\mathrm{H} := (H, \mathbf{R})$ is homogeneous if and only if it satisfies the following *mapping extension property*:

   If $\mathrm{F} := (F; \mathbf{R})$ *is an element of the age of* H *for which the substructure of* H *induced on* $H \cap F$ *is equal to the substructure of* F *induced on*

$H \cap F$ then there exists an embedding of F *into* H *which is the identity on $H \cap F$.*

3. Two countable homogeneous structures with the same age are isomorphic.

4. A class $\mathcal{D}$ of relational structures has the *amalgamation property* (in brief AP) if for every members A, B, C of $\mathcal{D}$, embeddings $f : A \to B$, $g : A \to C$, there is some member A' of $\mathcal{D}$ and embeddings $f' : B \to A'$, $g' : C \to A'$ such that $f' \circ f = g' \circ g$.

5. A homogeneous structure embeds any countable younger structure, i.e. any countable structure whose age is included in that of the homogeneous one.

6. A class $\mathcal{D}$ of finite relational structures is the age of a countable homogeneous structure if and only if it is non-empty, is closed under embeddability and has the amalgamation property.

7. A subset $S \neq \emptyset$ of H is an *orbit* of H if it is an orbit for the action of the automorphism group $Aut(\mathrm{H})$ of H which fixes pointwise a finite subset of H. That is to say that there exists a finite subset F of H, called a *socket* of the orbit S, so that for some $s \in H \setminus F$:

$$S := \{f(s) : f \in Aut(\mathrm{H}) \text{ and } f(y) = y \text{ for all } y \in F\}.$$

8. If H is a countable homogeneous structure, then a subset $S \subseteq H$ is an orbit of H if there is an $s \in H \setminus F$ and S is equal to the set of all elements $t \in H$ so that the function which fixes the socket F pointwise and maps s to t is an isomorphism of the substructure of H induced on $S \cup \{s\}$ on the substructure of H induced on $S \cup \{t\}$. That is, the orbit S is the set of all elements of H which are of the same "one-type" over F.

9. If H is a countable homogeneous structure, a subset $X \subseteq H$ induces an isomorphic copy of H if and only if $S \cap X \neq \emptyset$ for every orbit S of H with socket a subset of X.

10. Let κ be a cardinal and $\mathcal{A}_\kappa$(resp. $\mathcal{A}_{\kappa,<\omega,}$) be the collection of all (resp. finite) relational structures $\mathrm{B} := (B; \mathbf{R})$ where $\mathbf{R} := (R_i)_{i<\kappa}$ is a sequence of irreflexive and symmetric binary relations symbols for which for all $x, y \in B$ with $x \neq y$ there exists exactly one $i < \kappa$

with $R_i(x, y)$. The class $\mathcal{A}_{\kappa,<\omega}$ has the amalgamation property. If $\kappa \leq \omega$ then it is countable, therefore, this is the age of a countable homogeneous structure, that we denote H_κ. For example, H_2 is the well-known *Random graph* or *Rado graph.* Each such H_k is indivisible.

1.2 Metric spaces and relational structures

Let us recall a few standard notions. Given two metric spaces $\mathbb{M} := (M; d)$ and $\mathbb{M}' := (M'; d')$, a *local isometry* from $\mathbb{M}$ to $\mathbb{M}'$ is an isometry f from a subspace of $\mathbb{M}$ onto a subspace of $\mathbb{M}$, and this is an *isometric embedding* if the domain of f is M.
$\mathbb{M}$ is called *homogeneous* if every local isometry defined on M and with values in M extends to an isometry from $\mathbb{M}$ onto itself.
The *age* of $\mathbb{M}$ is the collection of finite metric spaces which embed into $\mathbb{M}$.
Finally the *spectrum* of $a \in M$ is the set $Spec(\mathbb{M}, a) = \{d(a, x) \mid x \in M\}$ and the *spectrum* of $(M; d)$ is the set $Spec(\mathbb{M}) := \cup\{Spec(\mathbb{M}, a) : a \in M\} = \{d(x, y) \mid x, y \in M\}$.

Metric spaces also fall under the realm of relational structures in various ways. To exemplify this association, consider a set I, a map $f : I \to \mathbb{R}^+$ and set $\mu := (n_i)_{i\in I}$, where $n_i := 2$ for all $i \in I$. To a metric space $\mathbb{M} := (M; d)$ associate two relational structures, namely $\mathbb{M}_{f,\leq} := (M; \mathbf{R})$ and $\mathbb{M}_{f,=} := (M; \mathbf{S})$ where $\mathbf{R} := (R_i)_{i\in I}$ and $\mathbf{S} := (S_i)_{i\in I}$ are defined by:

$$(x, y) \in R_i \Longleftrightarrow d(x, y) \leq f(i) \tag{1}$$

$$(x, y) \in S_i \Longleftrightarrow 0 \neq d(x, y) = f(i) \tag{2}$$

Using the above notation, the following result summarizes the connections we will need, and the straightforward proof is left to the reader.

Lemma 1.1. *1. Every local isometry of* $\mathbb{M}$ *is a local isomorphism of* $\mathbb{M}_{f,\leq}$ *and of* $\mathbb{M}_{f,=}$.

2. Every local isomorphism of $\mathbb{M}_{f,\leq}$ *(resp. of* $\mathbb{M}_{f,=}$*) is a local isometry of* $\mathbb{M}$ *if and only if :*

  (a) either the spectrum of $\mathbb{M}$ *contains at most a non-zero element,*

  (b) or the image of f *separates the spectrum of* $\mathbb{M}$ *in the sense that* $f(I) \cap [p, p') \neq \emptyset$, *(resp in the sense that* $f(I) \cap \{p, p'\} \neq \emptyset$*) for every* $p, p' \in Spec(\mathbb{M})$ *such that* $0 < p < p'$.

Conversely, every binary relational structure $\mathrm{B} := (B; \mathbf{R})$ can be viewed as a metric space, provided that the number of isomorphic types of induced substructures on two element subsets of B is not greater than the continuum. Indeed, let $a \in \mathbb{R}^+ \setminus \{0\}$ be given, we may define a one-to-one map $\varphi : [B]^2 \to [a, 2a]$ such that $\varphi(\{x, y\}) = \varphi(\{x', y'\})$ if and only if $\mathrm{B}_{\restriction\{x,y\}}$ and $\mathrm{B}_{\restriction\{x',y'\}}$ are isomorphic. The map $d : B \times B \to [a, 2a]$ defined by setting $d(x, y) := \varphi(\{x, y\})$ if $x \neq y$ and $d(x, y) := 0$ if $x = y$ is a distance. This is particularly the case if $\mathrm{B} \in \mathcal{A}_\kappa$. A map $f : \kappa \to [a, 2a]$ be given, we will set $m_f(\mathrm{B}) := (B, d)$ where $d(x, y) := f(i)$ if $R_i(x, y)$ and $d(x, y) := 0$ otherwise (if $x = y$).

1.3 Homogeneous metric spaces

Let V be a set such that $0 \in V \subseteq \mathbb{R}^+$. Let $\mathcal{M}_V$ (resp. $\mathcal{M}_{V,<\omega}$) be the collection of metric spaces (resp. finite metric spaces) $\mathbb{M}$ whose spectrum is included into V. We may note that any such V is in fact a spectrum, indeed $V = Spec(\mathbb{M})$ where $\mathbb{M} := (V; d)$ and $d(x, y) := Max(\{x, y\})$.

Lemma 1.2. *Let V be a set such that $0 \in V \subseteq \mathbb{R}^+$. The class $\mathcal{M}_{V,<\omega}$ is the age of a metric space whose spectrum is V and is closed under embeddability. Further it has the amalgamation property if and only if it satisfies the following condition:*

> *Given any $u, u', v, v', w \in V\{0\}$, if $|u - u'| \leq v + v'$ and $|v - u| \leq w \leq u + v$ (and $|v' - u'| \leq w \leq u' + v'$), then $V \cap [u - u', v + v']$ is non-empty.*

Proof. First, there is a family $(\mathbb{M}_i)_{i\in I}$ of at most $\kappa := |V| \cdot \aleph_0$ members of $\mathcal{M}_{V,<\omega}$ such that every member of $\mathcal{M}_{V,<\omega}$ embeds into one of the $\mathbb{M}_i$'s. Pick an element $0_i \in M_i$ for each $i \in I$. Set $M := \{\overline{x} := (x_i)_{i\in I} : x_i \in M_i$ for all $i \in I$ and $\{i \in I : x_i \neq 0_i\}$ is finite $\}$, set $d(\overline{x}, \overline{y}) := Max\{d(x_i, y_i) : i \in I\}$ and set $\mathbb{M} := (M, d)$. Then the age of $\mathbb{M}$ is $\mathcal{M}_{V,<\omega}$. Since every subset of $\mathbb{R}^+$ containing 0 is a spectrum, the spectrum of $\mathbb{M}$ is V. The fact that $\mathcal{M}_{V,<\omega}$ is closed under embeddability is obvious. We prove that $\mathcal{M}_{V,<\omega}$ satisfies a stronger form of the amalgamation property (namely, the so-called *disjoint* amalgamation property). Let $\mathbb{M}_1 := (M_1; d), \mathbb{M}_2 := (M_2; d) \in \mathcal{M}_{V,<\omega}$ such that d_1 and d_2 coincide on $X := M_1 \cap M_2$.
Claim $M_1 \cup M_2$ can be endowed with a distance d which coincides with d_1 on M_1 and with d_2 on M_2.

The proof goes by induction on the number m of elements of $M_1 \setminus M_2$. If $m = 0$, $\mathbb{M}_1$ is a subspace of $\mathbb{M}_2$ and there is nothing to prove.
Case 1. $m = 1$. Let x_1 be the unique member of $M_1 \setminus M_2$. We argue by a descending induction on the number n of elements of X. If $n = M_2$, $M_2 \subset M_1$ and there is nothing to prove. If not, let $x_2 \in M_2 \setminus M_1$. Set $\mathbb{M}'_1 := (M'_1, d'_1)$ where d'_1 coincide with d_1 on $X \cup \{x_1\}$ and with d_2 on $X \cup \{x_2\}$, $d'_1(x_1, x_2) := r$ with $r \in V \cap [a, b]$, $[a, b] := \mathbb{R}^+$ if $n = 0$ and $a := Max\{|d_1(x_1, z) - d_2(x_2, z)| : z \in X\}$, $b := Min\{d_1(x_1, z) + d_2(x_2, z) : z \in X\}$ otherwise. The definition of d'_1 only requires $a \leq b$, that is $|d_1(x_1, z_1) - d_2(x_2, z_1)| \leq d_1(x_1, z_2) + d_2(x_2, z_2)$ for every $z_1, z_2 \in X$, an inequality which readily follows from the triangular inequality. With this definition, d'_1 is a distance. If we set $\mathbb{M}'_2 := \mathbb{M}_2$, then $\mathbb{M}'_1$ and $\mathbb{M}'_2$ coincide on $X' := M'_1 \cap M'_2$ which has $n + 1$ element. This proves our claim in this case.
Case 2 $m > 1$. Pick $x_1 \in M_1 \setminus M_2$. Case 1 applied to $\mathbb{M}'_1 := \mathbb{M}_{1 \restriction X \cup \{x_1\}}$, $\mathbb{M}'_2 := \mathbb{M}_2$ leads to a metric space $\mathbb{M}''_2 := (M_2 \cup \{x_2\}, d''_2)$. The two metric spaces $\mathbb{M}''_1 := \mathbb{M}_1$ and $\mathbb{M}''_2$ coincide on $M''_1 \cap M''_2$ and $M''_1 \setminus M''_2 = m - 1$. This proves our claim. □

Fraïssé's theorem (Point 6 above) gives immediately:

Theorem 1.3. *If V is countable then there is a countable homogeneous space $\mathbb{U}_V$ whose age is $\mathcal{M}_{V,<\omega}$.*

We call the space $\mathbb{U}_V$ the *Urysohn space with spectrum V*. If $V := \mathbb{Q}$ then $\mathbb{U}_{\mathbb{Q}}$ (resp.$\mathbb{U}_{\mathbb{Q},\leq \ell}$) is the homogeneous metric space whose age is the set of all finite metric spaces whose spectrum is a subset of the set of rationals (resp. of rationals in the interval $[0, \ell]$). The Cauchy completion of $\mathbb{U}_{\mathbb{Q}}$ is the famous space discovered by Urysohn [11].

Here is a straightforward example, see [8] for others.

Example 1.4. *Suppose that for some $a \in \mathbb{R}^+ \setminus \{0\}$, $V \setminus \{0\} \subseteq [a, 2a]$. Then $\mathbb{U}_V = m_f(\mathrm{H}_\kappa)$ where $\kappa := |V \setminus \{0\}|$ and $f : \kappa \to V \setminus \{0\}$ is a bijective map (see Point 10 of Section 1.1).*

All metric spaces with age $\mathcal{M}_{V,<\omega}$ satisfy a weaker version of the indivisibility property.

Theorem 1.5. *Let $\mathbb{M}$ be a metric space with age $\mathcal{M}_{V,<\omega}$.*

1. *For every partition of M into two parts X and Y one of the induced metric spaces $\mathbb{M}_{\restriction X}$ and $\mathbb{M}_{\restriction Y}$ has the same age as $\mathcal{M}_{V,<\omega}$.*

2. *If V is countable and bounded then there is an indivisible metric space with age $\mathcal{M}_{V,<\omega}$*

Proof. This result is due to the fact that $\mathcal{M}_{V,<\omega}$ is closed under finite product. The key ingredients of the proof is the Hales-Jewett 's theorem [3].

Claim For every $\mathbb{F} \in \mathcal{M}_{V,<\omega}$ there is some $\mathbb{G} \in \mathcal{M}_{V,<\omega}$ such that for every partition of G into two parts X and Y, one of the spaces induced by $\mathbb{G}$ embeds $\mathbb{F}$. Recall that a *combinatorial line* of a finite cartesian power N^n of N is a set of the form $L_{\overline{l}} := \{\overline{x} := (x_i)_{i<n} \in N^n : x_i = l_i$ for all $i \in K$ and $x_i = x_j$ for all $i, j \notin K\}$ where $l := (l_i)_{i\in K}$ and $K \subset n$. According to Hales-Jewett 's theorem, if n is large enough then for every partition of N^n into two parts X, Y one of the parts contains a combinatorial line. Thus, if we equip F^n with the "sup-distance", the resulting space $\mathbb{G}'$ satisfies the conclusion of the claim.

To prove part 1, let $\mathbb{M}$ be metric space with age $\mathcal{M}_{V,<\omega}$ and let X, Y be a partition of M. Assume for a contradiction that the ages $\mathcal{A}$ of $\mathbb{M}_{\restriction X}$ and $\mathcal{B}$ of $\mathbb{M}_{\restriction Y}$ are distinct from $\mathcal{M}_{V,<\omega}$, and thus let $\mathbb{M}_X \in \mathcal{M}_{V,<\omega} \setminus \mathcal{A}$ and $\mathbb{M}_Y \in \mathcal{M}_{V,<\omega} \setminus \mathcal{B}$. Select $A, B \subseteq M$ such that $\mathbb{M}_{\restriction A}$ and $\mathbb{M}_{\restriction B}$ are an isometric copy of $\mathbb{M}_X$ and $\mathbb{M}_Y$ respectively. For $\mathbb{F} := \mathbb{M}_{\restriction A\cup B}$ there is no $\mathbb{G}$ satisfying the conclusion of the claim, a contradiction.

Now to prove 2, let $a \in V$ such that $2a$ is an upper-bound of V. Let $(\mathbb{F}_n)_{n<\omega}$ be an enumeration of the members of $\mathcal{M}_{V,<\omega}$. According to the Claim above, there is a sequence $(\mathbb{G}_n)_{n<\omega}$ such that $\mathbb{G}_{n+1}$ contains an isometric copy of $\mathbb{F}_{n+1}$ and for every partition of G_{n+1} into two parts one of the part contains an isometric copy of $\mathbb{G}_n$. Let G be the disjoint union of the G_n's and $d : G \times G \to V$ be defined by $d(x, y) := d_n(x, y)$ if $x, y \in G_n$ for some n and $d(x, y) := a$. Then $\mathbb{G} := (G, d)$ is an indivisible metric space with age $\mathcal{M}_{V,<\omega}$. □

Theorem 3.12 asserts that the condition that V is bounded is necessary.

1.4 Indivisibility of Urysohn spaces

Here is a short summary of indivisibility results regarding the Urysohn spaces.

- $\mathbb{U}_V$ is indivisible if $V \subseteq \{0\} \cup [a, 2a]$ for some $a > 0$. Indeed in this case $\mathbb{U}_V = m_f(\mathrm{H}_\kappa)$, $\kappa \leq \omega$. Since H_κ is indivisible, it follows that $\mathbb{U}_V$ is indivisible as well.
- Let R and S be two relational structures with the same signature. Write $R \preceq S$ if there exists a partition of R into finitely many parts $R_0, R_1, \ldots, R_{n-1}$

so that for all $i \in n$ there is an embedding of R_i into S. A necessary condition for a homogeneous structure to be indivisible is that any two orbits of it are related under $\preceq$, see [1]. Hence we obtain from Item 1 of Theorem 1.5 that if a homogeneous metric space is indivisible then it satisfies the *chain condition*, that is the ages of any two orbits are related under $\subseteq$.

It follows from results in [1] and [9] that homogeneous binary structures with finite signature whose age has free amalgamation are indivisible if and only if they satisfy the chain condition. (It seems to one of us, Sauer, that this result could be extended without too much of a problem to homogeneous structures with free amalgamation and infinite binary signature.)

Ages of homogeneous metric spaces satisfy the weaker notion of strong amalgamation. (See the appendix of [2] for the definitions of free and strong amalgamation.) According to Corollary 4.8 the space $\mathbb{U}_V$ satisfies the chain condition but is not indivisible for all subsets V of non negative reals. Hence we obtain a large class of examples of divisible homegeneous structures satisfying strong amalgamation and the chain condition.

On the other hand a sufficient condition for indivisibility given there is satisfied if $V := \{0, 1, 2, 3\}$. If $V := \{0, 1, 3, 4\}$ we do not know if $\mathbb{U}_V$ is indivisible.

- If $\mathbb{U}_V$ is indivisible then for every $a \in \mathbb{R}^+ \setminus \{0\}$, $[0, a] \cap V$ is not dense in $[0, a]$(Theorem 4.1). In particular neither $\mathbb{U}_{\mathbb{Q}}$ nor $\mathbb{U}_{\mathbb{Q},\leq \ell}$ is indivisible.
- If $\mathbb{U}_V$ is indivisible then V must be bounded (Theorem3.12). In this case let $a := SupV$, must then $V \cap [0, \frac{a}{2}]$ be dually well-founded?
- If $\mathbb{U}_V$ is indivisible then there is a map $u : V \to \mathbb{R}^+$ such that $u(v) \leq v$ for every $v \in V$ and $d^* := u \circ d$ is an ultrametric distance on U. It follows that $\mathbb{U}^*_V := (U, d^*)$ is homogeneous and indivisible (see Theorem 3.10).

2 Ultrametric spaces and homogeneous ultrametric spaces

A metric space is an *ultrametric space* if it satisfies the strong triangle inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. See [5] for example. Note that a space is an ultrametric space if and only if $d(x, y) \geq d(y, z) \geq d(x, z)$ implies $d(x, y) = d(y, z)$. What we did in the previous section for general metric spaces work for ultrametric spaces.

Let V be a set such that $0 \in V \subseteq \mathbb{R}^+$. Let $\mathcal{M}ult_V$ (resp. $\mathcal{M}ult_{V,<\omega}$) be the collection of ultrametric metric spaces (resp. finite ultrametric spaces) $\mathbb{M}$ whose spectrum is included into V. Then $\mathcal{M}ult_{V,<\omega}$ is the age of a metric space whose spectrum is V; it is closed under embeddability and has the amalgamation property. If V is countable then there is a countable homogeneous ultrametric space $\mathbb{U}ult_V$ whose age is $\mathcal{M}ult_{V,<\omega}$ and has spectrum V; we call it the *Urysohn ultrametric space with spectrum* V.

For a given set V, $\mathbb{U}_V$ and $\mathbb{U}ult_V$ are in general different, except if $V = \{a_n : n \in D\}$ where D is an interval of the set $\mathbb{Z}$ of integers and $2a_{i+1} < a_i$ for all $i, i+1 \in D$.

Homogeneous ultrametric are easy to describe. In fact ultrametric spaces can be described by means of real-valued trees. An ordered set P is a *forest* if for every $x \in P$ the set $\downarrow x := \{y \in P : y \leq x\}$ is a chain; this is a *tree* if in addition every pair of elements of P has a lower bound. If every pair $x, y \in P$ has an infimum, denoted $x \wedge y$, we will say that P is a *meet-tree.* We say that P is *ramified* if for every $x, y \in P$ such that $x < y$ there is some $y' \in P$ such that $x < y'$ and y' incomparable to y. In the sequel, we consider ramified meet-trees such that every element is below some maximal element. These posets are meet-semilattices generated by their coatoms. We will need the following property

Lemma 2.1. *Let P be a ramified meet-tree such that every element is below some maximal element. For every $x \in P \setminus Max(P)$ there is a subset $X \subseteq Max(P)$ of maximum cardinality such that $x = a \wedge b$ for every pair of distinct elements a, b of X*

Proof. For two elements a and b above an element x, set $a \equiv b$ if $x \neq a \wedge b$. Observe that this is an equivalence relation. A set X which meets each equivalence classe has maximum size. □

The cardinality of X, denoted $d_P(x)$, is the *degree* of x. For $x \in Max(P)$ we set $d_P(x) := 0$. If P is finite or well-founded, this the ordinary notion of degree, that is the number of upper-covers of x.

Two meet-tree P, P' are *isomorphic* if they are isomorphic as posets; in particular, an isomorphism f from P to P' preserves meets, that is $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in P$. A *positive real-valued meet-tree* , valued meet-tree for short, is a pair (P, v) where P is a meet-tree and v a map from P to $\mathbb{R}^*$. Two valued meet-trees (P, v), (P, v') are *isomorphic* if there is an isomorphism f from P onto P' such that $v' \circ f = v$. A *subtree* of a meet-tree P is a subset P' of P such that the meet of two arbitrary elements of P' belongs to P'; a *valued subtree* of a valued meet-tree (P, v) is a pair (P', v') where P' is a subtree and $v' := v_{\restriction P'}$. The *age* of a valued meet-tree (P, v) is the collection of finite valued meet-trees which are isomorphic to some valued subtree of P.

Let $\mathbb{M} = (M, d)$ be a metric space, $r \in R^+$ and $a \in M$, the *closed ball of center a, radius s* is the set $\mathcal{B}_a(s) := \{x \in M \mid d(a, x) \leq s\}$. The *diameter* of a subset B of E is $\delta(B) := sup\{d(x, y) : x, y \in B\}$. We denote by $\mathcal{B}all(\mathbb{M})$ be the collection of closed balls of M and by $Nerv(\mathbb{M}) := \{\mathcal{B}_a(s) : a \in M, s \in Spec(\mathbb{M}, a)\}$.
Notice that $\delta(B_a(s)) = s$ whenever $s \in Spec(\mathbb{M}, a)$, but more genarally let us recall the following fact.

Lemma 2.2. *If M is an ultrametric space then for every $B \in \mathcal{B}all(\mathbb{M})$ and $a \in B$, $B = \mathcal{B}_a(s)$ where $s := \delta(B)$,*

We give below a description of ultrametric spaces in terms of valued trees. A very close description is given by Lemin [6] (who instead of $Nerv(\mathbb{M})$ considered $Ball(\mathbb{M})$).

Theorem 2.3.

1. *Let $\mathbb{M} := (M, d)$ be an ultrametric space, then the pair (P, v), where $P := (Nerv(\mathbb{M}), \supseteq)$, v is the diameter function, is a valued ramified meet-tree such that every element is below some maximal element and the map $\delta : Nerv(\mathbb{M}) \rightarrow Spec(\mathbb{M})$ is strictly decreasing, $\delta(X) = 0$ for every $X \in M' := Max(P)$ and $d(x, y) = \delta(\{x\} \wedge \{y\})$ for every $x, y \in M$.*

2. *Conversely, let (P, v)a valued ramified meet-tree such that every element is below some maximal element of P and the map $v : P \rightarrow \mathbb{R}^+$ is strictly decreasing with $v(x) := 0$ for each maximal element of P. Then the map d defined on $M' := Max(P)$ by $d(x, y) := v(x \wedge y)$ is an ultrametric distance and $Nerv(\mathbb{M}') = up(P)_{\restriction M'}$ where $up(P)_{\restriction M'} := \{M' \cap \uparrow x : x \in P\}$.*

3. *The two correspondences are inverse of each other.*

Proof. 1) According to Lemma 2.2, balls are disjoint or comparable w.r.t. inclusion, hence P is a tree. Since $\{x\} \in P$ for every $x \in M$, P is ramified and every element is below some maximal element. Let $B, B' \in P$. Pick $a \in B$, $a' \in B'$ and set $r := d(a, a')$. It is easy to se that $\mathcal{B}_a(r) = B \wedge B'$, hence P is a meet-tree. The properties of δ follows from Lemma 2.2.

2) a) d is an ultrametric distance : Let $x \in M'$. We have $d(x, x) := v(x \wedge x) = v(x) = 0$. If $x \neq y$ then, since v is strictly decreasing, $d(x, y) := v(x \wedge y) > v(x) = 0$. Clearly $d(x, y) = d(y, x)$. Let $x, y, z \in M'$. Since P is a tree, $x \wedge z$ and $y \wedge z$ are comparable. Suppose $x \wedge z \leq y \wedge z$. Then $x \wedge z \leq x \wedge y$. Since v is decreasing, we have $d(x, y) \leq d(x, z) \leq Max\{d(x, z), d(y, z)\}$.

b) $Nerv(\mathbb{M}') = up(P)_{\restriction M'}$:

Let $B := M' \cap \uparrow x \in up(P)_{\restriction M'}$, $r := v(x)$ and $y \in B$.

Claim 1. $B = \mathcal{B}_y(r)$ and $r \in Spec(\mathbb{M}', y)$. Thus $B \in Nerv(\mathbb{M}')$.

Indeed, let $z \in B(y, r)$, that is $v(y \wedge z) \leq r$. Since $x \leq y$ and $y \wedge z \leq y$, x and $y \wedge z$ are comparable, since v is strictly decreasing $x \leq y \wedge z$ hence $z \in B$. Conversely, if $z \in B$ then $x \leq y \wedge z$ thus, since v is strictly decreasing, $d(y, z) := v(y \wedge z) \leq v(x) = r$ proving $z \in \mathcal{B}_y(r)$. Thus $B = \mathcal{B}_y(r)$ as claimed. Since P is ramified and every element of P is below some element of M', there is some $z \in M'$ such that $x = y \wedge z$. Clearly, $z \in B$ and $r = d(y, z)$ thus $r \in Spec(\mathbb{M}', y)$.

Let $B := B(y, r) \in Nerv(\mathbb{M}')$ with $r \in Spec(\mathbb{M}', y)$

Claim 2. $B \in up(P)_{\restriction M'}$.

Indeed, since $r \in Spec(\mathbb{M}', y)$ there is some $z \in M'$ such that $d(y, z) = r$. Let $x := y \wedge z$. Since $v(x) = r$ we get $B = \uparrow x \cap M' \in up(P)_{\restriction M'}$ from the previous claim.

3) We simply note that if $P := (Nerv(\mathbb{M}), \supseteq)$ then, for $M' := Max(P)$, P is isomorphic to $(up(P)_{\restriction M'}, \supseteq)$; moreover, if $v : P \to \mathbb{R}^+$ is the diameter function associated to $\mathbb{M}$, then $v(x) = \delta'(M' \cap \uparrow x)$ where δ is the diameter function associated to the metric defined on M' in part 2. $\square$

Lemma 2.4. *Two ultrametrics spaces have the same age if and only if the corresponding valued trees have the same age.*

The verification is immediate.

The *reduced valued tree associated to* an ultrametric space $\mathbb{M}$ is the pair (P', v') where $P' := P \setminus Max(P)$ and $v' := v_{\restriction P'}$. The age of the reduced valued tree does not determine the age of the tree, because the information about the degree, in P, of terminal nodes in P' is missing. With this information added, we have easily:

Lemma 2.5. *If two reduced valued trees are isomorphic via a map which preserves the degree of the original trees then the ultrametric spaces have the same age.*

Let λ be a chain and let $\overline{a} := (a_\mu)_{\mu \in \lambda}$ such that $2 \leq a_\mu \leq \omega$. Set $\omega^{[\overline{a}]} := \{\overline{b} := (b_\mu)_{\mu \in \lambda} : \mu \in \lambda \Rightarrow b_\mu < a_\mu$ and $supp(\overline{b}) := \{\mu < \omega : b_\mu \neq 0\}$ is finite $\}$. If $a_\mu = \omega$ for every $\mu \in \lambda$, the set $\omega^{[\overline{a}]}$ is usually denoted $\omega^{[\lambda]}$. Add a largest element, denoted ∞ to λ. Given $\overline{b}, \overline{c} \in \omega^{[\overline{a}]}$, set $\Delta(\overline{b}, \overline{c}) := \infty$ if $\overline{b} = \overline{c}$, otherwise $\Delta(\overline{b}, \overline{c}) := \mu$ where μ is the least member of λ such that $b_\mu \neq c_\mu$.

Suppose λ be countable. Let $w : \lambda \cup \{\infty\} \to \mathbb{R}^+$ be a strictly decreasing map such that $w(\infty) = 0$, let $d_w := w \circ \Delta$ and let V be the image of w. For $\mu \in \lambda \cup \{\infty\}$ set $\downarrow^* \mu := \downarrow \mu \setminus \{\mu\}$. Let $P' := \{f_{\restriction \downarrow^* \mu} : f \in \omega^{[\overline{a}]}, \mu \in \lambda \cup \{\infty\}\}$ ordered by extension and let $v'(f_{\restriction \downarrow^* \mu}) := w(\mu)$.

We have the following property, which is easy to check.

Lemma 2.6. *The pair* $\mathbb{M} := (\omega^{[\overline{a}]}, d_w)$ *is an ultrametric space,* $Spec(\mathbb{M}) = V$ *and the valued tree associated to* $\mathbb{M}$ *is isomorphic to* (P', v').

We say that $\mathbb{M}$ is *point-homogeneous* if the automorphism group of $\mathbb{M}$ acts transitively on $\mathbb{M}$.

Theorem 2.7. *Let* $\mathbb{M}$ *be a countable ultrametric space,* $P := (Nerv(\mathbb{M}), \supseteq)$, $v : P \to \mathbb{R}^+$ *where* $v(B) := \delta(B)$, $M' := Max(P)$. *The following properties are equivalent:*

(i) $\mathbb{M}$ *is isometric to some* $(\omega^{[\overline{a}]}, d_w)$.

(ii) $\mathbb{M}$ *is homogeneous;*

(iii) $\mathbb{M}$ *is point-homogeneous;*

(iv) *(a)* $v(x) = v(y) \Rightarrow d_P(x) = d_P(y)$ *for every* $x, y \in P$*;*
(b) $v[\downarrow x] = v[\downarrow y]$ *for every* $x, y \in Max(P)$.

Proof. $(i) \Rightarrow (iv)$ Let $\mathbb{M} := (\omega^{[\overline{a}]}, d_w)$. According to Lemma 2.6, the valued tree associated to $\mathbb{M}$ is isomorphic to (P', v'). Condition $(b)(iv)$ immediately follows. Let $x := f_{\restriction \downarrow^* \mu} \in P'$; if $\mu = \infty$ then $d_{P'}(x) = 0$, otherwise $d_{P'}(x) = a(\mu)$. Thus Condition $(a)(iv)$ holds too.

$(ii) \Rightarrow (iii)$ Trivial

$(iii) \Rightarrow (iv)$ Suppose $\mathbb{M}$ point homogeneous. First, Condition $(b)(iv)$ holds. Indeed, let $x, y \in M' := Max(P)$. Then $x := \{x'\}$ and $y := \{y'\}$, with $x', y' \in \mathbb{M}$. Let f be an isometry from $\mathbb{M}$ onto itself such that $f(x') = y'$. Then $Spec(x', \mathbb{M}) = Spec(y', \mathbb{M})$ and the result follows. Next, Condition $(a)(iv)$ holds. Let $x := B \in P, y := C \in P$ and $r := v(x) = v(y)$. Pick $x' \in B$, $y' \in C$. Let f be an isometry from $\mathbb{M}$ onto itself such that $f(x') = y'$. Then $f(B) = C$. For two elements x', y' of B, set $x' \equiv y'$ if $d(x', y') < r$. This relation is an equivalence relation. whose number of classes is the degree of $x := B$ in the poset $P := Nerv(\mathbb{M})$. The desired conclusion follows.

$(iv) \Rightarrow (ii)$ Let f be an isometry from a finite subset A of M onto a subset B of M. Let $x \in M \setminus A$. We prove that f extends to an isometry defined on $A \cup \{x\}$. If A is empty, we may send x onto any element b of M. If A is non-empty, set $r := Min(\{d(x, y) : y \in A\})$. In order to extend f we only need to send x onto some $b \in M$ such that $f(B(x, r)) = B(b, r) \cap f(A)$. There is some $u \in P$ such that $x \wedge x' = u$ for all $x' \in B(x, r) \cap A$ and moreover $v(u) = r$. Select $y \in f(B(x, r))$. Since $v[\downarrow x] = v[\downarrow y]$ there is some $u' \in \downarrow y$ such that $v(u') = r$. Since $d_P(u) = d_P(u')$, there is $b \in M$ such that $y' \wedge b = u'$ for all $y' \in f(A)$. Such an element will do.

$(ii) \Rightarrow (i)$. Let $\lambda := Spec(\mathbb{M}) \setminus \{0\}$ ordered with the dual of the order induced by the natural order on $\mathbb{R}$, let $w : \lambda \cup \{\infty\} \rightarrow \mathbb{R}^+$ with $w(x) := x$ for $x \in \lambda$ and $w(\infty) := 0$ and let $\overline{a} : \lambda \rightarrow \omega + 1$ such that $\overline{a} \circ w = d_P$ (such a map exists because of (iv) Condition 1).

Claim $\mathbb{M}$ is isometric to $(\omega^{[\overline{a}]}, d_w)$. According to the implications $(i) \Rightarrow (iv) \Rightarrow (ii)$ proved above, $(\omega^{[\overline{a}]}, d_w)$ is homogeneous. Since $\mathbb{M}$ is homogeneous, it suffices to prove that $(\omega^{[\overline{a}]}, d_w)$ and $\mathbb{M}$ have the same age to get the desired conclusion. From the implication $(iii) \Rightarrow (iv)$, the reduced valued trees associated to $(\omega^{[\overline{a}]}, d_w)$ and $\mathbb{M}$ are isomorphic by an isomorphism which preserves the degree. From Lemma 2.5, $(\omega^{[\overline{a}]}, d_w)$ and $\mathbb{M}$ have the same age. $\square$

Proposition 2.8. *The space* $(\omega^{[\lambda]}, d_w)$ *is the countable homogeneous ultrametric space* $\mathbb{U}lt_V$ *associated with* V*.*

Proof. We only neeed to prove that every finite ultrametric space $\mathbb{M} := (M, d)$ with spectrum included into V embeds isometrically into $(\omega^{[\lambda]}, d_w)$. We argue by induction on the number n of elements of M. If $n \leq 1$, the result is obvious. Suppose $n \geq 2$. Let $x \in M$. We may suppose that there is an isometric embedding f of $\mathbb{M}_{-x} := \mathbb{M}_{\restriction M \setminus \{x\}}$ into $(\omega^{[\lambda]}, d_w)$. We prove that f extends to $\mathbb{M}$. Set $r := Min(\{d(x, y) : y \in M \setminus \{x\}\})$ and $\mu \in \lambda$ such that $w(\mu) = r$. In order to extend f we only need to find some element $b \in \omega^{[\lambda]}$ such that $f(B(x, r)) = B(b, r) \cap f(M \setminus \{x\})$. For every $\overline{b}', \overline{b}'' \in f(B(x, r))$ we have $b'_{\mu'} = b''_{\mu'}$ for all $\mu' < \mu$. Select $b \in \omega^{[\lambda]}$ such that $b_{\mu'} = b'_{\mu'}$ for all $\mu' < \mu$ and $b_\mu \in \omega \setminus \{b'_\mu : \overline{b}' \in f(B(x, r)\}$. $\square$

2.1 Indivisible ultrametric spaces

Definition 2.9. *Let* $\mathbb{M} := (M; d)$ *be a metric space,* $a \in M$ *and* $0 \leq r < s$*. Then*

$$\mathcal{R}_a(r, s) := \{x \in M \mid r \leq d(a, x) < s\}.$$

Lemma 2.10. *Let $\mathbb{M} = (M, d)$ be an indivisible ultrametric space. Then the spectrum of every element of M is dually well founded.*

Proof. Let $a \in M$. Suppose for a contradiction that $r_0 = 0 < r_1 < r_2 < r_3 < \ldots$ is an infinite sequence of reals in the spectrum of a. Let s be its supremum. Cover M by a family $\mathcal{B} := \{\mathcal{R}_{a_\alpha}(0, s) : \alpha < \kappa\}$ of open balls of radius s such that $a_\alpha \notin M_\alpha := \cup\{\mathcal{R}_{a_\beta}(0, s) : \beta < \alpha\}$ (with the convention that if $s = \infty$ then $\mathcal{B}$ consists of M). Since d is an ultrametric distance, these balls are pairwise disjoint and therefore, the rings $\mathcal{R}_{a_\alpha}(r_i, r_{i+1})$ make-up a partition of M. Let:

$$\mathcal{E} := \bigcup_{\alpha<\kappa, i\in\omega} \mathcal{R}_{a_\alpha}(r_{2i}, r_{2i+1}) \text{ and } \mathcal{O} := \bigcup_{\alpha<\kappa, i\in\omega} \mathcal{R}_{a_\alpha}(r_{2i+1}, r_{2i+2})$$

and let f be an isometry of M into M. Let $\alpha < \kappa$ and $i \in \omega$ so that $f(a) \in \mathcal{R}_{a_\alpha}(r_i, r_{i+1})$. Let $b \in M$ with $d(a, b) = r_{i+1}$.

Then $d(f(a), f(b)) = r_{i+1}$ and because $d(f(a), a_\alpha) < r_{i+1}$ it follows that that $d(a_\alpha, f(b)) = r_{i+1} < s$. Thus $f(b) \in \mathcal{R}_{a_\alpha}(r_{i+1}, r_{i+2})$. □

Corollary 2.11. *If an ultrametric space is indivisible then the collection of balls, once ordered by inclusion, is dually well-founded and the diameter is attained.*

Proof. Let $(B_n)_{n<\omega}$ be an increasing sequence of balls of an ultrametric space $\mathbb{M} := (M, d)$. Pick $a \in \cap\{B_n : n \in\}$. Since $\mathbb{M}$ is ultrametric, a is the center of each B_n thus their radii belong to the spectrum of a. If $\mathbb{M}$ is indivisible, then from Lemma 2.10 above $Spec(\mathbb{M}, a)$ is dually well-founded, thus the sequence is stationnary. Let s be the maximum of $Spec(\mathbb{M}, a)$. Let $x, y \in M$. We have $d(x, y) \leq Max(\{d(x, a), d(y, a)\}) \leq s$, hence s is the maximum of the spectrum of $\mathbb{M}$, that is the diameter of $\mathbb{M}$. □

Theorem 2.12. *Let $\mathbb{M}$ be a denumerable ultrametric space. The following properties are equivalent:*

- *(i)* $\mathbb{M}$ *is isometric to some* $\mathbb{U}lt_V$*, where* V *is dually well-ordered;*
- *(ii)* $\mathbb{M}$ *is point-homogeneous,* $P := (Nerv(\mathbb{M}), \supseteq)$ *is well founded and the degree of every non maximal element is infinite;*
- *(iii)* $\mathbb{M}$ *is homogeneous and indivisible;*

Proof. $(i) \Rightarrow (ii)$ By definition, $\mathbb{U}lt_V$ is homogeneous, hence point-homogeneous. In fact, according to Proposition 2.8, $\mathbb{M}$ is isometric to some $(\omega^{[\lambda]}, d_w)$ where λ is a well-ordered chain. Thus, from Lemma 2.6, $P := (Nerv(\mathbb{M}), \supseteq)$ is well-founded and the degree of every non maximal element is infinite.

$(iii) \Rightarrow (i)$ Suppose that (iii) holds. Theorem 2.7 asserts that $\mathbb{M}$ is isometric to some $(\omega^{[\overline{a}]}, d_w)$. Since $\mathbb{M}$ is indivisible, it follows from Lemma 2.10 that $V := Spec(\mathbb{M})$ is well-founded, hence we may suppose that λ is an ordinal. To conclude it suffices to prove that $a_\mu = \omega$ for every $\mu < \lambda$. Let $\mu < \lambda$; set $r := w(\mu)$. First, observe that $M = \cup\mathcal{B}$ where $\mathcal{B}$ is a collection of pairwise disjoint balls, all of diameter r. Next, each member B of $\mathcal{B}$ is the union of a_μ balls B_i each of smaller diameter than r. Indeed, since $\mathbb{M}$ is point-homogeneous, all balls having the same radius are isometric spaces, thus it suffices to prove this property for the ball $B := \mathcal{B}_0(r)$, where 0 is the ordinal sequence which only takes value 0. This is easy: set $\overline{x}_i := (b_\nu)_{\nu<\lambda}$ where $i < a_n$, $b_\nu = 0$ if $\nu \neq \mu$ and $b_\mu := i$ otherwise, set $r^+ := w(\mu^+)$ where $\mu^+ := \mu + 1$ if $\mu + 1 < \lambda$ and $\mu^+ := \infty$ otherwise, then $B := \cup\{B(\overline{x}_i, r^+) : i < a_\mu\}$. With these two observations we have $M = \cup\{M_i : i < a_\mu\}$ where $M_i := \cup\{B_i : B \in \mathcal{B}\}$. Clearly, there is no isometry from $\mathbb{M}$ into an $\mathbb{M}_i$ hence if $a_\mu < \omega$, $\mathbb{M}$ cannot be indivisible.

$(ii) \Rightarrow (iii)$ According to Theorem 2.7, $\mathbb{M}$ is homogeneous. Let us show that it is indivisible. Let $f : M \to 2$ be a partition of M into two parts. Set $\mathcal{F}_0$ be the set of balls $B \in Nerv(\mathbb{M})$ such that there is some isometry φ_B from B into$B \cap f^{-1}(0)$ and let $M_0 := \cup\mathcal{F}_0$.

Claim 1 There is an isometry from M_0 to $M_0 \cap f^{-1}(0)$.

Indeed, let $\mathcal{F}'_0$ be the subset of $\mathcal{F}_0$ made of its maximal members (w.r.t. inclusion). Let $\varphi := \cup\{\varphi_B : B \in \mathcal{F}'_0\}$. Since balls are either disjoint or comparable, φ is a map and, since $P := (Nerv(\mathbb{M}), \supseteq)$ is well-founded, $M_0 = \cup\mathcal{F}'_0$, hence the domain of φ is M_0.

For B in $Nerv(\mathbb{M})$, set $Pred(B) := Max(\{B' : B' \subset B, B' \in Nerv(\mathbb{M})\})$.

Claim 2 If $B \notin \mathcal{F}_0$ then $Pred(B) \cap \mathcal{F}_0$ is finite.

Indeed, suppose not. Then, since the space is point-homogeneous, all members of $Pred(B)$ have the same radius and there is an isometry ψ from B into B which transforms each member of $Pred(B)$ to a member of $Pred(B) \cap \mathcal{F}_0$. Let $\varphi := \cup\{\varphi_{B'} : B' \in Pred(B) \cap \mathcal{F}_0\}$. Then φ is an isometry from $\cup(Pred(B) \cap \mathcal{F}_0)$ into $B \cap f^{-1}(0)$. Consequently, $\varphi \circ \psi$ is an isometry from B into $B \cap f^{-1}(0)$, thus $B \in \mathcal{F}_0$, a contradiction.

Suppose that $M \notin \mathcal{F}_0$. We construct an isometry h from M into $f^{-1}(1) \setminus M_0$ as follows. We start with an enumeration $(x_n)_{n<\omega}$ of the elements of M. According to Claim 1, $M \setminus M_0 \neq \emptyset$. We may also suppose that it contains an element of $f^{-1}(1)$ (otherwise the union of the identity map on $M \setminus M_0$ and an isometry as constructed in Claim 1, is an isometry from M into $f^{-1}(0)$). Let y_0 such an element. We set $h(x_0) := y_0$.

Suppose h be defined for all m, $m < n$. Let $p := Min(\{d(x_m, x_n) : m < n\})$. Let $I := \{i, i < n : d(x_i, x_n) := p\}$. Let $B := \mathcal{B}_{h(i)}(p)$ for $i \in I$. This set does not depend upon the choice of i. Since $h(i) \in f^{-1}(1) \setminus M_0$, $B \notin \mathcal{F}_0$. For each $i \in I$ let B'_i such that $h(i) \in B'_i \in Pred(B)$. According to Claim 2, there is some $B'' \in Pred(B) \setminus \mathcal{F}_0$ which is distinct from all the $B'_i s$. As in our first step, $B'' \setminus M_0$ is nonempty and in fact contains an element, say y_n of $f^{-1}(1)$. We set $h(x_n) := y_n$.

□

3 Divisibility of metric spaces

The sequence $a_0, a_1, \ldots, a_{n-1}, a_n$ of elements in a metric space $\mathbb{M} := (M; d)$ is an *ϵ-chain joining a_0 and a_n* if $d(a_i, a_{i+1}) \leq \epsilon$ for all $i \in n$. The space $\mathbb{M}$ is *Cantor connected* if any two of its elements can be joined by an ϵ-chain for any $\epsilon > 0$. The *Cantor connected component of an element* $a \in M$ is the largest Cantor connected subset of M containing a. The space $\mathbb{M}$ is *totally Cantor disconnected* if the Cantor connected component of every a reduce to a. See [5] for more details and references.

For $a \in M$ let $\lambda_\epsilon(a)$ be the supremum of all reals $l \leq 1$ for which there exists an ϵ-chain $a_0, a_1, \ldots, a_{n-1}, a_n$ with $d(a_0, a_n) \geq l$ containing a. (The condition $l \leq 1$ saves us from having to consider the special case ∞.) Let

$$\lambda(a) := \sup\{l \in \mathbb{R} \mid \forall \epsilon > 0 \, (\lambda_\epsilon(a) \geq l)\}.$$

A space $(M; d)$ is *restricted* if $\lambda(a) = 0$ for all $a \in M$. It follows that every restricted space is totally Cantor disconnected. There are totally Cantor disconnected spaces which are not restricted. Here is an example with bounded diameter:

Example 3.1. Let $(M; d)$ be the metric space so that:

1. $M = \{a\} \cup \{(m, n) \mid 0 < m \in \omega \text{ and } 0 < n \in \omega\}$.

2. $d(a, (m, n)) = \sum_{1 \leq i \leq n} \frac{1}{m 2^i}$.

3. If $n_1 < n_2$ then $d((m, n_1), (m, n_2)) = \sum_{n_1 < i \leq n_2} \frac{1}{m 2^i}$.

4. $d((m_1, n_1), (m_2, n_2)) = \sum_{1 \leq i \leq n_1} \frac{1}{m_1 2^i} + \sum_{1 \leq i \leq n_2} \frac{1}{m_2 2^i}$.

Lemma 3.2. *Let $c \in M$ and $0 \leq r_0 < r_1 < r_2 < r_3$ and $a \in \mathcal{R}_c(r_0, r_1)$ and $b \in \mathcal{R}_c(r_2, r_3)$ then:*

1. *$d(a, b) > r_2 - r_1$.*
2. *$d(x, y) < 2r_2$ for all $x, y \in \mathcal{R}_c(r_1, r_2)$.*
3. *If $0 < \epsilon < \min\{r_1 - r_0, r_3 - r_2\}$ and $x_0, x_1, x_2, \ldots, x_{n-1}$ is an ϵ-sequence with $x_i \notin R_c(r_0, r_1) \cup R_c(r_2, r_3)$ for all $i \in n$ but with $x_i \in \mathcal{R}_c(r_1, r_2)$ for at least one $i \in n$, then $x_i \in R_c(r_1, r_2)$ for all $i \in n$.*
4. *Let f be an isometry of M with $f[M] \cap (\mathcal{R}_c(r_0, r_1) \cup \mathcal{R}_c(r_2, r_3)) = \emptyset$ and let $z \in M$ with $\lambda(z) > 2r_2$. Then $f(z) \notin \mathcal{R}_c(r_1, r_2)$.*

Proof. Items 1 and 2 follow from the triangle inequality. Item 3 follows from item 1 and item 4 follows from items 2 and 3. □

Definition 3.3. *Let $c \in M$ and $0 < l$. Then*

$$\mathcal{E}_c(l) := \bigcup_{n \geq 2,\, n \text{ even}} \mathcal{R}_c\left(\frac{l(n-1)}{n}, \frac{ln}{n+1}\right)$$

and

$$\mathcal{O}_c(l) := \bigcup_{n \text{ odd}} \mathcal{R}_c\left(\frac{l(n-1)}{n}, \frac{ln}{n+1}\right)$$

.

Theorem 3.4. *Let $\mathbb{M} = (M; d)$ be a countable metric space. If there exists an element $a \in M$ with $\lambda(a) > 0$ then $\mathbb{M}$ is divisible.*

Proof. Since M is countable, it can be covered by a family of pairwise disjoint open balls with radius less that $\frac{\lambda(a)}{2}$. In fact, there exists a subset C of M and for every $c \in C$ a positive real l_c so that:

1. $l_c \neq d(x, y)$ for every $c \in C$ and $x, y \in M$.
2. $2l_c < \lambda(a)$ for every $c \in C$.
3. For every element $x \in M$ there is one and only one element $c \in C$ with $x \in \mathcal{R}_c(0, l_c)$.

(After enumerating M into an ω-sequence $m_0, m_1, m_2, m_3, \dots$ such a set C and function l can be constructed step by step exhausting all of the elements of M.)

Let

$$\mathcal{E} := \bigcup_{c \in C} \mathcal{E}_c(l_c) \text{ and } \mathcal{O} := \bigcup_{c \in C} \mathcal{O}_c(l_c).$$

Then $\mathcal{E} \cup \mathcal{O} = M$ and $\mathcal{E} \cap \mathcal{O} = \emptyset$.

Assume for a contradiction that there is an isometry f which maps M into $\mathcal{E}$. Then there is a $c \in C$ so that $f(a) \in \mathcal{E}_c(l_c)$. But this is not possible according to Lemma 3.2 item 4. Similarly it is not possible that f maps M into $\mathcal{O}$. □

Corollary 3.5. *A countable metric space which is indivisible is restricted and hence totally Cantor disconnected.*

The second part of the conclusion of the corollary above extends to uncountable metric spaces.

Theorem 3.6. *Let $\mathbb{M}$ be a metric space and d be a positive real, then there is a partition into two parts A_0 and A_1 which contains no Cantor connected component of diameter larger than d.*

Lemma 3.7. *Le $\mathbb{M}$ be a metric space and d be a positive real number. Then there is a sequence $(E_\mu)_{\mu<\lambda}$ such that:*

1. *$E_0 = \emptyset$ and each E_μ is open in M*
2. *the sequence is strictly increasing and continuous, that is E_μ is the union of E_ν for $\nu < \mu$ if μ is a limit ordinal,*
3. *the union covers M*
4. *$F_\mu := E_{\mu+1} \setminus E_\mu$ decomposes into two sets $A_{\mu,0}$ and $A_{\mu,1}$ such that for each Cantor connected component Y of $A_{\mu,i}$:*
   - *(a) the diameter of Y is at most d,*
   - *(b) there is some ϵ_Y such that the set of elements of $M \setminus E_\mu$ at distance at most ϵ_Y of Y is included into F_μ and disjoint from all other Cantor connected components of $A_{\mu,i}$.*

Proof. Suppose the sequence defined for all ν, $\nu < \mu$. If μ is a limit ordinal, set $E_\mu := \bigcup\{E_\nu : \nu < \mu\}$. If μ is a successor, say $\mu := \nu + 1$, pick $x \in E' := M \setminus E_\nu$, set $\mathcal{R}'_x(0, d/2) := \{y \in E' : d(x, y) < d/2\}$ and set $E_\mu := E_\nu \cup R'_x(0, d/2)$. Decompose $\mathcal{R}'_x(0, d/2)$ into countably many crowns $\mathcal{R}'{}_x\big(\frac{d(n-1)/2}{n}, \frac{dn/2}{n+1}\big)$ as in the proof of Theorem 3.4 , the union of the even ones gives $A_{\nu,0}$, the rest gives $A_{\nu,1}$. □

Proof of Theorem 3.6 Let $A_i := \bigcup\{A_{\mu,i} : \mu < \lambda\}$. Then A_i contains no subset X which is Cantor- connected and has diameter larger than d.

Indeed, suppose the contrary. Let μ be minimum such that E_μ meets X. Clearly μ is a successor, say $\mu = \nu + 1$. Let $x \in X_\nu := X \cap F_\nu$. Let Y be a Cantor connected component of x in F_ν.

Claim $X \subseteq Y$. Indeed suppose not, let $y \in X \setminus Y$. Let $x_0 := x, \ldots, x_k, \ldots, x_n = y$ be an ϵ_Y path contained in X. Let ℓ be least index such that $x_\ell \notin Y$. Due to the definition of ϵ_Y, from $x_{\ell-1} \in Y$, we get that $x_\ell \in F_\nu$ and belongs to none of the other components. A contradiction.

¿From this claim the diameter of X is at most d. A contradiction.

Definition 3.8. *Let* $\mathbb{M} := (M; d)$ *be totally Cantor disconnected. Then*

$$d^*(x, y) := \inf\{\epsilon > 0 \mid \text{ there exists an } \epsilon\text{-sequence containing } x \text{ and } y\}.$$

Lemma 3.9. *Let* $\mathbb{M} := (M; d)$ *be totally Cantor disconnected. Then* $\mathbb{M}^* := (M; d^*)$ *is an ultrametric space.*

Proof. Let $x, y, z \in M$ with $d^*(x, y) \geq d^*(x, z) \geq d^*(y, z)$. Then there exists a $d^*(x, z)$-sequence containing x and y. Hence $d^*(x, z) = d^*(x, y)$. □

(See [5], Theorem 1 and Lemma 8.)

Theorem 3.10. *Let* $\mathbb{M} := (M; d)$ *be a countable homogeneous indivisible metric space then* $\mathbb{M}^*$ *is an homogeneous indivisible ultrametric space.*

Proof. Since $\mathbb{M}$ is indivisible it is totally Cantor disconnected, hence d^* is well defined. Since $\mathbb{M}$ is homogeneous then $d(x, y) = d(x', y')$ implies $d^*(x, y) = d^*(x', y')$ for all $x, y, x', y' \in M$. From this property, every local isometry of $\mathbb{M}$ is a local isometry of $\mathbb{M}^*$. Hence, since M is indivisible, M^* is indivisible. Since every automorphism of M is an automorphism of M^*, M^* is point-homogeneous. According to Theorem 2.7, M^* is homogeneous. □

Theorem 3.11. *Let $\mathbb{M}$ be a homogeneous metric space and $V := Spec(\mathbb{M})$. If $\mathbb{M}$ is totally Cantor disconnected and every three element metric space $\mathbb{T}$ with $Spec(\mathbb{T}) \subseteq V$ embeds into $\mathbb{M}$ then the set $V \setminus \{0\}$ is either contained into an interval of the form $[a \rightarrow +\infty)$ for some $a \in \mathbb{R}^+ \setminus \{0\}$ or into an union of intervals of the form $\cup\{[a_{2(n+1)}, a_{2n+1}] : n < \omega\} \cup [a_0 \rightarrow +\infty)$ where $\{a_n : n < \omega\}$ is a sequence such that $a_{2n+1} \leq \frac{a_{2n}}{2}$.*

Proof. **Claim** For every $w \in V^* := Spec(\mathbb{M}^*)$, $]\frac{w}{2}, w[\cap V = \emptyset$.

Suppose the contrary. Pick $r \in]\frac{w}{2}, w[\cap V = \emptyset$. Since $w \in V^*$, we may find x, y such that $d^*(x, y) = w$. Let $n < \omega$ and $\epsilon := 2r$, then there is an ϵ-sequence $x_0, \ldots, x_n$ containing x, y. For $i < n$, let $\mathbb{T}_i := (\{x_i, x_{i+1}, z_i\}, d_i)$ where $d_i(x_i, x_{i+1}) := d(x_i, x_{i+1})$, $d_i(x_i, z_i) = d_i(x_{i+1}) := r$. Each $\mathbb{T}_i$ is a metric space whith spectrum included into V, hence can be isometrically embedded into $\mathbb{M}$. Since M is homegenous, we may suppose that $z_i \in M$ and that the embedding is the inclusion. By adding the z_i''s to the $x_i' s$ we get a r-sequence containing x and y. Since $r < w$ this gives a contradiction.

Since every element of V^* is the infimum of elements of V it also follows that $]\frac{w}{2}, w[\cap V^* = \emptyset$.

Let $\alpha := Inf(V \setminus \{0\})$. If $\alpha \neq 0$ set $a := \alpha$; in this case $V \setminus \{0\} \subseteq [a \rightarrow +\infty)$. If $\alpha = 0$ then, since every element de $V \setminus \{0\}$ majorizes some element of $V^* \setminus \{0\}$ it follows that $Inf(V^* \setminus \{0\}) = 0$ too. Let $\{a_{2n} : n < \omega\}$ be a strictly decreasing sequence of elements of V^* which converges to 0. Set $a_{2n+1} := a_{2n}$. From the Claim $]\frac{a_{2n}}{2}\, a_{2n}[\cap V = \emptyset$, hence $a_{2n+2} \leq a_{2n+1}$. The rest follows.

□

Theorem 3.12. *Every unbounded metric space is divisible.*

Proof. Let $\mathbb{M} := (M; d)$ be an unbounded metric space. Construct a sequence of reals $r_0, r_1, r_2, \ldots$ and a sequence $a_0, a_1, a_2, \ldots$ of elements of M so that:

1. $d(a_0, a_{i+1}) > 2r_i$.
2. $d(a_0, a_{i+1}) + r_i < r_{i+1}$.

Let $r_0 := 1$ and $a_0 \in M$ be arbitrary. Suppose that $\{r_i \mid i \in n\}$ and $\{a_i \mid i \in n\}$ have already be constructed. From the fact that $\mathbb{M}$ is unbounded, we may find $a_n \in M$ be such that $d(a_0, a_n) > 2r_{n-1}$. Next, choose $r_n > d(a_0, a_n) + r_{n-1}$. Note that the set $\{r_i \mid i \in \omega\}$ such constructed is unbounded.

Let

$$\mathcal{E} := \bigcup_{i\in\omega} \mathcal{R}_{a_0}(r_{2i}, r_{2i+1}) \text{ and } \mathcal{O} := \bigcup_{i\in\omega} \mathcal{R}_{a_0}(r_{2i+1}, r_{2i+2}).$$

We prove that there is no isometric embedding of M into $\mathcal{E}$ or into $\mathcal{O}$.

Let f be an isometric embedding of M into M. Let i be minimal so that $d(a_0, f(a_0)) < r_i$ that is $f(a_0) \in \mathcal{R}_{a_0}(r_{i-1}, r_i)$. We have:

$$\begin{aligned} d(a_0, f(a_{i+1})) &\geq d(f(a_0), f(a_{i+1})) - d(a_0, f(a_0)) = \\ &= d(a_0, a_{i+1}) - d(a_0, f(a_0)) > 2r_i - r_i = r_i. \end{aligned}$$

Also:

$$d(a_0, f(a_{i+1})) \leq d(f(a_0), f(a_{i+1})) + d(a_0, f(a_0)) \leq d(a_0, a_{i+1}) + r_i < r_{i+1}.$$

It follows that $f(a_{i+1}) \in \mathcal{R}_{a_0}(r_i, r_{i+1})$. Therefore $f[M]$ intersects both $\mathcal{E}$ and $\mathcal{O}$.

$\square$

4 Divisibility of the bounded Urysohn space

In [10], Hjorth shows that the Urysohn space $\mathbb{U}_{\mathbb{Q}}$ is divisible, and asks whether the corresponding bounded space has the same property. We show that it does, and in fact this generalizes to all such bounded spaces.

Proposition 4.1. *Let V be a countable subset of $\mathbb{R}^+$ containing 0. If for some $r > 0$, $V \cap [0, r]$ is dense in $[0, r]$ then the diameter of each Cantor connected component of $\mathbb{U}_V$ is at least r.*

Proof. Let $a \in U_V$ and $\ell \in V \cap]0, r]$. Let $b \in U$ such that $d(a, b) = \ell$. For any $n \in \omega$ choose successively $a_0, \ldots, a_n$ such that:

$a_0 := \frac{\ell}{n}$, $a_k := \frac{\ell}{n}k + \epsilon_k \in V \cap [\frac{\ell}{n}(k-1), \frac{a_{k-1}}{2}]$ for $1 \leq k < n$ and $a_n := \ell$.

Let $x_0 = a$, $x_n = b$, $x_1, x_2, \ldots, x_{n-1}$ be elements not in U and $X := \{x_0, x_1, x_2, \ldots, x_n\}$. Let $d' : X \times X \to V$ defined by $d'(x_i, x_{i+k}) = a_k$ for $1 \leq k \leq n$ and $d'(x_i, x_i) := 0$. With our choice, $\epsilon_{i+j} \leq \epsilon_i + \epsilon_j$ for all i, j such that $i + j \leq n$, thus $\mathbb{X} := (X, d') \in \mathcal{M}_V$.

Hence we can use the mapping extension property of the Urysohn space and obtain an embedding f of the space $\mathbb{X}$ into U which is the identity map on x_0 and x_n.
Since this can be done for any n we conclude that the Cantor connected component of a contains b, hence its diameter is at least ℓ. Since this holds for every $\ell \in V \cap]0, r]$, this diameter is at least r. $\square$

Corollary 4.2. *The countable homogeneous metric space* $\mathbb{U}_{\mathbb{Q},\leq 1} := (U; d)$ *having all rational numbers less than or equal to one as distances is Cantor connected.*

Theorem 4.3. *Let* V *be a countable subset of* $\mathbb{R}^+$ *containing* 0. *If for some* $r > 0$, $V \cap [0, r]$ *is dense in* $[0, r]$ *then* $\mathbb{U}_V$ *is divisible.*

Proof. Follows from Proposition 4.1 and from Corollary 3.5. $\square$

In particular, we have:

Theorem 4.4. *The Urysohn space* $\mathbb{U}_{\mathbb{Q},\leq 1}$ *is divisible.*

In the remainder of this section, we investigate certain conditions which guarantee that the bounded Urysohn space $\mathbb{U}_{\mathbb{Q},\leq 1}$ isometrically embeds into "large" parts of itself. These give some measure of the indivisibility of the space.

We first wish to extend the notion of an orbit and its socket. Indeed notice that if S is an orbit of the Urysohn space $\mathbb{U}_V = (U; d)$ with socket $B = \{b_i \mid i \in n\}$ and s and t elements in S, then $d(b_i, s) = d(b_i, t)$ for all $i \in n$ because there exists an isometry which fixes B element-wise and maps s to t. Hence we are led to the following definition.

Definition 4.5. *A* distance-socket *(or simply* d-socket*)* $\mathfrak{B}$ *of* $\mathbb{U}_V$ *is a sequence of the form*

$$\langle (b_0, d_0), (b_1, d_1), \ldots, (b_{n-1}, d_{n-1}) \rangle$$

so that for all $i, j \in n$:

1. $b_i \in U$ *and* $d_i \in V$.
2. $d_i + d_j \geq d(b_i, b_j)$.
3. $d_i + d(b_i, b_j) \geq d_j$.

The set vert($\mathfrak{B}$) *of* vertices *of* $\mathfrak{B}$ *is the set* $\{b_0, b_1, \dots, b_{n-1}\}$*, and the set of* distances *of* $\mathfrak{B}$ *is the set* $\{d_0, d_1, \dots, d_{n-1}\}$*.*

An orbit therefore naturaly gives rise to a corresponding socket and d-socket. But it also follows that given a d-socket

$$\mathfrak{B} = \langle (b_0, d_0), (b_1, d_1), \dots, (b_{n-1}, d_{n-1}) \rangle,$$

the set of all $s \in U$ so that $d(s, b_i) = d_i$ for all $s \in S$ and $i \in n$ is an orbit of $\mathbb{U}_V$ with socket $B = \{b_i \mid i \in n\}$ (Conditions 1. 2. and 3. of the definition ensure that the set is not empty).

We first show that under certain conditions an orbit itself contains an isometric copy of the bounded Urysohn space.

Lemma 4.6. *Let* S *be an orbit of the Urysohn space* $\mathbb{U}_V = (U; d)$ *with corresponding d-socket* $\mathfrak{B} = \langle (b_0, d_0), (b_1, d_1), \dots, (b_{n-1}, d_{n-1}) \rangle$*.*

If $\ell := \min\{d_i \mid i \in n\}$*, then the metric subspace of* $\mathbb{U}_V$ *induced by* S *is an isometric copy of the Urysohn space* $\mathbb{U}_{V \cap [0, 2\ell]}$*.*

Proof. Let $i \in n$ be such that $\ell = d_i$. Then $d(s, b_i) = \ell$ for every element $s \in S$ and hence it follows from the triangle inequality that $d(s, t) \leq 2\ell$ for any two elements s and t of S.

Let $\mathrm{F} := (F, d')$ be an element in the age of $\mathbb{U}_{V \cap [0, 2\ell]}$ so that $F \cap U \subseteq S$ and the metric subspace of $\mathbb{U}_{V \cap [0, 2\ell]}$ induced by $F \cap S$ is equal to the metric subspace of F induced by $F \cap S$. According to the mapping extension property, it suffices to show that there exists an embedding of F into S.

Let $\mathrm{G} := (\{b_i \mid i \in n\} \cup F, \overline{d})$ be the metric space for which $\overline{d}$ agrees with d on $F \cap U$ and $\overline{d}$ agrees with d' on $F \setminus U$, and $\overline{d}(x, b_i) = d_i$ for all $x \in F \setminus S$ and all $i \in n$. The function $\overline{d}$ satisfies the triangle inequality and hence G is an element of the age of $\mathbb{U}_V$.
It follows from the mapping extension property of $\mathbb{U}_V$ that there exists an embedding f of G into U which fixes the elements of $G \cap U$. It follows from the condition that $\overline{d}(x, b_i) = d_i$ for all $x \in F \setminus S$ and all $i \in n$, that the elements of F are mapped by f into S. $\square$

If V' is an initial segment of V then $\mathbb{U}'_V$ embeds into $\mathbb{U}_V$. Hence, if we compare orbits of an Urysohn space by isometric embedding, it follows from Lemma 4.6 above that they form a chain. This is important as you may recall (see [1] and [9]) that a necessary condition for indivisbility is that the ages of the orbits of an homogeneous structure H form a chain.

Corollary 4.7. *The orbits of an Urysohn metric space form a chain.*

Corollary 4.8. *Let S be an orbit of the Urysohn space* $\mathbb{U}_{V\cap[0,\ell]} := (U; d)$ *with corresponding d-socket* $\mathfrak{B} = \langle (b_0, d_0), (b_1, d_1), \ldots, (b_{n-1}, d_{n-1})\rangle$.

If $\frac{\ell}{2} \leq \min\{d_i \mid i \in n\}$*, then the metric subspace of* $\mathbb{U}_{V\cap[0,\ell]}$ *induced by S is an isometric copy of the Urysohn space* $\mathbb{U}_{V\cap[0,\ell]}$*. In particular if* $\ell' \geq \frac{\ell}{2}$ *and* $a \in U$*, then the subspace of* $\mathbb{U}_{V\cap[0,\ell]}$ *induced by the set* $\{x \in U \mid d(a, x) = \ell'\}$ *is an isometric copy of* $\mathbb{U}_{V\cap[0,\ell]}$.

¿From now on we consider the Urysohn space $\mathbb{U}_{\mathbb{Q},\leq 1} = (U; d)$. From Lemma 4.6 we get.

Corollary 4.9. *Let S be an orbit of the Urysohn space* $\mathbb{U}_{\mathbb{Q},\leq 1}$ *so that the subspace of* $\mathbb{U}_{\mathbb{Q},\leq 1}$ *induced by S has diameter ℓ.*
Then if $r \leq \ell$ is any positive rational and $x \in S$, then there exists an element $y \in S$ with $d(x, y) = r$.

There are certain small subsets of the Urysohn space that can be avoided by an isometrical embedding.

Definition 4.10. *Let X be a subset of the base set U of the Urysohn space* $\mathbb{U}_{\mathbb{Q},\leq 1}$*. An element $x \in X$ is called* semi-isolated in X *if for every $\epsilon > 0$ there exists a positive rational number $r < \epsilon$ so that* $\{y \in U \mid d(x, y) = r\} \cap X = \emptyset$.

Lemma 4.11. *Let X be a subset of the base set U of the Urysohn space* $\mathbb{U}_{\mathbb{Q},\leq 1}$ *and I the set of semi-isolated points of X in X.*
If $S \subset X$ is an orbit of $\mathbb{U}_{\mathbb{Q},\leq 1}$ *with socket disjoint from I, then S itself is also disjoint from I.*

Proof. Let $S \subset X$ be an orbit of $\mathbb{U}_{\mathbb{Q},\leq 1}$ with socket disjoint from I.
According to Lemma 4.6, there is a number ℓ so that the metric space induced by S is the space $\mathbb{U}_{\mathbb{Q},\leq \ell}$.
If there exists $s \in S \cap I$, then s is a semi-isolated point of X. But by Corollary 4.9 for each $r \leq \ell$ there is a $t \in S$ at distance exactly r, so $s \notin I$, a contradiction. This completes the proof. □

Corollary 4.12. *Let X be a subset of the base set U of the Urysohn space* $\mathbb{U}_{\mathbb{Q},\leq 1}$*, and I the set of semi-isolated points of X in X.*
Then $U \setminus I$ induces an isometric copy of $\mathbb{U}_{\mathbb{Q},\leq 1}$.

Proof. This follows from Lemma 4.11 and Point 9 of Section 1.1. □

Corollary 4.13. *Let X be a subset of the base set U of the Urysohn space $\mathbb{U}_{\mathbb{Q},\leq 1}$ so that for every $\epsilon > 0$ there exists a positive rational r which is not in the spectrum of the metric subspace of $\mathbb{U}_{\mathbb{Q},\leq 1}$ induced by X.*
Then $U \setminus X$ induces an isometric copy of $\mathbb{U}_{\mathbb{Q},\leq 1}$.

Theorem 4.14. *Let X be a subset of the base set U of the Urysohn space $\mathbb{U}_{\mathbb{Q},\leq 1}$. Then there exists an isometric copy V of U in U for which $V \cap X$ does not contain any semi-isolated points in $V \cap X$.*

Proof. For any subset $W \subseteq U$, denote by I_W the set of semi-isolated points of $X \cap W$ in $X \cap W$.
We construct a sequence W_α of subsets of U by first defining $W_0 := U$. For a successor ordinal $\beta = \alpha + 1$, define $W_\beta = W_\alpha \setminus I_{W_\alpha}$, and if β is a limit ordinal, then define $W_\beta := \bigcap_{\alpha \in \beta} W_\alpha$.
Notice that $U \setminus X \subseteq W_\alpha$ for any α.

The process has to end when $W'_\beta = W_\beta := V$ for some ordinal β, and fix β minimal with this condition. We show by induction that the substructure of $\mathbb{U}_{\mathbb{Q},\leq 1}$ induced by W_α is an isomorphic copy of $\mathbb{U}_{\mathbb{Q},\leq 1}$ for every $\alpha \leq \beta$. Corollary 4.12 shows this is the case for all successor ordinals. Let $\alpha \leq \beta$ therefore be a limit ordinal. Using point 5. of the list of properties of homogeneous structures, we have to show that every orbit of $\mathbb{U}_{\mathbb{Q},\leq 1}$ with socket in W_α contains an element of W_α. So let S be an orbit of $\mathbb{U}_{\mathbb{Q},\leq 1}$ with socket in W_α and diameter ℓ. Assuming for a contradiction that S does not contain an element of W_α, fix $s \in S$. There is an $\delta < \alpha$ so that $s \in S \cap W_\delta$ and $s \notin S \cap W_{\delta+1}$. Hence S is a semi-isolated point of $X \cap W_\delta$ in $X \cap W_\delta$. Hence there is a positive rational $r < \ell$ so that $\{y \in W_\delta \mid d(s,y) = r\} \cap X \cap W_\delta = \emptyset$. There is an element $y \in S$ with $d(s,y) = r$ and hence $y \notin X$ is an element of S. It follows from the Remarkcomment above that $y \in W_\alpha$. $\square$

Our final goal is to show that an isometric embedding can avoid a set containing elements close to a sequence of relatively far elements.

Lemma 4.15. *Let $\mathfrak{B} = \langle (b_0, d_0), (b_1, d_1), \ldots, (b_{n-1}, d_{n-1}) \rangle$ be a d-socket with orbit S. Let $a \in U$, $r \in \mathbb{Q} \cap [0,1]$, and $x \in S$ such that $d(a,x) \leq r$.*
Then if $d(a, b_i) \geq r$ for all $i \in n$, there exists an element $y \in S$ with $d(a,y) = r$.

Proof. It suffices to show that $\mathfrak{B}' := \langle (b_0, d_0), (b_1, d_1), \ldots, (b_{n-1}, d_{n-1})(a, r) \rangle$ is again a d-socket.
To fulfill the requirements of Definition 4.5, it remains to verify that for every $i \in n$ the following inequalities hold:

1. $r + d_i \geq d(a, b_i)$.
2. $d_i + d(a, b_i) \geq r$.
3. $r + d(a, b_i) \geq d_i$.

But by the triangle inequality with x we have $d(a, b_i) \leq d(a, x) + d_i \leq r + d_i$. Trivially we have $r \leq d(a, b_i) \leq d_i + d(a, b_i)$, and finally $d_i \leq d(a, x) + d(a, b_i) \leq r + d(a, b_i)$. □

Theorem 4.16. *Let* $\mathbb{U}_{\mathbb{Q},\leq 1} = (U; d)$ *be the bounded Urysohn space. Fix* $R := \{r_i \mid i \in \omega\}$ *a set of rationals in the interval* $(0, 1]$, $A := \{a_i \mid i \in \omega\}$ *a subset of* U *so that* $d(a_i, a_j) \geq r_i + r_j$ *for all* $i, j \in \omega$ *with* $i \neq j$, *and let*

$$X := \bigcup_{i \in \omega} \{x \mid d(a_i, x) < r_i\}.$$

Then the metric subspace of $\mathbb{U}_{\mathbb{Q},\leq 1}$ *induced by* $U \setminus X$ *is an isometric copy of* $\mathbb{U}_{\mathbb{Q},\leq 1}$.

Proof. Notice that if y is any element at distance from some a_i greater than or equal to r_i for some $i \in \omega$, then $y \in U \setminus X$.
Let S be an orbit of $\mathbb{U}_{\mathbb{Q},\leq 1}$ with socket $F \subseteq U \setminus X$. Given Point 9 of Section 1.1, let us check that S meets $U \setminus X$. Let $s \in S$. If $s \notin X$ then there is nothing to prove. Otherwise there exists an $i \in \omega$ so that $d(a_i, s) < r_i$. But then it follows from Lemma 4.15 that there is an element $y \in S$ with $d(a_i, y) = r_i$, and hence $y \in U \setminus X$. This completes the proof. □

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