THE URYSOHN SPHERE IS OSCILLATION STABLE.

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Abstract. We solve the oscillation stability problem for the Urysohn sphere, an analog of the distortion problem for ℓ_2 in the context of the Urysohn space $\mathbf{U}$. This is achieved by solving a purely combinatorial problem involving a family of countable homogeneous metric spaces with finitely many distances.

1. Introduction.

This note can be seen as the continuation of the paper [11] where a combinatorial approach is proposed for an analog of the distortion problem for ℓ_2. This latter problem can be formulated as follows: Let $\mathbb{S}^\infty$ denote the unit sphere of the Hilbert space ℓ_2. Is it true that if $\varepsilon > 0$ and $f : \mathbb{S}^\infty \longrightarrow \mathbb{R}$ is bounded and uniformly continuous, then there is a closed infinite-dimensional subspace V of ℓ_2 such that

$$\sup\{|f(x) - f(y)| : x, y \in V \cap \mathbb{S}^\infty\} < \varepsilon?$$

Equivalently, for a metric space $\mathbf{X} = (X, d^{\mathbf{X}})$, a subset $Y \subset X$ and $\varepsilon > 0$, let

$$(Y)_\varepsilon = \{x \in X : \exists y \in Y \quad d^{\mathbf{X}}(x, y) \leqslant \varepsilon\}.$$

Then the distortion problem for ℓ_2 asks: Given a finite partition γ of $\mathbb{S}^\infty$, is there always $A \in \gamma$ such that $(A)_\varepsilon$ includes an isometric copy of $\mathbb{S}^\infty$? That problem appeared in the early seventies when Milman's work led to the following reformulation of Dvoretzky's theorem: for $N \in \omega$ strictly positive, let $\mathbb{S}^N$ denote the unit sphere of the $(N+1)$-dimensional Euclidean space. Then:

Theorem (Milman [12]). *Let γ be a finite partition of $\mathbb{S}^\infty$. Then for every $\varepsilon > 0$ and every $N \in \omega$, there is $A \in \gamma$ and an isometric copy $\widetilde{\mathbb{S}}^N$ of $\mathbb{S}^N$ in $\mathbb{S}^\infty$ such that $\widetilde{\mathbb{S}}^N \subset (A)_\varepsilon$.*

In that context, the distortion problem for ℓ_2 really asks whether this result has an infinite dimensional analog. It is only a long time after Milman's theorem was established that the distortion problem for ℓ_2 was solved by Odell and Schlumprecht in [14]:

Theorem (Odell-Schlumprecht [14]). *There is a finite partition γ of $\mathbb{S}^\infty$ and $\varepsilon > 0$ such that no $(A)_\varepsilon$ for $A \in \gamma$ includes an isometric copy of $\mathbb{S}^\infty$.*

This result is traditionally stated in terms of the Banach space theoretic concept of oscillation stability, but can also be stated thanks to a new concept of oscillation

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stability for topological groups due to Kechris, Pestov and Todorcevic and introduced in [8] (cf [16] for a detailed exposition). In this latter formalism, the theorem of Odell and Schlumprecht is equivalent to the fact that the standard action of $\mathrm{iso}(\mathbb{S}^\infty)$ on $\mathbb{S}^\infty$ is not oscillation stable. On the other hand, in the context of isometry groups of complete separable homogeneous metric spaces, oscillation stability for topological groups coincides with the Ramsey-theoretic concept of approximate indivisibility. Recall that a metric space is called *homogeneous* (or sometimes *ultrahomogeneous*) when every isometry between finite metric subspaces of $\mathbf{X}$ can be extended to an isometry of $\mathbf{X}$ onto itself. For $\varepsilon \geqslant 0$, call a metric space $\mathbf{X}$ *ε-indivisible* when for every strictly positive $k \in \omega$ and every $\chi : \mathbf{X} \longrightarrow k$, there is $i < k$ and $\widetilde{\mathbf{X}} \subset \mathbf{X}$ isometric to $\mathbf{X}$ such that

$$\widetilde{\mathbf{X}} \subset (\overleftarrow{\chi}\{i\})_\varepsilon.$$

Then $\mathbf{X}$ is *approximately indivisible* when $\mathbf{X}$ is ε-indivisible for every $\varepsilon > 0$, and $\mathbf{X}$ is *indivisible* when $\mathbf{X}$ is 0-indivisible.

Using this terminology, the theorem of Odell and Schlumprecht states that the sphere $\mathbb{S}^\infty$ is not approximately indivisible. However, because the proof is not based on the intrinsic geometry of ℓ_2, the impression somehow persists that something is still missing in our understanding of the metric structure of $\mathbb{S}^\infty$. That fact was one of the motivations for [11] as well as for the present paper: our hope is that understanding the indivisibility problem for another remarkable space, namely the Urysohn sphere $\mathbf{S}$, will help to reach a better grasp of $\mathbb{S}^\infty$. The space $\mathbf{S}$ is defined as follows: Up to isometry, it is the unique complete separable homogeneous metric space with diameter 1 into which every separable metric space with diameter less or equal to 1 embeds isometrically. Equivalently, it is also the sphere of radius 1/2 in the so-called universal Urysohn space $\mathbf{U}$ constructed in 1927 by Urysohn.

Apart from the fact that both $\mathbb{S}^\infty$ and $\mathbf{S}$ are complete, separable and homogeneous, the study of $\mathbf{S}$ is believed to be relevant for the distortion problem for ℓ_2 because, from a Ramsey-theoretic point of view, the spaces $\mathbb{S}^\infty$ and $\mathbf{S}$ behave in a very similar way. For example, the following analog of Milman's theorem holds for $\mathbf{S}$:

Theorem (Pestov [15])**.** *Let γ be a finite partition of $\mathbf{S}$. Then for every $\varepsilon > 0$ and every compact $K \subset \mathbf{S}$, there is $A \in \gamma$ and an isometric copy $\widetilde{K}$ of K in $\mathbf{S}$ such that $\widetilde{K} \subset (A)_\varepsilon$.*

In fact, since the work of Gromov and Milman [7] and of Pestov [15], it is known that this analogy is only the most elementary form of a very general Ramsey-theoretic theorem. It is also known that this latter result has a very elegant reformulation at the level of the surjective isometry groups $\mathrm{iso}(\mathbb{S}^\infty)$ and $\mathrm{iso}(\mathbf{S})$ (seen as topological groups when equipped with the pointwise convergence topology). Call a topological group G *extremely amenable* when every continuous action of G on a compact space admits a fixed point. Then on the one hand:

Theorem (Gromov-Milman [7])**.** *The group* $\mathrm{iso}(\mathbb{S}^\infty)$ *is extremely amenable.*

While on the other hand:

Theorem (Pestov [15])**.** *The group* $\mathrm{iso}(\mathbf{S})$ *is extremely amenable.*

Actually, even more is known as both $\mathrm{iso}(\mathbb{S}^\infty)$ and $\mathrm{iso}(\mathbf{S})$ are known to satisfy the so-called *Lévy property* (cf Lévy [10] for $\mathrm{iso}(\mathbb{S}^\infty)$ and Pestov [17] for $\mathrm{iso}(\mathbf{S})$), a property shown to be stronger than extreme amenability by Gromov and Milman in [7]. In this note, we prove that:

Theorem 1. *The Urysohn sphere* $\mathbf{S}$ *is approximately indivisible.*

In other words, for every finite partition γ of $\mathbf{S}$ and $\varepsilon > 0$, there is $A \in \gamma$ such that $(A)_\varepsilon$ includes an isometric copy of $\mathbf{S}$. Or equivalently, in terms of oscillation stability for topological groups, the standard action of $\mathrm{iso}(\mathbf{S})$ on $\mathbf{S}$ is oscillation stable. Theorem 1 therefore exhibits an essential Ramsey-theoretic distinction between $\mathbb{S}^\infty$ and $\mathbf{S}$. At the level of $\mathrm{iso}(\mathbb{S}^\infty)$ and $\mathrm{iso}(\mathbf{S})$, it highlights a deep topological difference which, for the reasons mentioned previously, was not at all apparent until now.

Our proof here is combinatorial and rests on a discretization method largely inspired from the proof by Gowers in [6] of the stabilization theorem for the positive sphere of c_0. More precisely, it builds on the following result proved in [11] and involving a family $(\mathbf{U}_m)_{m\geqslant 1}$ of countable metric spaces. For $m \geqslant 1$, the space $\mathbf{U}_m$ is defined as follows: up to isometry it is the unique countable homogeneous metric space with distances in $\{1, \ldots, m\}$ into which every countable metric space with distances in $\{1, \ldots, m\}$ embeds isometrically. Then:

Theorem (Lopez-Abad - Nguyen Van Thé [11])**.** *The following are equivalent:*

(i) $\mathbf{S}$ *is oscillation stable (equivalently, approximately indivisible).*
(ii) For every strictly positive $m \in \omega$, $\mathbf{U}_m$ *is indivisible.*

For an explanation as of why the spaces $\mathbf{U}_m$ are relevant, see [11] or [13]. In the present paper, we show that:

Theorem 2. *Let* $m \in \omega$, $m \geqslant 1$*. Then* $\mathbf{U}_m$ *is indivisible.*

The basic methods used in the proof of our results have been developed in the sequence of papers [2], [3], [19], [4], [20] dealing with partition results of so-called countable homogeneous structures with free amalgamation. However, because the spaces $\mathbf{U}_m$ do not enter the framework provided by free amalgamation, substantial modifications were needed to prove Theorem 2.

The paper is organized as follows. Section 2 corresponds to a short presentation of the partition theory of countable homogeneous structures with free amalgamation. In section 3, the essential ingredients, the main technical results (Lemma 2 and Lemma 3) as well as the general outline of the proof of Theorem 2 are presented. Finally, the proof of Lemma 2 is presented in section 4.

2. Partition theory of countable homogeneous structures with free amalgamation.

In this section, we present a brief outline of the general theory on which the proof of Theorem 2 is based. For the undefined notions and for a general introduction to the partition theory of countable homogeneous structures see [18]. As mentioned in the introduction, the essential techniques were developed in a series of papers dealing with vertex partition results of countable homogeneous structures with free amalgamation starting with [2] (The partition theory for sets of substructures other than vertices is much more complicated, see [9] and [21]). In [3] it is proven that if a countable homogeneous structure is indivisible then the stabilizers of finite subsets form a chain, which in the binary case is a chain under embedding. This then led to [19] in which it is shown, in the case of directed graphs, that if the partial order of the stabilizers is finite then the Ramsey degree is equal to its maximal antichain. In [4] the finiteness condition was removed in the case that the partial order is a chain. [20] contains the most general result from which it follows that the Ramsey degree of a binary countable homogeneous structure with free amalgamation is equal to the size of the maximal antichain of the partial order of finite set stabilizers under embedding if this partial order is finite. Hence if this partial order is a chain then the homogeneous structure is indivisible.

It is not difficult to see that in the case of metric spaces the partial order of stabilizers of finite subsets forms a chain under isometric embedding and if $m \leqslant 3$ that $\mathbf{U}_m$ has free amalgamation. Hence it follows from [20] and actually also from [19], with a not too difficult modification of the proof, that if $m \leqslant 3$ then $\mathbf{U}_m$ is indivisible. This allowed to prove that $\mathbf{S}$ is 1/6-indivisible in [11]. However, if $m > 3$ then $\mathbf{U}_m$ has strong amalgamation but not free amalgamation. This fact complicates the situation considerably and requires some more elaborate arguments at some stages of the proof showing that the metric spaces $\mathbf{U}_m$ are indivisible.

3. Notations and definitions.

In this section, we present the notions and objects that will play a central role throughout the paper.

3.1. **Katětov maps and orbits.** Given a metric space $\mathbf{X} = (X, d^{\mathbf{X}})$, a map $f : X \longrightarrow]0, +\infty[$ is *Katětov over* $\boldsymbol{X}$ when

$$\forall x, y \in X, \quad |f(x) - f(y)| \leqslant d^{\mathbf{X}}(x, y) \leqslant f(x) + f(y).$$

Equivalently, one can extend the metric $d^{\mathbf{X}}$ to $X \cup \{f\}$ by defining, for every x, y in X, $\widehat{d^{\mathbf{X}}}(x, f) = f(x)$ and $\widehat{d^{\mathbf{X}}}(x, y) = d^{\mathbf{X}}(x, y)$. The corresponding metric space is then written $\mathbf{X} \cup \{f\}$. The set of all Katětov maps over $\mathbf{X}$ is written $E(\mathbf{X})$. For a metric subspace $\mathbf{X}$ of $\mathbf{Y}$, a Katětov map $f \in E(\mathbf{X})$ and a point $y \in \mathbf{Y}$, then y *realizes* f *over* $\boldsymbol{X}$ if

$$\forall x \in \mathbf{X} \quad d^{\mathbf{Y}}(y, x) = f(x).$$

The set of all $y \in \mathbf{Y}$ realizing f over $\mathbf{X}$ is then written $O(f, \mathbf{Y})$ and is called the *orbit of* f *in* $\boldsymbol{Y}$. When $\mathbf{Y}$ is clear from the context, the set $O(f, \mathbf{Y})$ is simply written $O(f)$. Here, the concepts of Katětov map and orbit are relevant because of the following standard reformulation of the notion of homogeneity, which will be used extensively in the sequel:

Lemma 1. *Let* $\boldsymbol{X}$ *be a countable metric space. Then* $\boldsymbol{X}$ *is homogeneous iff for every finite subspace* $\boldsymbol{F} \subset \boldsymbol{X}$ *and every Katětov map* f *over* $\boldsymbol{F}$*, if* $\boldsymbol{F} \cup \{f\}$ *embeds into* $\boldsymbol{X}$*, then* $O(f, \boldsymbol{X}) \neq \emptyset$.

For a proof of that fact in the general context of relational structures, see for example [5]. For a proof in the particular context of metric spaces, see [13].

3.2. $\mathbb{P}$ **and** ψ_i**.** For metric spaces $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, write $\mathbf{X} \cong \mathbf{Y}$ if there is an isometry from $\mathbf{X}$ onto $\mathbf{Y}$ and define the set $\binom{\mathbf{Z}}{\mathbf{X}}$ as

$$\binom{\mathbf{Z}}{\mathbf{X}} = \{\widetilde{\mathbf{X}} \subset \mathbf{Z} : \widetilde{\mathbf{X}} \cong \mathbf{X}\}.$$

Let q be a fixed strictly positive integer and $\chi : \mathbf{U}_q \longrightarrow \{0, 1\}$.

Definition 1. *The set $\mathbb{P}$ is the set of all ordered pairs of the form $p = (f_p, \boldsymbol{C}_p)$ where*

(i) $\boldsymbol{C}_p \in \binom{\boldsymbol{U}_q}{\boldsymbol{U}_q}$.

(ii) f_p *is a map with finite domain* $\mathrm{dom} f_p \subset \boldsymbol{C}_p$ *and with values in* $\{1, \ldots, q\}$.

(iii) $f_p \in E(\mathrm{dom} f_p)$, *ie* f_p *is Katětov on its domain.*

The set $\mathbb{P}$ is partially ordered by the relation $\leqslant$ defined by

$$\forall p, r \in \mathbb{P} \quad r \leqslant p \leftrightarrow (\mathrm{dom} f_p \subset \mathrm{dom} f_r \subset \boldsymbol{C}_r \subset \boldsymbol{C}_p \ \text{ and } \ f_r \restriction \mathrm{dom} f_p = f_p)\,.$$

Finally, if $k \in \omega$, then $r \leqslant_k p$ stands for

$$r \leqslant p \ \text{ and } \ \min f_r = \max(\min f_p - k, 1).$$

Observe that if $p \in \mathbb{P}$, then the homogeneity of $\mathbf{U}_q$ ensures that the set $O(f_p, \mathbf{C}_p)$ is not empty. Observe also that there is $r \in \mathbb{P}$ such that $r \leqslant_1 p$.

Definition 2. *Let $i \in \{0, 1\}$, $p \in \mathbb{P}$. The formula $\psi_i(p)$ is defined recursively by:*
If $\min f_p = 1$, *then $\psi_i(p)$ holds iff*

$$\forall r \leqslant_0 p \ \ |O(f_r) \cap \boldsymbol{C}_r \cap \overleftarrow{\chi}\{i\}| = \aleph_0.$$

If $\min f_p > 1$, *then $\psi_i(p)$ holds iff*

$$\forall r \leqslant_0 p \ \ \exists s \leqslant_1 r \quad \psi_i(s).$$

The idea behind the definition of ψ_i is that if $\psi_i(p)$ holds, then inside $\mathbf{C}_p$ the map χ should take the value i on a substantial part of the orbit of f_p. This intuition is made precise by the following Lemma 2:

Lemma 2. *Let $m, q \in \omega$ with $q \geqslant m$ and $i \in \{0, 1\}$. Let f be a Katětov map with finite domain* $\mathrm{dom} f \subset \boldsymbol{U}_q$, *and* $F \subset \mathrm{dom} f$ *such that* $\min f \restriction F = \min f = m$. *Assume that $\psi_i(f, \boldsymbol{U}_q)$. Then there exists an isometric copy $\boldsymbol{C}$ of $\boldsymbol{U}_q$ inside $\boldsymbol{U}_q$ such that:*

(i) $\mathrm{dom} f \cap \boldsymbol{C} = F$.

(ii) $O(f \restriction F) \cap \boldsymbol{C} = O(f) \cap \boldsymbol{C} \subset \overleftarrow{\chi}\{i\}$.

In words, Lemma 2 means that up to a change of the ambient space, it is possible to ensure that the whole orbit of f has color i. The requirement $\min f \restriction F = \min f$ guarantees that the orbit of f in the new ambient space has the same metric structure as the orbit of f in the original space. The proof of Lemma 2 represents the core of the proof of Theorem 2 and is detailed in section 4. The second crucial fact about $\mathbb{P}$ and ψ_i lies in:

Lemma 3. *Let $p \in \mathbb{P}$ be such that $\neg\psi_i(p)$. Then there is $r \leqslant_0 p$ such that $\psi_{1-i}(r)$ holds.*

Proof. We proceed by induction on $\min f_p$. If $\min f_q = 1$, then there is $r \leqslant_0 p$ such that

$$|O(f_r) \cap \mathbf{C}_r \cap \overleftarrow{\chi}\{i\}| < \aleph_0.$$

It is then clear that $\psi_{1-i}(r)$ holds. On the other hand, if $\min f_p > 1$, then there is $r \leqslant_0 p$ such that

$$\forall s \leqslant_1 r \quad \neg\psi_i(s).$$

We claim that $\psi_{1-i}(r)$ holds: Let $s \leqslant_0 r$. We want to find $t \leqslant_1 s$ such that $\psi_{1-i}(t)$ holds. Let $u \leqslant_1 s$. Then $u \leqslant_1 r$ and it follows that $\neg\psi_i(u)$. By induction hypothesis, since $\min f_u < \min f_r = \min f_q$ there is $t \leqslant_0 u$ such that $\psi_{1-i}(t)$ holds. Additionnally $t \leqslant_1 s$. Thus t is as required. $\square$

When combined, Lemma 2 and Lemma 3 lead to Theorem 2 as follows: Take $m = q$. According to Lemma 3, there are $i \in \{0,1\}$ and f Katětov with finite domain $\mathrm{dom} f \subset \mathbf{U}_q$ and $\min f = q$ such that $\psi_i(f, \mathbf{U}_q)$. Applying Lemma 2 for $F = \emptyset$, we obtain a copy $\mathbf{C}$ of $\mathbf{U}_q$ inside $\mathbf{U}_q$ such that $\mathrm{dom} f \subset \mathbf{C}$ and $O(f) \cap \mathbf{C} \subset \overleftarrow{\chi}\{i\}$. Observe that $O(f) \cap \mathbf{C}$ is isometric to $\mathbf{U}_q$. $\square$

The remaining part of this article is therefore devoted to a proof of Lemma 2.

4. Proof of Lemma 2.

From now on, the integer q is a fixed strictly positive integer and $\chi : \mathbf{U}_q \longrightarrow \{0,1\}$. We proceed by induction on m and prove that for every strictly positive $m \in \omega$ with $m \leqslant q$ the following statement $\mathcal{H}_m$ holds:

$\mathcal{H}_m$: "For every Katětov map f with finite domain $\mathrm{dom} f \subset \mathbf{U}_q$, every $F \subset \mathrm{dom} f$ such that $\min f \restriction F = \min f = m$ and every $i \in \{0,1\}$, if $\psi_i(f, \mathbf{U}_q)$ holds, then there exists an isometric copy $\mathbf{C}$ of $\mathbf{U}_q$ inside $\mathbf{U}_q$ such that:

(i) $\mathrm{dom} f \cap \mathbf{C} = F$.
(ii) $O(f \restriction F) \cap \mathbf{C} = O(f) \cap \mathbf{C} \subset \overleftarrow{\chi}\{i\}$."

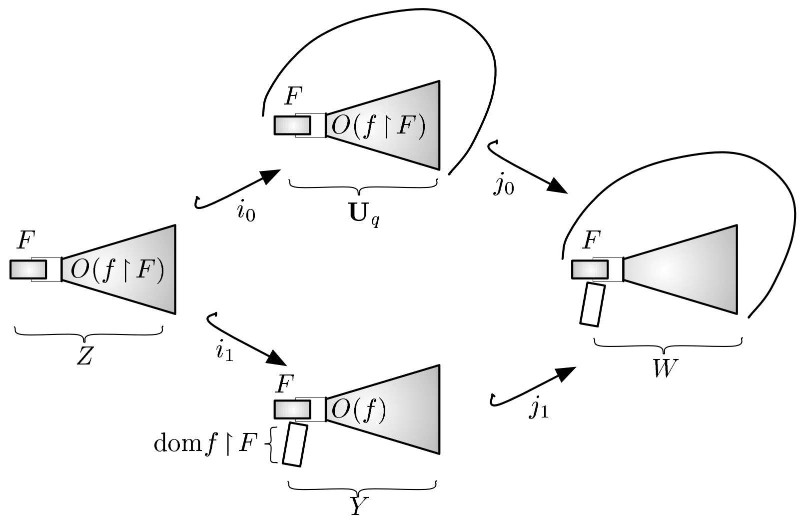
4.1. **Proof of Lemma 2 for $m = 1$.** The idea is to obtain $\mathbf{C}$ by reconstituting an isometric copy of $\mathbf{U}_q$ around $O(f) \cap \overleftarrow{\chi}\{i\}$. This is achieved by considering an enumeration $\{x_n : n \in \omega\}$ of $\mathbf{U}_q$ admitting F as an initial segment, and by constructing $\{\varphi(x_n) : n \in \omega\}$ inductively such that φ is distance preserving. At each step, we make sure that if x_n realizes $f \restriction F$ over F, then $\varphi(x_n)$ realizes f over $\mathrm{dom} f$ and is such that $\chi(\varphi(x_n)) = i$. Thus, even though at the end of the construction $\mathrm{dom} f$ will not be entirely included in $\mathbf{C}$, we will still need to have control of the distances of the elements of $\mathbf{C}$ with respect to those of the whole domain of f, and not only $F = \mathrm{dom} f \cap \mathbf{C}$. When x_n realizes $f \restriction F$ over F, this is not a problem as f tells us what the distances between $\varphi(x_n)$ and the elements of $\mathrm{dom} f \smallsetminus F$ should be. However, if x_n does not realize $f \restriction F$ over F, then f seems to provide no information of that sort. Still, we need to make sure that the distances that $\varphi(x_n)$ takes with respect to the elements of $\mathrm{dom} f \smallsetminus F$ will not block us at a later stage of the construction. This issue is solved thanks to the following preliminary construction:

Let $\mathbf{Y}$ and $\mathbf{Z}$ be the metric subspaces of $\mathbf{U}_q$ supported by $\mathrm{dom} f \cup O(f)$ and $F \cup O(f \restriction F)$ respectively. Let $i_0 : \mathbf{Z} \longrightarrow \mathbf{U}_q$ be the isometric embedding provided by the identity. Observe that because $\min f \restriction F = \min f$, the space $\mathbf{Z}$ is isometric to $F \cup O(f)$ via an isometry ϕ that fixes F. We can therefore consider the metric space $\mathbf{W}$ obtained by gluing $\mathbf{U}_q$ and $\mathbf{Y}$ via an identification of $\mathbf{Z} \subset \mathbf{U}_q$ and $F \cup O(f) \subset \mathbf{Y}$. The space $\mathbf{W}$ really looks like $\mathbf{U}_q$ with an extra piece isometric to $\mathrm{dom} f \smallsetminus F$ sticking out of it.

Formally, the space $\mathbf{W}$ can be constructed as follows: The isometry ϕ provides an isometric embedding $j_0 : \mathbf{Z} \longrightarrow \mathbf{Y}$ that fixes F. Using strong amalgamation for countable metric spaces with distances in $\{1, \ldots, q\}$, we can find a countable metric space $\mathbf{W}$ and isometric embeddings $i_1 : \mathbf{U}_q \longrightarrow \mathbf{W}$ and $j_1 : \mathbf{Y} \longrightarrow \mathbf{W}$ such that $i_1 \circ i_0 = j_1 \circ j_0$, $\mathbf{W} = i_1'' \mathbf{U}_q \cup j_1'' \mathbf{Y}$, $i_1'' \mathbf{U}_q \cap j_1'' \mathbf{Y} = (i_1 \circ i_0)'' \mathbf{Z} = (j_1 \circ j_0)'' \mathbf{Z}$, and for every $x \in \mathbf{U}_q$ and $y \in \mathbf{Y}$:

$$\begin{aligned} d^{\mathbf{W}}(i_1(x), j_1(y)) &= \min\{d^{\mathbf{W}}(i_1(x), i_1 \circ i_0(z)) + d^{\mathbf{W}}(j_1 \circ j_0(z), j_1(y)) : z \in \mathbf{Z}\} \\ &= \min\{d^{\mathbf{U}_q}(x, i_0(z)) + d^{\mathbf{Y}}(j_0(z), y) : z \in \mathbf{Z}\} \\ &= \min\{d^{\mathbf{U}_q}(x, z) + d^{\mathbf{Y}}(j_0(z), y) : z \in \mathbf{Z}\}. \end{aligned}$$

Observe that in $\mathbf{W}$, every $x \in i_1''\mathbf{U}_q$ realizing $f \restriction F$ over $i_1''F$ also realizes f over $j_1''\mathrm{dom}f$.



Using $\mathbf{W}$, we show how $\mathbf{C}$ can be constructed inductively: Consider an enumeration $\{x_n : n \in \omega\}$ of $i_1''\mathbf{U}_q$ admitting $i_1''F$ as an initial segment. Assume that the points $\varphi(x_0), \ldots, \varphi(x_n)$ are constructed so that the map φ is an isometry, $\mathrm{dom}\varphi \subset i_1''\mathbf{U}_q$, $\mathrm{ran}\varphi \subset \mathbf{U}_q$, $\varphi(i_1(x)) = x$ whenever $x \in F$, $d^{\mathbf{U}_q}(\varphi(x_k), z) = d^{\mathbf{W}}(x_k, j_1(z))$ whenever $z \in \mathrm{dom}f$ and $k \leqslant n$, and $\varphi(x_k) \in \overleftarrow{\chi}\{i\}$ whenever $\varphi(x_k)$ realizes f over F.

We want to construct $\varphi(x_{n+1})$. Consider the map h defined on $\{\varphi(x_k) : k \leqslant n\} \cup \mathrm{dom}f$ by:

$$\begin{cases} \forall k \leqslant n \quad h(\varphi(x_k)) = d^{\mathbf{W}}(x_k, x_{n+1}), \\ \forall z \in \mathrm{dom}f \quad h(z) = d^{\mathbf{W}}(j_1(z), x_{n+1}). \end{cases}$$

Observe that the metric subspace of $\mathbf{W}$ given by $\{x_k : k \leqslant n+1\} \cup j_1''\mathrm{dom}f$ witnesses that h is Katětov. It follows that the set H of all $y \in \mathbf{U}_q$ realizing h over $\{\varphi(x_k) : k \leqslant n\} \cup \mathrm{dom}f$ is not empty and $\varphi(x_{n+1})$ can be chosen in H. Additionnally, observe that if $h \restriction F = f \restriction F$, then $h \restriction \mathrm{dom}f = f$. The fact that $\psi_i(f, \mathbf{U}_q)$ holds then guarantees that h can be realized by a point in $\overleftarrow{\chi}\{i\}$. We can therefore choose $\varphi(x_{n+1})$ to be one of those points. After ω steps, the subspace $\mathbf{C}$ of $\mathbf{U}_q$ supported by $\{\varphi(x_n) : n \in \omega\}$ is as required. This finishes the proof in the case $m = 1$. $\square$

4.2. **Induction step.** Assume that Lemma 2 holds up to $m - 1 \geqslant 1$. We are going to show that it also holds for m. We start with an observation related to the metric space $\mathbf{W}$ constructed in the previous subsection. First, note that if h and g are Katětov over F and $x \in \mathbf{U}_q$ realizing g over F, then

$$\min\{d^{\mathbf{U}_q}(x, z) : z \in O(h)\} = \min\{|g(w) - h(w)| : w \in F\} \quad (*).$$

Let then $y \in \mathrm{dom} f$. It turns out that when $x \in \mathbf{U}_q$, the equation $(*)$ can be used to provide an explicit computation of $d^{\mathbf{W}}(i_1(x), j_1(y))$ in terms of g. This is achieved by going back to the definition of $d^{\mathbf{W}}(i_1(x), j_1(y))$ and by breaking the evaluation of the minimum over $\mathbf{Z}$ by two evaluations, over F and $O(f \restriction F)$ respectively. Observe then that for $z \in F$,

$$\begin{cases} d^{\mathbf{U}_q}(x, z) = g(z) \\ d^{\mathbf{U}_q}(j_0(z), y) = d^{\mathbf{Y}}(z, y). \end{cases}$$

On the other hand, if $z \in O(f \restriction F)$, then

$$d^{\mathbf{U}_q}(j_0(z), y) = f(y).$$

It follows that $d^{\mathbf{W}}(i_1(x), j_1(y))$ is actually equal to

$$\min\left(\min\{g(z) + d^{\mathbf{Y}}(z, y) : z \in F\}, f(y) + \min\{d^{\mathbf{U}_q}(x, z) : z \in O(f \restriction F)\}\right).$$

Applying the equation $(*)$ to $f \restriction F = h$, this quantity can be written

$$\min\left(\min\{g(z) + d^{\mathbf{Y}}(z, y) : z \in F\}, f(y) + \min\{|g(w) - h(w)| : w \in F\}\right).$$

Thus, the distance $d^{\mathbf{W}}(i_1(x), j_1(y))$ does not depend on the particular point x that we chose to realize g, and we can denote it by $g^*(y)$. More generally, given a finite subspace $\mathbf{X}$ of $\mathbf{U}_q$ such that $\mathbf{X} \cap \mathrm{dom} f = \mathbf{X} \cap F$, this property provides a way to extend every $g \in E(\mathbf{X})$ to g^* defined and Katětov over $\mathbf{X}^* := \mathbf{X} \cup \mathrm{dom} f$. In particular, it follows that $(f \restriction F)^* = f$ and if $g \restriction F = f \restriction F$, then g^* extends f. Observe that in that latter case, $\min g^* = \min g$. Now we turn to the proof of Lemma 2 for m. The required copy $\mathbf{C}$ is constructed inductively thanks to the following lemma. Recall that for a metric subspace $\mathbf{X}$ of $\mathbf{U}_q$ and $\varepsilon > 0$, the set $(\mathbf{X})_\varepsilon$ is defined by:

$$(\mathbf{X})_\varepsilon = \{y \in \mathbf{U}_q : \exists x \in \mathbf{X} \quad d^{\mathbf{U}_q}(y, x) \leqslant \varepsilon\}.$$

Lemma 4. *Let $\boldsymbol{X}$ be a finite subspace of $\boldsymbol{U}_q$ and $\boldsymbol{A} \in \binom{\boldsymbol{U}_q}{\boldsymbol{U}_q}$ such that:*

(i) $\boldsymbol{X} \cap \mathrm{dom} f = F$, $\boldsymbol{X}^* \subset \boldsymbol{A}$.

(ii) $(\boldsymbol{X})_{m-1} \cap \boldsymbol{A} \cap O(f) \subset \overleftarrow{\chi}\{i\}$.

(iii) $\forall g \in E(\boldsymbol{X}) \quad g \restriction F = f \restriction F \to \psi_i(g^*, \boldsymbol{A})$.

Then for every $h \in E(\boldsymbol{X})$, there are $\boldsymbol{B} \in \binom{\boldsymbol{A}}{\boldsymbol{U}_q}$ and $x^* \in \boldsymbol{B}$ realizing h^* over $\boldsymbol{X}^*$ such that:

(i') $(\boldsymbol{X} \cup \{x^*\}) \cap \mathrm{dom} f = F$, $\boldsymbol{X}^* \subset \boldsymbol{B}$.

(ii') $((\boldsymbol{X} \cup \{x^*\}))_{m-1} \cap \boldsymbol{B} \cap O(f) \subset \overleftarrow{\chi}\{i\}$.

(iii') $\forall g \in E(\boldsymbol{X} \cup \{x^*\}) \quad g \restriction F = f \restriction F \to \psi_i(g^*, \boldsymbol{B})$.

Claim 1. *Lemma 4 implies $\mathcal{H}_m$, and therefore Lemma 2.*

Proof. The required copy of $\mathbf{C}$ can be constructed inductively as follows: Fix an enumeration $\{x_n : n \in \omega\}$ of $\mathbf{U}_q$ such that $F = \{x_0, \ldots, x_k\}$. For every $i \leqslant k$, set $\tilde{x}_i = x_i$. Set also $\mathbf{A}_k = \mathbf{U}_q$. To construct $\tilde{x}_{k+1}$, consider h_{k+1} defined on $\{\tilde{x}_0, \ldots, \tilde{x}_k\}$ by:

$$\forall i \leqslant k \quad h_{k+1}(\tilde{x}_i) = d^{\mathbf{U}_q}(x_{k+1}, x_i).$$

Then h_{k+1} is Katětov over $\{\tilde{x}_0, \ldots, \tilde{x}_k\}$ and Lemma 4 can be applied to the subspace of $\mathbf{U}_q$ supported by $\{\tilde{x}_0, \ldots, \tilde{x}_k\}$, the copy $\mathbf{A}_k$ and the Katětov map h_{k+1}. It produces x^* and $\mathbf{B}$, and we set $\tilde{x}_{k+1} = x^*$ and $\mathbf{A}_{k+1} = \mathbf{B}$. In general, assume that $\tilde{x}_0, \ldots, \tilde{x}_l$ and $\mathbf{A}_k, \ldots, \mathbf{A}_l$ are constructed so that $\mathbf{A}_l$ and the subspace of $\mathbf{U}_q$ supported by $\{\tilde{x}_0, \ldots, \tilde{x}_l\}$ satisfy the hypotheses of Lemma 4. Consider h_{l+1} defined on $\{\tilde{x}_0, \ldots, \tilde{x}_l\}$ by:

$$\forall i \leqslant l \quad h_{l+1}(\tilde{x}_i) = d^{\mathbf{U}_q}(x_{l+1}, x_i).$$

Then h_{l+1} is Katětov over $\{\tilde{x}_0, \ldots, \tilde{x}_l\}$, Lemma 4 can be applied to produce x^* and $\mathbf{B}$, and we set $\tilde{x}_{l+1} = x^*$ and $\mathbf{A}_{l+1} = \mathbf{B}$. After ω steps, we are left with $\mathbf{C} = \{\tilde{x}_n : n \in \omega\}$ isometric to $\mathbf{U}_q$, as required. □

The remaining part of this section is consequently devoted to a proof of Lemma 4. First, an observation: If x^* and $\mathbf{B}$ satisfy the first two requirements of Lemma 4, then the third one is automatically satisfied. Indeed, let $g \in E(\mathbf{X} \cup \{x^*\})$ be such that $g \restriction F = f \restriction F$. If $\min g \geqslant m$, then $(g^*, \mathbf{B}) \leqslant_0 (f, \mathbf{U}_q)$. Since $\psi_i(f, \mathbf{U}_q)$ holds, it follows that $\psi_i(g^*, \mathbf{B})$ holds. On the other hand, if $\min g \leqslant m-1$, then $\psi_i(g^*, \mathbf{B})$ holds since

$$(O(g^*) \cap \mathbf{B}) \subset \big((\mathbf{X} \cup \{x^*\})_{m-1} \cap \mathbf{B} \cap O(f) \big) \subset \overleftarrow{\chi}\{i\}.$$

With this observation in mind, we define

$$K = \{\phi \in E(\mathbf{X} \cup \{h\}) : \phi \restriction F = f \restriction F \ \text{ and } \ \phi(h) \leqslant m-1\}.$$

The reason for which K is relevant here lies in the following fact:

Claim 2. *Assume that $\boldsymbol{B} \in \binom{\boldsymbol{A}}{\boldsymbol{U}_q}$ and $x^* \in \boldsymbol{B}$ are such that:*

(i) $\boldsymbol{X}^* \subset \boldsymbol{B}$.
(ii) x^* realizes h^* over $\boldsymbol{X}^*$.
(iii) $\chi(x^*) = i$ if $h \restriction F = f \restriction F$ (that is if $x^* \in O(f)$).
(iv) For every $\phi \in K$, every point in $\boldsymbol{B}$ realizing ϕ^* over $\boldsymbol{X}^* \cup \{x^*\} \cong \boldsymbol{X}^* \cup \{h\}$ is in $\overleftarrow{\chi}\{i\}$.

Then x^* and $\boldsymbol{B}$ satisfy the requirements of Lemma 4.

Proof. Let $y \in (\mathbf{X} \cup \{x^*\})_{m-1} \cap \mathbf{B} \cap O(f)$. We need to prove that $\chi(y) = i$. If $y \in (\mathbf{X})_{m-1}$, then y is actually in $(\mathbf{X})_{m-1} \cap \mathbf{A} \cap O(f) \subset \overleftarrow{\chi}\{i\}$ and we are done. Otherwise, $y \in (\{x^*\})_{m-1}$. If $y = x^*$, there is nothing to do: Since y is in $O(f)$, so is x^*. Thus, by hypothesis, $\chi(x^*) = i$, that is $\chi(y) = i$. Otherwise, let ϕ be defined on $\mathbf{X} \cup \{h\}$ by

$$\phi(z) = \begin{cases} d^{\mathbf{U}_q}(z, y) & \text{if } z \in \mathbf{X}, \\ d^{\mathbf{U}_q}(x^*, y) & \text{if } z = h. \end{cases}$$

One can then check that ϕ is in K: First, the metric space $\mathbf{X} \cup \{x^*, y\}$ witnesses that ϕ is Katětov over $\mathbf{X} \cup \{h\}$. Next, $y \in O(f)$ hence $\phi \restriction F = f \restriction F$. Finally, $\phi(h) = d^{\mathbf{U}_q}(x^*, y) \leqslant m-1$ since $y \in (\{x^*\})_{m-1}$. Observe that y realizes ϕ^* over $\mathbf{X}^* \cup \{x^*\}$: This is so because y realizes both ϕ over $\mathbf{X} \cup \{x^*\}$ and f over $\mathrm{dom} f$. According to the hypotheses of the claim, it follows that y is in $\overleftarrow{\chi}\{i\}$. □

The strategy to construct $\mathbf{B}$ and x^* is the following one. Let $\{\phi_\alpha : \alpha < |K|\}$ be an enumeration of K so that the sequence $(\phi_\alpha(h))_{\alpha < |K|}$ is nondecreasing. We first construct $(x_\alpha)_{\alpha < |K|}$ and $(\mathbf{D}_\alpha)_{\alpha < |K|}$ so that for every $\beta < \alpha < |K|$:

(i) $x_\alpha \in \mathbf{D}_\alpha$.
(ii) $\mathbf{X}^* \subset \mathbf{D}_\alpha$.
(iii) x_α realizes h^* over $\mathbf{X}^*$.
(iv) Every point in $\mathbf{D}_\alpha$ realizing ϕ_β over $\mathbf{X}^* \cup \{x_\alpha\} \cong \mathbf{X}^* \cup \{h^*\}$ is in $\overleftarrow{\chi}\{i\}$.

The details of this construction are provided in section 4.3. Once this is done, call $x' = x_{|K|-1}$, $\mathbf{B}' = \mathbf{D}_{|K|-1}$. The point x' and the copy $\mathbf{B}'$ are almost as required except that $\chi(x')$ may not be i. If $h \restriction F \neq f \restriction F$, this is not a problem and setting $x^* = x'$ and $\mathbf{B} = \mathbf{B}'$ works. On the other hand, if $h \restriction F = f \restriction F$, then some extra work is required and we proceed as follows.

Pick $x^* \in \mathbf{B}'$ such that $d^{\mathbf{U}_q}(x, x') = 1$. We are going to construct $\mathbf{B} \in \binom{\mathbf{B}'}{\mathbf{U}_q}$ so that that $(\mathbf{X}^* \cup \{x', x^*\}) \cap \mathbf{B} = \mathbf{X}^* \cup \{x^*\}$ and for every $\phi \in K$, every point in $\mathbf{B}$ realizing ϕ^* over $\mathbf{X}^* \cup \{x^*\}$ will realize ϕ^* over $\mathbf{X}^* \cup \{x'\}$. Here is how $\mathbf{B}$ is constructed:

Recall that for $\phi \in K$, $O(\phi^*)$ denotes the set of all elements of $\mathbf{B}'$ realizing ϕ^* over $\mathbf{X}^* \cup \{x^*\}$. Let $H(\phi^*)$ be the set of all elements of $\mathbf{B}'$ realizing ϕ^* both over $\mathbf{X}^* \cup \{x^*\}$ and $\mathbf{X}^* \cup \{x'\}$, let $\mathbf{Z}$ be the metric subspace of $\mathbf{B}'$ supported by $\mathbf{X}^* \cup \{x^*\} \cup \bigcup \{O(\phi^*) : \phi \in K\}$, let $i_0 : \mathbf{Z} \longrightarrow \mathbf{U}_q$ be the isometric embedding provided by the identity map and let $\mathbf{Y}$ be the metric subspace of $\mathbf{B}'$ supported by $\mathbf{X}^* \cup \{x^*, x'\} \cup \bigcup \{H(\phi^*) : \phi \in K\}$. One can then check that $\mathbf{Z}$ is isometric to $\mathbf{Y} \smallsetminus \{x'\}$ via an isometry j_0 that fixes $\mathbf{X}^* \cup \{x^*\}$. From now on, the proof mimics the proof that was carried out in subsection 4.1: Using strong amalgamation for countable metric spaces with distances in $\{1, \ldots, q\}$, we can find a countable metric space $\mathbf{W}$ and isometric embeddings $i_1 : \mathbf{U}_q \longrightarrow \mathbf{W}$ and $j_1 : \mathbf{Y} \longrightarrow \mathbf{W}$ such that $i_1 \circ i_0 = j_1 \circ j_0$, $\mathbf{W} = i_1'' \mathbf{U}_q \cup j_1'' \mathbf{Y}$, $i_1'' \mathbf{U}_q \cap j_1'' \mathbf{Y} = (i_1 \circ i_0)'' \mathbf{Z} = (j_1 \circ j_0)'' \mathbf{Z}$, and for every $x \in \mathbf{U}_q$ and $y \in \mathbf{Y}$:

$$\begin{aligned} d^{\mathbf{W}}(i_1(x), j_1(y)) &= \min\{d^{\mathbf{W}}(i_1(x), i_1 \circ i_0(z)) + d^{\mathbf{W}}(j_1 \circ j_0(z), j_1(y)) : z \in \mathbf{Z}\} \\ &= \min\{d^{\mathbf{U}_q}(x, i_0(z)) + d^{\mathbf{Y}}(j_0(z), y) : z \in \mathbf{Z}\} \\ &= \min\{d^{\mathbf{U}_q}(x, z) + d^{\mathbf{Y}}(j_0(z), y) : z \in \mathbf{Z}\}. \end{aligned}$$

Observe that in $\mathbf{W}$, given $\phi \in K$, every $x \in i_1'' \mathbf{U}_q$ realizing ϕ^* over $i_1'' (\mathbf{X}^* \cup \{x^*\})$ also realizes ϕ^* over $j_1'' (\mathbf{X}^* \cup \{x'\})$. Using $\mathbf{W}$, we show how $\mathbf{B}$ can be constructed inductively: Consider an enumeration $\{x_n : n \in \omega\}$ of $i_1'' \mathbf{U}_q$ admitting $i_1'' (\mathbf{X}^* \cup \{x^*\})$ as an initial segment. Assume that the points $\varphi(x_0), \ldots, \varphi(x_n)$ are constructed so that:

(i) φ is an isometry.
(ii) $\mathrm{dom}\varphi \subset i_1'' \mathbf{U}_q$.
(iii) $\mathrm{ran}\varphi \subset \mathbf{U}_q$.
(iv) $\varphi(i_1(x)) = x$ whenever $x \in \mathbf{X} \cup \{x^*\}$.
(v) $d^{\mathbf{U}_q}(\varphi(x_k), x') = d^{\mathbf{W}}(x_k, j_1(x'))$ whenever $k \leqslant n$.
(vi) $\varphi(x_k) \in \overleftarrow{\chi}\{i\}$ whenever $\varphi(x_k)$ realizes ϕ^* over $\mathbf{X}^* \cup \{x^*\}$.

We want to construct $\varphi(x_{n+1})$. This is done by considering the map e defined on the set $\{\varphi(x_k) : k \leqslant n\} \cup \{x'\}$ by:

$$\begin{cases} e(\varphi(x_k)) = d^{\mathbf{W}}(x_k, x_{n+1}) & \text{whenever } k \leqslant n, \\ e(x') = d^{\mathbf{W}}(j_1(x'), x_{n+1}). \end{cases}$$

Observe that the metric subspace of $\mathbf{W}$ given by $\{x_k : k \leqslant n+1\} \cup \{j_1(x')\}$ witnesses that e is Katětov. It follows that the set E of all $y \in \mathbf{U}_q$ realizing e over $\{\varphi(x_k) : k \leqslant n\} \cup \{x'\}$ is non empty and $\varphi(x_{n+1})$ can be chosen in E. Additionnally, observe that if there is $\phi \in K$ such that $e \restriction \mathbf{X}^* = \phi^* \restriction \mathbf{X}^*$ and $e(x) = \phi(h)$, then $e(x') = \phi(h)$. By construction, the map Katětov e can then be realized by a point in $\overleftarrow{\chi}\{i\}$, and we can choose $\varphi(x_{n+1})$ to be one of those points. After ω steps, the subspace $\mathbf{B}$ of $\mathbf{B}'$ supported by $\{\varphi(x_n) : n \in \omega\}$ is as required. $\square$

4.3. **Construction of the sequences $(x_\alpha)_{\alpha<|K|}$ and $(\mathbf{D}_\alpha)_{\alpha<|K|}$.** It should be clear that the construction of the sequences $(x_\alpha)_{\alpha<|K|}$ and $(\mathbf{D}_\alpha)_{\alpha<|K|}$ can be carried out if we prove the following lemma:

Lemma 5. *Let $\mathcal{F} \subset K$ and $\boldsymbol{D} \in \binom{\boldsymbol{U}_q}{\boldsymbol{U}_q}$ be such that $\boldsymbol{X}^* \subset \boldsymbol{D}$. Assume that $u \in \boldsymbol{D}$ realizes h^* over $\boldsymbol{X}^*$ and is such that for every $\phi \in \mathcal{F}$, every point in $\boldsymbol{D}$ realizing ϕ^* over $\boldsymbol{X}^* \cup \{u\} \cong \boldsymbol{X}^* \cup \{h^*\}$ is in $\overleftarrow{\chi}\{i\}$. Let $s \in K \setminus \mathcal{F}$ be such that*

$$\forall \phi \in K \quad \phi(h) > s(h) \to \phi \in \mathcal{F}.$$

Then there are $\boldsymbol{E} \in \binom{D}{\boldsymbol{U}_q}$ and $v \in \boldsymbol{E} \cap \overleftarrow{\chi}\{i\}$ realizing h^* over $\boldsymbol{X}^*$ such that $\boldsymbol{X}^* \subset \boldsymbol{E}$ and for every $\phi \in \mathcal{F} \cup \{s\}$, every point in $\boldsymbol{E}$ realizing ϕ^* over $\boldsymbol{X}^* \cup \{v\} \cong \boldsymbol{X}^* \cup \{h^*\}$ is in $\overleftarrow{\chi}\{i\}$.

Proof. We start with the case where $s(h) \geqslant \min s \restriction \mathbf{X}$. The map s being in K, $s(h) \leqslant m-1$ and so $\min s \restriction \mathbf{X} \leqslant m-1$. Then, from (ii),

$$O(s \restriction \mathbf{X}) \cap \mathbf{D} \subset \big((\mathbf{X})_{m-1} \cap \mathbf{D} \cap O(f)\big) \subset \overleftarrow{\chi}\{i\}.$$

Observe now that every point in $\mathbf{D}$ realizing s^* over $\mathbf{X}^* \cup \{u\}$ is in $O(s \restriction \mathbf{X}) \cap \mathbf{D}$. Thus, according to the previous inclusion, any such point is also in $\overleftarrow{\chi}\{i\}$. So in fact, there is nothing to do: $v = u$ and $\mathbf{E} = \mathbf{D}$ works.

From now on, we consequently suppose that $s(h) < \min s \restriction \mathbf{X}$. Let s_1 be defined on $\mathbf{X} \cup \{u\}$ by

$$s_1(x) = \begin{cases} s(x) & \text{if } x \in \mathbf{X}, \\ s(h)+1 & \text{if } x = u. \end{cases}$$

Claim 3. *The map s_1 is Katětov.*

Proof. The map s is Katětov over $\mathbf{X}$. Hence, it is enough to prove that for every $x \in \mathbf{X}$

$$|s_1(u) - s_1(x)| \leqslant d^{\mathbf{U}_q}(x,u) \leqslant s_1(u) + s_1(x).$$

That is

$$|s(h) + 1 - s(x)| \leqslant h(x) \leqslant s(h) + 1 + s(x).$$

Because s is Katětov over $\mathbf{X} \cup \{h\}$, it is enough to prove that

$$s(h) + 1 - s(x) \leqslant h(x).$$

But this holds true since $s(h) < \min s \restriction \mathbf{X}$. $\square$

Claim 4. *$\psi_i(s_1^*, \boldsymbol{D})$ holds.*

Proof. If $s(h) = m-1$, then $\min s_1 = m = \min f$ and so $(s_1^*, \mathbf{D}) \leqslant_0 (f, \mathbf{U}_q)$. Since $\psi_i(f, \mathbf{U}_q)$ holds, so does $\psi_i(s_1^*, \mathbf{D})$. On the other hand, if $s(h) < m-1$, then $s_1 \in K$ and it follows from the hypothesis on $\mathcal{F}$ that $s_1 \in \mathcal{F}$. In particular, every point in $\mathbf{D}$ realizing s_1^* over $\mathbf{X}^* \cup \{u\}$ is in $\overleftarrow{\chi}\{i\}$, and it follows that $\psi_i(s_1^*, \mathbf{D})$ holds. $\square$

Consequently, there is $(S_2, \mathbf{D}_2) \leqslant_1 (s_1^*, \mathbf{D})$ such that $\psi_i(S_2, \mathbf{D}_2)$ holds. Observe that if we take s_2 to be the restriction of S_2 to $\mathrm{dom} S_2 \setminus (\mathrm{dom} f \setminus F)$, then $s_2^* = S_2$. Hence $\psi_i(s_2^*, \mathbf{D}_2)$ holds. Pick $w \in O(s_2^*) \cap \mathbf{D}_2$, and let s_3 be defined on $\mathrm{dom} s_2 \cup \{w\}$ by

$$s_3(x) = \begin{cases} s_2(x) & \text{if } x \in \mathrm{dom} s_2, \\ s(h) & \text{if } x = w. \end{cases}$$

Claim 5. *The map s_3 is Katětov.*

Proof. The map s_2 is Katětov over $\mathrm{dom} s_2$ so it is enough to prove that for every $x \in \mathrm{dom} s_2$

$$|s_3(w) - s_3(x)| \leqslant d^{\mathbf{U}_q}(w, x) \leqslant s_3(w) + s_3(x).$$

That is

$$|s(h) - s_2(x)| \leqslant s_2(x) \leqslant s(h) + s_2(x).$$

For that to be satisfied, it suffices to prove that

$$s(h) - s_2(x) \leqslant s_2(x).$$

But this is true since

$$s(h) = \min s_1 - 1 = \min s_2 \leqslant s_2(x). \qquad \square$$

Observe that $(s_3^*, \mathbf{D}_2) \leqslant_0 (s_2^*, \mathbf{D}_2)$. Hence $\psi_i(s_3^*, \mathbf{D}_2)$ holds. Observe also that $\min s_3 = \min s_3 \restriction \mathbf{X}^* \cup \{u, w\} = \min s \leqslant m - 1$. Thus, one can apply $\mathcal{H}_{\min s}$ inside $\mathbf{D}_2$ to s_3^* and $\mathbf{X}^* \cup \{u, w\}$ to obtain $\mathbf{D}_3 \in \binom{\mathbf{D}_2}{\mathbf{U}_q}$ such that $\mathrm{dom} s_3^* \cap \mathbf{D}_3 = \mathbf{X}^* \cup \{u, w\}$ and $O(s_3^* \restriction \mathbf{X}^* \cup \{u, w\}) \cap \mathbf{D}_3 \subset \overleftarrow{\chi}\{i\}$. Observe that in $\mathbf{D}_3$, the point w realizes s^* over $\mathbf{X}^*$. Consider the map h_1 defined on $\mathbf{X} \cup \{u, w\}$ by

$$h_1(x) = \begin{cases} h(x) & \text{if } x \in \mathbf{X}. \\ 1 & \text{if } x = u. \\ s(h) & \text{if } x = w. \end{cases}$$

Claim 6. *The map h_1 is Katětov.*

Proof. The metric space $(\mathbf{X} \cup \{h\}) \cup \{s\}$ witnesses that $h_1 \restriction \mathbf{X} \cup \{w\}$ is Katětov. Next, $h_1 \restriction \mathbf{X} \cup \{u\}$ is also Katětov: Let $x \in \mathbf{X}$. Then

$$|h_1(x) - h_1(u)| = h(x) - 1 \leqslant h(x) = d^{\mathbf{U}_q}(x, u) \leqslant h(x) + 1 = h_1(x) + h_1(u).$$

The only thing we still need to show is therefore

$$|h_1(u) - h_1(w)| \leqslant d^{\mathbf{U}_q}(u, w) \leqslant h_1(u) + h_1(w).$$

But this inequalities hold as they are equivalent to

$$|1 - s(h)| \leqslant s(h) + 1 \leqslant 1 + s(h). \qquad \square$$

Let now $v \in \mathbf{D}_3$ realizing h_1^* over $\mathbf{X}^* \cup \{u, w\}$. Let s_4 be defined on $\mathbf{X} \cup \{u, v, w\}$ by

$$s_4(x) = \begin{cases} s_3(x) & \text{if } x \in \mathbf{X} \cup \{u, w\}, \\ s(h) & \text{if } x = v. \end{cases}$$

Claim 7. *The map s_4 is Katětov.*

Proof. The map $s_4 \restriction \mathbf{X} \cup \{u, w\}$ is Katětov because s_3 is, and $s_4 \restriction \mathbf{X} \cup \{v\}$ is Katětov because $\mathbf{X} \cup \{v\} \cong \mathbf{X} \cup \{h\}$, the map s is Katětov over $\mathbf{X} \cup \{h\}$ and s_4 is essentially equal to s on $\mathbf{X} \cup \{v\}$. Therefore, it suffices to show that

$$|s_4(v) - s_4(u)| \leqslant d^{\mathbf{U}_q}(v, u) \leqslant s_4(v) + s_4(u) \quad (1)$$

$$|s_4(v) - s_4(w)| \leqslant d^{\mathbf{U}_q}(v, w) \leqslant s_4(v) + s_4(w) \quad (2)$$

The inequality(1) holds as it is equivalent to

$$|s(h) - (s(h) + 1)| \leqslant 1 \leqslant s(h) + s(h) + 1.$$

As for (2), it holds as it is equivalent to

$$|s(h) - s(h)| \leqslant s(h) \leqslant s(h) + s(h). \qquad \square$$

Observe that $(s_4^*, \mathbf{D}_3) \leqslant_0 (s_3^*, \mathbf{D}_3)$ and that consequently $\psi_i(s_4^*, \mathbf{D}_3)$ holds. Observe also that $\min s_4 = \min s_4 \upharpoonright \mathbf{X}^* \cup \{u,v\} = \min s \leqslant m-1$. Thus, we can apply $\mathcal{H}_{\min s}$ again inside $\mathbf{D}_3$ to s_4^* and $\mathbf{X}^* \cup \{u,v\}$ to obtain $\mathbf{D}_4 \in \binom{\mathbf{D}_3}{\mathbf{U}_q}$ such that $\mathrm{dom} s_4^* \cap \mathbf{D}_4 = \mathbf{X}^* \cup \{u,v\}$ and $O(s_4^* \upharpoonright \mathbf{X}^* \cup \{u,v\}) \cap \mathbf{D}_4 \subset \overleftarrow{\chi}\{i\}$. Note that u and v both realize h^* over $\mathbf{X}^*$. Note also that if $\phi \in \mathcal{F}$, then every point in $\mathbf{D}_4$ realizing ϕ^* over $\mathbf{X}^* \cup \{u\}$ is in $\overleftarrow{\chi}\{i\}$.

We are now going to construct $\mathbf{E} \in \binom{\mathbf{D}_4}{\mathbf{U}_q}$ such that:

(i) $(\mathbf{X}^* \cup \{u,v\}) \cap \mathbf{E} = \mathbf{X}^* \cup \{v\}$.
(ii) For every $\phi \in \mathcal{F}$, every point in $\mathbf{E}$ realizing ϕ^* over $\mathbf{X}^* \cup \{v\}$ realizes ϕ^* over $\mathbf{X}^* \cup \{u\}$.
(iii) Every point in $\mathbf{E}$ realizing s^* over $\mathbf{X}^* \cup \{v\}$ realizes s_4^* over $\mathbf{X}^* \cup \{u,v\}$.

For $\phi \in \mathcal{F} \cup \{s\}$, denote by $O(\phi^*)$ the set of all elements of $\mathbf{D}_3$ realizing ϕ^* over $\mathbf{X}^* \cup \{v\}$. Let $\mathbf{Z}$ be the metric subspace of $\mathbf{D}_3$ supported by $\mathbf{X}^* \cup \{v\} \cup \bigcup\{O(\phi^*) : \phi \in \mathcal{F} \cup \{s\}\}$. Let g be defined on $\mathbf{Z}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbf{X}^*, \\ 1 & \text{if } x = v, \\ d^{\mathbf{U}_q}(x,v) & \text{if } x \in O(\phi^*) \text{ with } \phi \in \mathcal{F}, \\ d^{\mathbf{U}_q}(x,v) + 1 & \text{if } x \in O(s^*). \end{cases}$$

Claim 8. *The map g is Katětov.*

Proof. On $\mathbf{X}^* \cup \{v\}$ there is no problem: the metric space $\mathbf{X} \cup \{u,v\}$ witnesses the fact that g is Katětov. For $x, y \in O(\phi^*)$ and $\phi \in \mathcal{F} \cup \{s\}$, there is no problem either. On the other hand, let $x \in O(s^*)$ and $y \in O(\phi)$ for some $\phi \in \mathcal{F}$. Then what we need is equivalent to

$$\left|d^{\mathbf{U}_q}(x,v) + 1 - d^{\mathbf{U}_q}(y,v)\right| \leqslant d^{\mathbf{U}_q}(x,y) \leqslant d^{\mathbf{U}_q}(x,v) + 1 - d^{\mathbf{U}_q}(y,v).$$

To prove that, it is enough to check that

$$d^{\mathbf{U}_q}(x,v) + 1 - d^{\mathbf{U}_q}(y,v) \leqslant d^{\mathbf{U}_q}(x,y).$$

Observe that since $x \in O(s^*)$, $d^{\mathbf{U}_q}(x,v) = s(v)$. Similarly $y \in O(\phi^*)$ and hence $d^{\mathbf{U}_q}(y,v) = \phi(v)$. It then follows from the assumption on $\mathcal{F}$ that $s(v) \leqslant \phi(v)$. Thus, the left hand side of the inequality is bounded above by 1 and the required inequality holds. If $x \in \mathbf{X}^*$ and $y \in O(\phi^*)$ for some $\phi \in \mathcal{F}$, then we need:

$$|f(x) - \phi(v)| \leqslant f(x) \leqslant f(x) + \phi(v).$$

The right hand side of the inequality clearly holds. As for the left hand side, observe that $\phi(v) \leqslant m-1 \leqslant \min f$. It follows that the left hand side is satisfied as it is equivalent to

$$f(x) - \phi(v) \leqslant f(x).$$

If $x \in \mathbf{X}^*$ and $y \in O(s^*)$, then the inequality we have to prove is the same as the previous one except that $\phi(v)$ is replaced by $s(v)+1$, and the same argument works because $s(v)+1 \leqslant m \leqslant \min f$. Finally, we need to check that no problem occurs when $x = v$ and $y \in O(\phi^*)$ for some $\phi \in \mathcal{F} \cup \{s\}$. If $\phi \in \mathcal{F}$, there is no problem because what needs to be proven turns out to be equivalent to:

$$|1 - \phi(v)| \leqslant \phi(v) \leqslant 1 + \phi(v).$$

On the other hand, if $\phi = s$, then there is no problem either since what we need to check is equivalent to

$$|1 - (s(v) + 1)| \leqslant s(v) \leqslant 1 + s(v) + 1. \qquad \square$$

For $\phi \in \mathcal{F}$, denote by $J(\phi^*)$ the set of all elements of $\mathbf{D}_3$ realizing ϕ^* both over $\mathbf{X}^* \cup \{v\}$ and $\mathbf{X}^* \cup \{u\}$. Let also $J(s^*)$ be the set of all elements in $\mathbf{D}_3$ realizing s_4^* over $\mathbf{X}^* \cup \{u, v\}$. Finally, let $\mathbf{Y}$ be the metric subspace of $\mathbf{D}_3$ supported by $\mathbf{X}^* \cup \{u, v\} \cup \bigcup\{J(\phi^*) : \phi \in \mathcal{F} \cup \{s\}\}$.

Claim 9. *The space* $\mathbf{Z}$ *embeds into* $\mathbf{Y} \smallsetminus \{u\}$ *via an isometry* j_0 *that fixes* $\mathbf{X}^* \cup \{v\}$.

Proof. The embedding j_0 is constructed inductively thanks to the metric space $\mathbf{Z} \cup \{g\}$. Let $\{z_n : n \in \omega\}$ be an enumeration of $\mathbf{Z}$ such that $\{z_n : n \leqslant k\} = \mathbf{X}^* \cup \{v\}$. Assume that $\{j(z_n) : n < l\}$ are constructed for some $l > k$ and such that $j_0(z_n) \in \mathbf{Y} \smallsetminus \{u\}$ and $d^{\mathbf{U}_q}(j_0(z_n), u) = g(z_n)$ for every $n \leqslant l$. We would like to construct $j_0(z_l)$. Consider the map p defined on $\{j(z_n) : n < l\} \cup \{u\}$ by

$$\begin{cases} p(j_0(z_n)) = d^{\mathbf{Z}}(z_n, z_l) & \text{whenever } n \leqslant l, \\ p(u) = g(z_l). \end{cases}$$

Then the metric subspace of $\mathbf{Z} \cup \{g\}$ supported by $\{z_n : n < l\} \cup \{g\}$ witnesses that p is Katětov over $\{j(z_n) : n < l\} \cup \{u\}$. For $j_0(z_l)$, pick any point realizing it in $\mathbf{D}_3$. Observe then that $j_0(z_l)$ is in $\mathbf{Y} \smallsetminus \{u\}$. $\square$

Applying once again the technique used in subsection 4.1, we obtain the required $\mathbf{E}$. More precisely: Consider $i_0 : \mathbf{Z} \longrightarrow \mathbf{U}_q$ the identity embedding. Using strong amalgamation for countable metric spaces with distances in $\{1, \ldots, q\}$, we can find a countable metric space $\mathbf{W}$ and isometric embeddings $i_1 : \mathbf{U}_q \longrightarrow \mathbf{W}$ and $j_1 : \mathbf{Y} \longrightarrow \mathbf{W}$ such that $i_1 \circ i_0 = j_1 \circ j_0$, $\mathbf{W} = i_1'' \mathbf{U}_q \cup j_1'' \mathbf{Y}$, $i_1'' \mathbf{U}_q \cap j_1'' \mathbf{Y} = (i_1 \circ i_0)'' \mathbf{Z} = (j_1 \circ j_0)'' \mathbf{Z}$, and for every $x \in \mathbf{U}_q$ and $y \in \mathbf{Y}$:

$$\begin{aligned} d^{\mathbf{W}}(i_1(x), j_1(y)) &= \min\{d^{\mathbf{W}}(i_1(x), i_1 \circ i_0(z)) + d^{\mathbf{W}}(j_1 \circ j_0(z), j_1(y)) : z \in \mathbf{Z}\} \\ &= \min\{d^{\mathbf{U}_q}(x, i_0(z)) + d^{\mathbf{Y}}(j_0(z), y) : z \in \mathbf{Z}\} \\ &= \min\{d^{\mathbf{U}_q}(x, z) + d^{\mathbf{Y}}(j_0(z), y) : z \in \mathbf{Z}\}. \end{aligned}$$

Observe that in $\mathbf{W}$, given $\phi \in \mathcal{F}$, every $x \in i_1'' \mathbf{U}_q$ realizing ϕ^* over $i_1''(\mathbf{X}^* \cup \{v\})$ also realizes ϕ^* over $j_1''(\mathbf{X}^* \cup \{u\})$. Observe also that every $x \in i_1'' \mathbf{U}_q$ realizing s^* over $i_1''(\mathbf{X}^* \cup \{v\})$ also realizes s_4^* over $j_1''(\mathbf{X}^* \cup \{u, v\})$. Using $\mathbf{W}$, we show how $\mathbf{E}$ can be constructed inductively: Consider an enumeration $\{x_n : n \in \omega\}$ of $i_1'' \mathbf{U}_q$ admitting $i_1''(\mathbf{X}^* \cup \{v\})$ as an initial segment. Assume that the points $\varphi(x_0), \ldots, \varphi(x_n)$ are constructed so that φ is an isometry, $\mathrm{dom}\varphi \subset i_1'' \mathbf{U}_q$, $\mathrm{ran}\varphi \subset \mathbf{U}_q$, $\varphi(i_1(x)) = x$ whenever $x \in \mathbf{X} \cup \{v\}$, $d^{\mathbf{U}_q}(\varphi(x_k), u) = d^{\mathbf{W}}(x_k, j_1(u))$ whenever $k \leqslant n$, and $\varphi(x_k) \in \overleftarrow{\chi}\{i\}$ whenever $\varphi(x_k)$ realizes ϕ^* over $\mathbf{X}^* \cup \{v\}$ for $\phi \in \mathcal{F} \cup \{s\}$. We want to construct $\varphi(x_{n+1})$. Consider the map t defined on $\{\varphi(x_k) : k \leqslant n\} \cup \{u\}$ by:

$$\begin{cases} t(\varphi(x_k)) = d^{\mathbf{W}}(x_k, x_{n+1}) & \text{whenever } k \leqslant n, \\ t(u) = d^{\mathbf{W}}(j_1(u), x_{n+1}). \end{cases}$$

Observe that the metric subspace of $\mathbf{W}$ given by $\{x_k : k \leqslant n+1\} \cup \{j_1(u)\}$ witnesses that t is Katětov. It follows that the set T of all $y \in \mathbf{U}_q$ realizing t over $\{\varphi(x_k) : k \leqslant n\} \cup \{u\}$ is not empty and $\varphi(x_{n+1})$ can be chosen in T. Additionnally, observe that if there is $\phi \in \mathcal{F}$ such that $t \restriction \mathbf{X}^* = \phi^* \restriction \mathbf{X}^*$ and $t(v) = \phi(h)$, then $t(u) = \phi(h)$. By construction, the map t can be realized by a point in $\overleftarrow{\chi}\{i\}$, and we can choose $\varphi(x_{n+1})$ to be one of those points. Finally, observe that if $t\restriction\mathbf{X}^* = s^*\restriction\mathbf{X}^*$ and $t(v) = s(h)$, then $t(u) = s_4(u) = s(h)+1$. By construction, the map t can be realized by a point in $\overleftarrow{\chi}\{i\}$ and we can choose $\varphi(x_{n+1})$ to be one of those points. After ω steps, the subspace $\mathbf{E}$ of $\mathbf{D}_3$ supported by $\{\varphi(x_n) : n \in \omega\}$ is as required. □

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