Asymptotic Behaviour of Mean-Quantile Efficient Portfolios
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Abstract. In this paper we investigate portfolio optimization in a Black-Scholes continuous-time setting under quantile based risk measures: value at risk, capital at risk and relative value at risk. We show that the optimization results are consistent with Merton's Two-Fund Separation Theorem, i.e., that every optimal strategy is a weighted average of the bond and Merton's portfolio. We present optimization results obtained under constained above risk measures, including a surprising and perhaps counterintuitive outcome: under value at risk, in better markets and during longer time horizons, it is optimal to invest less into the risky assets.

Keywords: Portfolio optimization, Merton's portfolio, quantile, value at risk, capital at risk

JEL codes/AMS codes: G11, C61/91B28, 93E20

1. Introduction

Over the last decade, downside risk measures have received considerable attention in the financial world. Their appealing feature is that they only capture negative returns relative to a given confidence level α, and are indifferent to the upside tail of the return's distribution.

Probably the most famous among these measures, value at risk (VaR) is a quantile based function defined by $VaR(X) = E[X] - \xi(\alpha)$, where X denotes wealth, $\alpha < 0.5$ is the risk level, and $\xi(\alpha)$ is the α-quantile of X. In this paper we also look at capital at risk (CaR), defined as the difference between the riskless wealth and the quantile, which was introduced in [9] and further investigated in [4], and relative value at risk ($RVaR$), defined as the ratio of value at risk to the expected wealth, which was introduced in [6].

We investigate these measures in the Black-Scholes setting, in continuous time, and with time dependent coefficients. By generalizing and extending the optimization approach of Emmer at al. (see [9]) to the continuous time setting, we show that every optimal portfolio

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is a weighted average of Merton's portfolio and the bond. We determine the proportions of wealth invested into the risky assets, subject to constrained *VaR*, *CaR* and *RVaR*, and examine their asymptotic behaviour.

The outline of the paper is as follows. In Section 2, we give the notation, market setting, the definitions of the portfolio process and *VaR*, *CaR* and *RVaR*. In Section 3, we derive analytical solutions for the maximal expected wealth subject to constrained *VaR*, *CaR* and *RVaR* problems. Section 4 is devoted to the investigation of the asymptotic behaviour of these solutions as the time horizon increases. Section 5 provides some numerical examples, and Section 6 concludes the paper.

2. Preliminaries

We use the following notation. The m-dimensional column vector with each component equal to 1 is denoted by $\mathbf{1}$, the Euclidean norm of a matrix or vector by $\|\cdot\|$, and the L^2 norm over the space $L^2([0,t],\mathbb{R}^n)$ of $\mathbb{R}^n$-valued, square-integrable functions defined on $[0,t]$, with its natural inner product $\langle f,g\rangle_t = \sum_{i=1}^n \int_0^t f_i(s)g_i(s)ds$, by $\|f\|_t = \sqrt{\langle f,f\rangle_t}$.

In relation to the market setting, we assume the following:

Assumption 2.1 *i)* $m+1$ *assets are traded continuously over a finite horizon* $[0,T]$ *in a frictionless market.*

ii) m *of these assets are stocks that follow the generalized geometric Brownian motion dynamics given by the system of stochastic differential equations*

$$dS_i(t)=S_i(t)\left(b_i(t)dt+\sum_{j=1}^m \sigma_{ij}(t)dW^j(t)\right), \quad t\in[0,T],\ S_i(0)>0, \tag{1}$$

where $b(t) := (b_1(t),\cdots,b_m(t))'$ *is the vector of stocks' drifts,* $\sigma(t) := (\sigma_{ij}(t))$ *is the volatility matrix, and* $W^j(t)$ *are independent Brownian motions.*

iii) One of the assets is a bond, whose price $S_0(t)$*,* $t\geq 0$*, evolves according to the differential equation*

$$dS_0(t) = r(t)S_0(t)dt, \quad t\in[0,T], \quad S_0(0)=S_0>0, \tag{2}$$

where $r(t)$ (>0) *is the interest rate of the bond. Throughout this work, we assume that the borrowing of the bond is unconstrained. We also introduce the notation*

$$R_0(t) = \exp\left(\int_0^t r(s)ds\right). \tag{3}$$

iv) The market coefficients $\sigma(t)$, $\sigma^{-1}(t)$, $b(t)$ *and* $r(t)$ *are deterministic, Borel measurable, bounded functions over* $[0,T]$*; then they are also square integrable belonging to appropriate* L^2 *spaces.*

v) $\sigma(t)$ *satisfies the non-degeneracy condition, i.e.*

$$x'\sigma(t)\sigma(t)'x \geq \delta x'x, \quad \forall t \in [0,T], \ \forall x \in R^m, \tag{4}$$

where $\delta > 0$ *is a given constant.*

We note that, under the above assumptions, the market is complete.

At any time t, $N_i(t)$ shares are held in the asset $S_i(t)$, leading to the wealth $X^\pi(t) = \sum N_i(t)S_i(t)$. The $m+1-$dimensional vector valued function $N(t) = (N_0(t), ..., N_m(t))'$ is called the *trading strategy.* We denote the fraction of the wealth $X^\pi(t)$ invested into the risky asset $S_i(t)$ by

$$\pi_i(t) = \frac{N_i(t)S_i(t)}{X^\pi(t)}, \ i = 1, \cdots, m,$$

and call $\pi(t) := (\pi_1(t), \cdots, \pi_m(t))' \in \mathbb{R}^m$ the *portfolio.* The fraction of wealth held in the bond is $\pi_0(t) = 1 - \pi(t)'\mathbf{1}$. Under the assumption that the trading strategy is *self-financing,* the wealth process (see [17]) follows the dynamics

$$dX^\pi(t) = X^\pi(t)\left((r(t) + B(t)'\pi(t))dt + \pi'(t)\sigma(t)dW(t)\right), \ X(0) = X_0, \tag{5}$$

where X_0 is the initial wealth, and

$$B(t) = b(t) - r(t)\mathbf{1}, \tag{6}$$

is the risk premium vector.

Proceeding as in [4], to ensure a minimal tractability of the optimization problems which we solve through the following sections, we restrict our attention in this work to the class of portfolios $\pi(t)$ which are Borel measurable, deterministic and bounded over $[0,T]$. Such portfolios are called *admissible.* Note that for an admissible portfolio $\pi(\cdot)$, equation (5) is guaranteed to have a strong solution $X^\pi(t)$ (see [15], Theorem 5.2.9). We denote by $\mathcal{Q}$ the set of admissible portfolios. We project the optimization problems considered in this paper onto the family of surfaces

$$\mathcal{Q}_\varepsilon = \left\{\pi(\cdot) \in L^2 : \int_0^T \|\sigma(t)'\pi(t)\|^2 dt = \varepsilon^2\right\} \quad , \quad \mathcal{Q} = \bigcup_{\varepsilon \geq 0} \mathcal{Q}_\varepsilon. \tag{7}$$

This decomposition will find its justification in the dimension reduction or portfolio compression procedure used throughout the paper. We note that we have generalized the approach of [9], in the sense that the coefficients $r(t), b(t), \sigma(t)$ as well as $\pi(t)$ are allowed to be time dependent, resulting in a much broader class of portfolios.

Finally, we denote *the market price of risk* by

$$\theta(t) = \sigma(t)^{-1} B(t).$$

It will be shown throughout this work that the magnitude of the L^2 norm of the market price of risk is the determining criterion for optimal investment strategies, which turn out to be weighted averages of the bond and *Merton's portfolio* defined by

$$\pi_M(t) = (\sigma(t)\sigma(t)')^{-1} B(t).$$

More specifically, as will be shown later, every optimal portfolio can be expressed in the form

$$\pi_\varepsilon = \frac{\varepsilon}{\|\theta\|_T} \pi_M(t),$$

where ε is given by (7). Since ε determines the proportion of the wealth invested into the stocks, we call ε the *wealth coefficient.*

In order to define *CaR*, *VaR* and *RVaR* for a given risk level α, we denote by z_α the corresponding α-quantile of the standard normal distribution. We assume that the given risk level α satisfies $\alpha < 0.5$, so that the corresponding α-quantile of the standard normal distribution z_α satisfies $z_\alpha < 0$, and proceed with the following definition.

DEFINITION 2.1. *Suppose that $F\left(X^\pi(t)\right)$ is the cumulative distribution function of the wealth process (5). For a risk level $\alpha \in (0, 0.5)$, at time $t \in [0, T]$, the α-quantile $\rho_\alpha(X_0, \pi, t)$ of $X^\pi(t)$ is defined by*

$$\rho_\alpha(X_0, \pi, t) := \inf\{x \in R \mid F(x) \geq \alpha\}.$$

It was shown in [4] that the α-quantile of the wealth process given by (5) is equal to

$$\rho_\alpha(X_0, \pi, t) = X_0 R_0\left(t\right) \exp\left(\langle B, \pi\rangle_t - \frac{1}{2}\|\sigma'\pi\|_t^2 - |z_\alpha| \|\sigma'\pi\|_t\right), \quad (8)$$

and that the expected value of the wealth process $X^\pi(t)$ is equal to

$$E\left[X^\pi\left(t\right)\right] = X_0 R_0\left(t\right) \exp\left(\langle B, \pi\rangle_t\right), \quad (9)$$

where $R_0(t)$ is defined in (3).

Assuming that the risk level $\alpha \in (0, 0.5)$, and the time horizon T are given, we can now give formal definitions of value at risk, capital at risk and relative value at risk at time $t \in [0, T]$.

DEFINITION 2.2. *The value at risk of the wealth process $X^\pi(t)$, which we denote by $VaR(X_0, \pi, t)$, is the difference between the expected value of wealth and the corresponding $\alpha-$quantile, i.e.,*

$$VaR(X_0, \pi, t) := E\left[X^\pi(t)\right] - \rho_\alpha(X_0, \pi, t). \tag{10}$$

The capital at risk (CaR) of the wealth process $X^\pi(t)$, which we denote by $CaR(X_0, \pi, t)$, is the difference between the riskless wealth and the corresponding $\alpha-$quantile, i.e.

$$CaR(X_0, \pi, t) := X_0 R_0(t) - \rho_\alpha(X_0, \pi, t). \tag{11}$$

The relative value at risk of the wealth process $X^\pi(t)$, which we denote by $RVaR(\pi, t)$, is the ratio of value at risk to the expected value of wealth, i.e.

$$RVaR(\pi, t) := \frac{E\left[X^\pi(t)\right] - \rho_\alpha(X_0, \pi, t)}{E\left[X^\pi(t)\right]}. \tag{12}$$

From (8) and (9) and the above definitions, we obtain analytic expressions for VaR, CaR and $RVaR$ for the given risk level $\alpha \in (0, 0.5)$, at time $t \in [0, T]$. For notational convenience we define

$$\Lambda(X_0, \pi, t) := \exp\left(-\frac{1}{2}\|\sigma'\pi\|_t^2 - |z_\alpha|\|\sigma'\pi\|_t\right). \tag{13}$$

Then we have

$$VaR(X_0, \pi, t) = X_0 R_0\,(t) \exp(\langle B, \pi\rangle_t)\big(1 - \Lambda(X_0, \pi, t)\big). \tag{14}$$

$$CaR(X_0, \pi, t) = X_0 R_0(t)\Big(1 - \exp(\langle B, \pi\rangle_t)\Lambda(X_0, \pi, t)\Big). \tag{15}$$

$$RVaR(\pi, t) = 1 - \Lambda(X_0, \pi, t). \tag{16}$$

3. Portfolio Optimization with Respect to Quantile Based Risk Measures

We start this section by reviewing the some of the results obtained in [4] where portfolio optimization with respect to CaR was investigated.

3.1. PORTFOLIO OPTIMIZATION WITH RESPECT TO *CaR*

We first look at minimal CaR. Consider the problem

$$\min_{\pi \in \mathcal{Q}} CaR(X_0, \pi, T). \tag{17}$$

The solution to problem (17) is given in the following theorem.

THEOREM 3.1. *i) If $\|\theta\|_T \leq |z_\alpha|$, the optimal strategy for (17) is $\pi(t) = 0$, with $CaR(X_0, \pi, T) = 0$, and the expected wealth $E\left[X^{\pi}(T)\right] = X_0 R_0(T)$.*

ii) If $\|\theta\|_T > |z_\alpha|$, the optimal strategy for (17) is

$$\pi_{\varepsilon_1}(t) = \tfrac{\varepsilon_1}{\|\theta\|_T}(\sigma(t)\sigma(t)')^{-1}B(t) \in \mathcal{Q}, \tag{18}$$

where $\varepsilon_1 = \|\theta\|_T - |z_\alpha|$. The minimum capital at risk is given by

$$CaR(X_0, \pi_{\varepsilon_1}, T) = X_0 R_0(T)\left(1 - \exp\left(\tfrac{1}{2}(\|\theta\|_T - |z_\alpha|)^2\right)\right), \tag{19}$$

and the corresponding expected wealth

$$E\left[X^{\pi_{\varepsilon_1}}(T)\right] = X_0 R_0(T) \exp\Big((\|\theta\|_T - |z_\alpha|)\|\theta\|_T\Big). \tag{20}$$

The proof of the theorem is given in [4].

REMARK 3.1. *i) We assume that $\|\theta\|_T > 0$, and that $b_i(t) \geq r(t)$ for all i and for all t.*

ii) The portfolio allocation criterion, determining the proportions of stocks and bonds, depends on the relationship between $\|\theta\|_T$ and $|z_\alpha|$, and changes over time. Since $\|\theta\|_T$ is an increasing function of the time horizon T, an increasing in the time horizon will imply an increased investment into stocks.

iii) When market value of risk is high enough so that the optimal investment strategy includes stocks, it is clear from expression (19) *that the $CaR(X_0, \pi_{\varepsilon_1}, T)$ is negative, which means that the α-quantile, or the threshold of low returns, is higher than the riskless investment. From the same expression we see that in this case CaR decreases as the time horizon increases.*

iv) Similarly, when the optimal investment strategy includes stocks, increasing the time horizon increases the expected wealth too, despite the fact that the increasing wealth is not an explicit feature of the objective function.

The minimum value of $CaR(X_0, \pi, T)$ given in the theorem above gives us a lower bound for the risk constant C in the following, constrained optimization problem

$$\begin{array}{c} \max\limits_{\pi(\cdot)\in\mathcal{Q}} E\left[X^{\pi}(T)\right] \\ \text{subject to } \; CaR(X_0, \pi, T) \leq C. \end{array} \tag{21}$$

The upper bound for CaR can be readily seen from its very expression, so that we have to impose the constraint

$$C < X_0 R_0(T). \tag{22}$$

Note that the constraint in (21) is then well-defined. Under condition (22), (21) is equivalent to

$$\begin{aligned} &\max_{\pi(\cdot)\in\mathcal{Q}} \langle B,\pi\rangle_T \\ &\text{subject to } \langle B,\pi\rangle_T - \tfrac{1}{2}\|\sigma'\pi\|_T^2 - |z_\alpha|\|\sigma'\pi\|_T) \geq c, \end{aligned} \tag{23}$$

where

$$c = \ln\Big(1 - \frac{C}{X_0 R_0\,(T)}\Big).$$

The solution to (21) is given in the following theorem.

THEOREM 3.2. *Suppose that the constant C satisfies the conditions*

$$\begin{array}{l} i)\ 0 \leq C < X_0 R_0\,(T)\,,\ \ if\ \|\theta\|_T \leq |z_\alpha|, \\ ii)\ 1 - \exp\Big(\frac{1}{2}(\|\theta\|_T - |z_\alpha|)^2\Big) \leq \frac{C}{X_0 R_0(T)} < 1,\ \ if\ \|\theta\|_T > |z_\alpha|. \end{array} \tag{24}$$

Then the optimal solution to problem (23) is

$$\pi_{\varepsilon_2}(t) = \tfrac{\varepsilon_2}{\|\theta\|_T}(\sigma(t)\sigma(t)')^{-1}B(t) \in \mathcal{Q},$$

where $\varepsilon_2 = \|\theta\|_T - |z_\alpha| + \sqrt{(\|\theta\|_T - |z_\alpha|)^2 - 2c}$, with the corresponding optimal wealth

$$E\left[X^{\pi_{\varepsilon_2}}(T)\right] = X_0 R_0\,(T) \exp\left(\varepsilon_2\,\|\theta\|_T\right),$$

and the corresponding capital at risk

$$CaR(X_0, \pi_{\varepsilon_2}, T) = C.$$

The proof of the theorem is given in [4].

REMARK 3.2. *i) The conditions for C in both cases guarantee that the expressions for $\pi_{\varepsilon_2}(t)$ and $E\left[\mathrm{X}^{\pi_{\varepsilon_2}}(\mathrm{T})\right]$ are well defined.*

ii) In the first case, the conditions $0 \leq C < X_0 R_0\,(T)$ transform into the condition $0 \geq \ln(1 - \frac{C}{X_0 R_0(T)}) = c$. If a risk averse investor decides to take no risk, i.e., chooses $C = 0$, then $c = 0$ and $\pi_{\varepsilon_2}(t) = 0$, so that the optimal strategy consists of investing the total wealth into the bond with $CaR(X_0, \pi_{\varepsilon_2}, T) = 0$. This is consistent with case i) of Theorem 3.1. However, a more risk tolerant investor will choose $C > 0$, i.e. $c < 0$ which leads to a higher proportion of the total wealth being invested into stocks and resulting in a higher expected return. Moreover, as the time horizon T increases, $E\left[\mathrm{X}^{\pi_{\varepsilon_2}}(\mathrm{T})\right]$ increases, while $CaR(X_0, \pi_{\varepsilon_2}, T)$ remains constant.

iii) In the second case, the conditions for C can be transformed into the condition $\frac{1}{2}(\|\theta\|_T - |z_\alpha|)^2) \geq \ln(1 - \frac{C}{X_0 R_0(T)}) = c$. If a risk *averse investor decides to take no risk, i.e. chooses* $C = X_0R_0(T)\Big(1 - \exp(\frac{1}{2}(\|\theta\|_T - |z_\alpha|)^2)\Big)$*, then* $c = \frac{1}{2}(\|\theta\|_T - |z_\alpha|)^2$*, giving the same portfolio and minimal* $CaR(X_0, \pi_{\varepsilon_2}, T)$ *as in Theorem 3.1. A more risk tolerant investor will choose a larger* C*, implying a smaller and negative* c*, leading to the larger proportion of the wealth invested in stocks. Thus, again, an increasing time horizon leads to increasing expected wealth* $E[\mathrm{X}^{\pi_{\varepsilon_2}}(\mathrm{T})]$*, and constant capital at risk* $CaR(X_0, \pi_{\varepsilon_2}, T)$*.*

iii) As previously, we see that neither the expected wealth nor capital at risk depend directly on the stocks; the dependence is indirect—through the L^2 *norm of the market price of risk* $\theta(t)$*. This is again an illustration of the two-fund separation theorem, in the sense that every mean-CaR efficient portfolio is a weighted average of Merton's portfolio and the bond, and that the weights depend on the investor's risk tolerance.*

We now turn to portfolio optimization with respect to *VaR*.

3.2. Portfolio Optimization with Respect to *VaR*

From the definition of *VaR* it is clear that $VaR(X_0, \pi, t) \geq 0$, and that $VaR(X_0, \pi, t)$ has a unique global minimum at $\pi(\cdot) = 0$, i.e., $VaR(X_0, 0, t) = 0$, so that it is pointless to look at the unconstrained minimization of *VaR*. We look at maximization of the expected wealth while keeping *VaR* constrained, i.e., at the problem

$$\max_{\pi\in\mathcal{Q}} E[X^\pi(T)] \text{ subject to } VaR(X_0, \pi, T) \leq C. \tag{25}$$

Using (9) and (14), (25) can be expressed as

$$\begin{gathered}\max_{\pi\in\mathcal{Q}} \langle B', \pi\rangle_T \\ \text{subject to } \exp(\langle B', \pi\rangle_T)\Big(1 - \exp\Big(-\tfrac{1}{2}\|\sigma'\pi\|_T^2 - |z_\alpha|\|\sigma'\pi\|_T\Big)\Big) \leq c,\end{gathered} \tag{26}$$

where

$$c = \frac{C}{X_0R_0(T)}. \tag{27}$$

The optimal solution of problem (26) is given by

THEOREM 3.3. *The optimal solution of problem (26) is*

$$\pi_{\varepsilon_3}(t) = \varepsilon_3 \frac{(\sigma(t)\sigma'(t))^{-1} B(t)}{\|\theta\|_T},$$

where ε_3 *is the unique solution of*

$$\exp(\varepsilon\|\theta\|_T)\left(1-\exp\left(\frac{-1}{2}\varepsilon^2-|z_\alpha|\varepsilon\right)\right)=c,$$

and c *is defined by (27). The corresponding expected wealth is*

$$E[X^{\pi_{\varepsilon_3}}(T)]=X_0R_0\,(T)\exp(\varepsilon_3\|\theta\|_T),$$

with corresponding value at risk

$$VaR(X_0,\pi_{\varepsilon_3},T)=C.$$

Proof of Theorem 3.3 . We now apply the dimension reduction procedure by rewriting (26) as

$$\begin{gathered}\max_{\varepsilon\geq 0}\max_{\pi\in\mathcal{Q}_\varepsilon}\langle B,\pi\rangle_T\\ \text{subject to }\exp\left(\langle B,\pi\rangle_T\right)\left(1-\exp\left(-\tfrac{1}{2}\|\sigma'\pi\|_T^2-|z_\alpha|\|\sigma'\pi\|_T\right)\right)\leq c.\end{gathered}\tag{28}$$

We deal with this problem in two stages. First, fixing ε reduces the problem to

$$\begin{gathered}\max_{\pi\in\mathcal{Q}_\varepsilon}\langle B,\pi\rangle_T\\ \text{subject to }\exp\left(\langle B,\pi\rangle_T\right)\left(1-\exp\left(-\tfrac{1}{2}\varepsilon^2-|z_\alpha|\varepsilon\right)\right)\leq c.\end{gathered}$$

If π_ε denotes the optimal solution to this problem, then optimizing over all $\varepsilon\geq 0$ results in the problem

$$\begin{gathered}\max_{\varepsilon\geq 0}\langle B,\pi_\varepsilon\rangle_T\\ \text{subject to }\exp\left(\langle B,\pi_\varepsilon\rangle_T\right)\times\left(1-\exp\left(\tfrac{-1}{2}\varepsilon^2-|z_\alpha|\varepsilon\right)\right)\leq c.\end{gathered}\tag{29}$$

Therefore, we can reduce the m-dimensional optimization problem (28) to the above one-dimensional problem.

Thus we will first solve the problem

$$\max_{\pi(\cdot)\in\mathcal{Q}_\varepsilon}\langle B,\pi\rangle_T,\tag{30}$$

and then show that the solution to (30) substituted into (29) provides the unique solution to (28). We state

PROPOSITION 3.1. *For a fixed* ε, *the optimal solution of problem* (30) *is*

$$\pi_\varepsilon(t)=\frac{\varepsilon}{\|\theta\|_T}(\sigma(t)\sigma(t)')^{-1}B(t)\in\mathcal{Q}_\varepsilon.\tag{31}$$

The proof of proposition 3.1 is given in [5]. We note that the proof includes a square completion technique and a saddle-point argument.

Substituting (31) into (29) transforms it into the problem

$$\begin{aligned} &\max_{\varepsilon \geq 0} \varepsilon \|\theta\|_T \\ \text{subject to } &\exp(\varepsilon\|\theta\|_T)\left(1 - \exp(\tfrac{-1}{2}\varepsilon^2 - |z_\alpha|\varepsilon)\right) \leq c. \end{aligned} \tag{32}$$

The solution to this problem is given in the following lemma.

LEMMA 3.1. *The optimal solution of problem (32) is ε_3, defined to be the unique solution of the equation*

$$\exp(\varepsilon\|\theta\|_T)\left(1 - \exp\left(\frac{-1}{2}\varepsilon^2 - |z_\alpha|\varepsilon\right)\right) = c. \tag{33}$$

Proof of Lemma 3.1. Since the objective function of problem (32) is a linear increasing function of ε, the optimal solution of the problem is the maximal ε for which the constraint is satisfied. In order to find such a value, we look at the function

$$\gamma(\varepsilon) := \exp(\varepsilon\|\theta\|_T)\left(1 - \exp\left(\frac{-1}{2}\varepsilon^2 - |z_\alpha|\varepsilon\right)\right) - c. \tag{34}$$

We have

$$\gamma(0) = -c < 0 \tag{35}$$

and

$$\begin{aligned} \gamma'(\varepsilon) = {} & \|\theta\|_T \exp(\varepsilon\|\theta\|_T)\left(1 - \exp\left(\frac{-1}{2}\varepsilon^2 - |z_\alpha|\varepsilon\right)\right) \\ & + \exp(\varepsilon\|\theta\|_T)\exp\left(\frac{-1}{2}\varepsilon^2 - |z|_\alpha\right)(\varepsilon + |z_\alpha|). \end{aligned}$$

Thus $\gamma'(\varepsilon) > 0$ for all $\varepsilon > 0$. Moreover,

$$\lim_{\varepsilon \to \infty} \gamma(\varepsilon) = +\infty,$$

so that (33) has a unique solution, ε_3. Clearly, ε_3 is the maximal value for which the constraint in (32) is satisfied, so that ε_3 is the optimal solution of problem (32). This ends the proof of the lemma.

To complete the proof of Theorem 3.3 it remains to show that $\pi_{\varepsilon_3}(t) = \frac{\varepsilon_3}{\|\theta\|_T}(\sigma(t)\sigma(t)')^{-1}B(t)$ is the optimal solution to problem (29).

Suppose that there exists $\widehat{\pi}(t)$ such that $\langle B, \widehat{\pi}\rangle_T > \langle B, \pi_{\varepsilon_3}\rangle_T$, and set $\widehat{\varepsilon} := \|\sigma'\widehat{\pi}\|_T$. It follows from Proposition 3.1 that there exists

$\pi_{\widehat{\varepsilon}}(t) = \frac{\widehat{\varepsilon}}{\|\theta\|_T}(\sigma(t)\sigma(t)')^{-1}B(t)$ satisfying $\langle B, \pi_{\widehat{\varepsilon}}\rangle_T \geq \langle B, \widehat{\pi}\rangle_T$. We now consider the following two cases:

i) $\widehat{\varepsilon} < \varepsilon_3$. In this case we have $\langle B, \widehat{\pi}\rangle_T \leq \langle B, \pi_{\widehat{\varepsilon}}\rangle_T = \widehat{\varepsilon}\|\theta\|_T < \varepsilon_3\|\theta\|_T = \langle B, \pi_{\varepsilon_3}\rangle_T$, which is a contradiction.

ii) $\widehat{\varepsilon} > \varepsilon_3$. In this case we have

$$\begin{aligned}\exp(\langle B, \widehat{\pi}\rangle_T)&\left(1 - \exp\left(\frac{-1}{2}\widehat{\varepsilon}^2 - |z_\alpha|\widehat{\varepsilon}\right)\right)\\ &> \exp(\langle B, \pi_{\varepsilon_3}\rangle_T)\left(1 - \exp\left(\frac{-1}{2}\varepsilon_3^2 - |z_\alpha|\varepsilon_3\right)\right)\\ &= \exp(\varepsilon_3\|\theta\|_T)\left(1 - \exp\left(\frac{-1}{2}\varepsilon_3^2 - |z_\alpha|\varepsilon_3\right)\right) = c.\end{aligned}$$

Thus, in this case the constraint in problem (29) is violated so that $\pi_{\varepsilon_3}(t)$ is the optimal solution to problem (29), which, substituted into the expression for $E[X^{\pi_{\varepsilon_3}}(T)]$ and $VaR(X_0, \pi_{\varepsilon_3}, T)$, gives

$$E[X^{\pi_{\varepsilon_3}}(T)] = X_0R_0(T)\exp(\varepsilon_3\|\theta\|_T),$$

and

$$VaR(X_0, \pi_{\varepsilon_3}, T) = X_0R_0(T)\exp(\varepsilon_3\|\theta\|_T)\left(1 - \exp\left(\frac{-1}{2}\varepsilon_3 - |z_\alpha|\varepsilon_3\right)\right).$$

Since ε_3 is the solution of equation (33), the above expression reduces to

$$VaR(X_0, \pi_{\varepsilon_3}, T) = X_0R_0(T)c = X_0R_0(T)\frac{C}{X_0R_0(T)} = C.$$

This completes the proof of Theorem 3.3.

We note that although we cannot give an explicit solution to equation (33), we can find bounds for ε_3, provided that appropriate bounds for C are given. For instance, it may be reasonable to restrict VaR to be smaller than the wealth resulting from investing everything in the bond, i.e., to put the constraint for C

$$C < X_0R_0(T). \tag{36}$$

Since we are not interested in the trivial case when $C = VaR = 0$, we also assume that $C > 0$, so that, under these assumptions we have

$$0 < c < 1. \tag{37}$$

Now we can give bounds for ε_3 in the following lemma.

LEMMA 3.2. *Suppose that the constant* $C > 0$ *satisfies condition (36). Then the optimal solution* ε_3 *of problem (32), satisfies*

$$0 \leq \varepsilon_3 < -|z_\alpha| + \sqrt{|z_\alpha|^2 + 2d}, \tag{38}$$

where $d = -\ln(1-c)$, *and* c *is given in (27).*

Proof of Lemma 3.2. The left hand side of (38) is obvious. For the right hand side, let γ be given by (34), and set $\overline{\varepsilon} = -|z_\alpha| + \sqrt{|z_\alpha|^2 + 2d}$. (Note that under conditions (36) and (37), d is well defined.) Since γ is an increasing function and $\gamma(\varepsilon_3) = 0$, it is enough for us to show that $\gamma(\overline{\varepsilon}) > 0$. Using the fact that $\overline{\varepsilon}^2 + 2|z_\alpha|\overline{\varepsilon} = 2d$, we find

$$\begin{aligned} \gamma(\overline{\varepsilon}) &= \exp(\overline{\varepsilon}\|\theta\|_T)\left(1 - \exp\left(\frac{-1}{2}\overline{\varepsilon}^2 - |z_\alpha|\overline{\varepsilon}\right)\right) - c \\ &= \exp(\overline{\varepsilon}\|\theta\|_T)(1 - \exp(-d)) - c \\ &= \exp(\overline{\varepsilon}\|\theta\|_T)(1 - \exp(\ln(1-c))) - c \\ &= \exp(\overline{\varepsilon}\|\theta\|_T)\,c - c. \end{aligned}$$

Under assumption (37), $d > 0$, which implies that $\overline{\varepsilon} > 0$ so that, since $c > 0$, we have

$$\exp(\overline{\varepsilon}\|\theta\|_T)\,c > c,$$

and then $\gamma(\overline{\varepsilon}) > 0$ as required.

REMARK 3.3. *i) It will be shown that* $-|z_\alpha| + \sqrt{|z_\alpha|^2 + 2d}$ *is the optimal wealth coefficient obtained under constrained RVaR optimization. Financial implications of this fact will be discussed below.*

ii) We see that, as in case of optimization with respect to CaR, neither the expected wealth nor VaR depend directly on the stocks. The dependence is again indirect—through the L^2 *norm of the market price of risk* $\theta(t)$, *which again illustrates the two-fund separation theorem.*

3.3. Portfolio Optimization with Respect to *RVaR*

It is clear from (16) that *RVaR* satisfies

$$0 \leq RVaR(\pi, T) < 1. \tag{39}$$

It also follows that *RVaR* has a unique minimum attained at $\pi(t) \equiv 0$. Therefore, we turn our attention to the following problem

$$\max_{\pi \in Q} E[X^\pi(T)] \text{ subject to } RVaR(\pi, T) \leq D. \tag{40}$$

From (39), we can assume that the constant D satisfies

$$0 \leq D < 1. \tag{41}$$

Using (16) and (9), (40) can be written in the form

$$\max_{\pi\in Q} \langle B,\pi\rangle_T \text{ subject to } 1-\exp\left(-\frac{1}{2}\|\sigma'\pi\|_T^2-|z_\alpha|\|\sigma'\pi\|_T\right)\le D,$$

which is equivalent to

$$\max_{\pi\in Q} \langle B,\pi\rangle_T \text{ subject to } \frac{1}{2}\|\sigma'\pi\|_T^2+|z_\alpha|\|\sigma'\pi\|_T\le d, \tag{42}$$

where $d=-\ln(1-D)$. Note that (41) guarantees that d is well defined, and nonnegative. Finally, using the dimension reduction procedure as in the previous section, we can write (42) as a one-parameter optimization problem:

$$\max_{\varepsilon\ge 0}\max_{\pi(\cdot)\in\mathcal{Q}_\varepsilon} \langle B,\pi\rangle_T \text{ subject to } \frac{1}{2}\|\sigma'\pi\|_T^2+|z_\alpha|\|\sigma'\pi\|_T\le d. \tag{43}$$

The solution to (43) is given in the following theorem.

THEOREM 3.4. *Suppose that constant D satisfies* (41)*. Then the optimal solution to (40) is*

$$\pi_{\varepsilon_4}(t)=\varepsilon_4\frac{(\sigma(t)\sigma'(t))^{-1}B(t)}{\|\theta\|_T},$$

where

$$\varepsilon_4:=-|z_\alpha|+\sqrt{|z_\alpha|^2+2d}, \tag{44}$$

and $d=-\ln(1-D)$. The corresponding expected wealth is

$$E[X^{\pi_{\varepsilon_4}}(T)]=X_0R_0(T)\exp\left(\varepsilon_4\|\theta\|_T\right),$$

and the corresponding relative value at risk is D.

Proof of Theorem 3.4. We apply the same approach as in the proof of Theorem 3.3. For a fixed ε, (43) reduces to

$$\max_{\pi\in Q_\varepsilon}\langle B,\pi\rangle_T,$$

which, according to Proposition 3.1, has the optimal solution

$$\pi_\varepsilon(t)=\frac{\varepsilon}{\|\theta\|_T}(\sigma(t)\sigma(t)')^{-1}B(t). \tag{45}$$

Substituting (45) into (43) transforms it into the problem

$$\max_{\varepsilon\ge 0}\varepsilon\|\theta\|_T \text{ subject to } \frac{1}{2}\varepsilon^2+|z_\alpha|\varepsilon-d\le 0. \tag{46}$$

To solve (46), we note that the objective function of (46) is linear, and the constraint is quadratic. Therefore, the optimal solution of (46) is the maximal ε for which the constraint in (46) is satisfied. This solution is the larger solution of the quadratic equation

$$\frac{1}{2}\varepsilon^2 + |z_\alpha|\varepsilon - d = 0,$$

which is ε_4 as defined in (44). Note that (41) guarantees that ε_4 is always real and nonnegative. Substituting $\pi_{\varepsilon_4}(t)$ into (9) and (16) we get

$$E[X^{\pi_{\varepsilon_4}}(T)] = X_0 R_0\,(T) \exp\left(\varepsilon_4 \|\theta\|_T\right),$$

and $RVaR = D$. Using the same arguments as in the proof of Theorem 3.3 we can conclude that any other feasible solution to (43) either gives a smaller value of the objective function, or violates the constraint, so that this ends the proof of the theorem.

This theorem, together with Lemma 3.2, has an immediate, important consequence, given in the following corollary.

COROLLARY 3.1. *Suppose that the constant* C *from (25) satisfies the condition*

$$0 < C < X_0 R_0\,(T)$$

and that constant D *from (40) satisfies* $D = \frac{C}{X_0 R_0(T)}$. *Then* $\varepsilon_3 < \varepsilon_4$.

REMARK 3.4. *i)* $\varepsilon_3 < \varepsilon_4$ *implies that* $E\left[X^{\pi_{\varepsilon_3}}(T)\right] < E\left[X^{\pi_{\varepsilon_4}}(T)\right]$. *Given that*

$$\frac{VaR(X_0, \pi_{\varepsilon_3}, T)}{X_0 R_0\,(T)} = D = RVaR(\pi_{\varepsilon_4}, T),$$

the implication is that VaR is a more conservative risk measure than RVaR, i.e. optimization with respect to VaR allocates less wealth into the stocks than optimization with respect to RVaR, resulting in smaller expected wealth, even if the normalized level of risk, $\frac{C}{X_0 R_0(T)}$, *is the same for both risk measures.*

ii) It follows from (44) that ε_4 *does not depend on* $\|\theta\|_T$, *but only on the risk level* α *and the risk constant* D, *which is indicative of the fact that RVaR is a relative risk measure and is independent of the market conditions.*

4. Asymptotic Behaviour of Optimal Solutions

In this section we look at asymptotic behaviour of the optimal wealth coefficients ε_i, and the corresponding proportions of wealth invested into risky assets—which are determined by $\frac{\varepsilon_i}{\|\theta\|_T}$. We first investigate the cases of minimal and constrained *CaR* in the following proposition.

PROPOSITION 4.1. *Let* ε_1 *and* ε_2 *be the optimal wealth coefficients for problems (17) and (21). If* $\lim\limits_{T\to\infty} \|\theta\|_T = \infty$*, and if* C *is fixed, then*

i) $\lim\limits_{T\to\infty} \varepsilon_1 = \infty$, $\lim\limits_{T\to\infty} \frac{\varepsilon_1}{\|\theta\|_T} = 1$.

ii) $\lim\limits_{T\to\infty} \varepsilon_2 = \infty$, $\lim\limits_{T\to\infty} \frac{\varepsilon_2}{\|\theta\|_T} = 2$.

Proof of Proposition 4.1 . The first pair of statements follow from the fact that $\varepsilon_1 = \|\theta\|_T - |z_\alpha|$, while the second pair can be deduced from the equation $\varepsilon_2 = \|\theta\|_T - |z_\alpha| + \sqrt{(\|\theta\|_T - |z_\alpha|)^2 - 2c}$ and the fact that $-2c = -2\ln\left(1 - \frac{C}{X_0 R_0(T)}\right) \to 0$, as $T \to \infty$.

REMARK 4.1. *The above results imply that, under constrained CaR, the optimal investment strategey is consistent with the idea that stocks, at least in the long run, outperform bonds.*

We now turn to the asymptotic behaviour of optimal portfolios under constrained *VaR*, and point out at one serious flaw of *VaR*. Namely ε_3, the unique solution of equation (33), is a decreasing function of the L^2-norm of the market price of risk, $\|\theta\|_T$. We state this formally in the following proposition.

PROPOSITION 4.2. *Suppose that* ε_3 *is the optimal wealth coefficient for problem (25). Then, for fixed* C*,* ε_3 *is a decreasing function of* $\|\theta\|_T$*.*

Proof of Proposition 4.2. Treating (33) as defining a functional relationship between ε_3 and $\|\theta\|_T$, we differentiate with respect to ε_3. Writing $\frac{d\|\theta\|_T}{d\varepsilon_3}$ as $\|\theta\|'_T$ we obtain

$$0 = \left(\|\theta\|_T + \varepsilon\,\|\theta\|'_T\right)\exp\left(\varepsilon\,\|\theta\|_T\right)\left(1 - \exp\left(-\frac{1}{2}\varepsilon_3^2 - |z_\alpha|\,\varepsilon_3\right)\right)$$
$$+ \exp\left(\varepsilon\,\|\theta\|_T\right)\left(\varepsilon + |z_\alpha|\right)\exp\left(-\frac{1}{2}\varepsilon_3^2 - |z_\alpha|\,\varepsilon_3\right).$$

Since $\varepsilon_3 > 0$, $\left(1 - \exp\left(-\frac{1}{2}\varepsilon_3^2 - |z_\alpha|\,\varepsilon_3\right)\right) > 0$, and so each of the terms on the right-hand side of the above equation, with the exception of $\|\theta\|'_T$, is strictly positive. It follows that $\|\theta\|'_T < 0$ so that $\|\theta\|_T$ is a decreasing function of ε_3. Moreover, this implies that ε_3, given by (33) for fixed C, is a well defined decreasing function of $\|\theta\|_T$, which ends the proof of the proposition.

REMARK 4.2. *i) The above proposition has a serious, undesirable consequence. Namely, as illustrated in Figure 4 (below), increasing* $\|\theta\|_T$ *leads to decreasing* ε_3*, which leads to an even sharper decrease of* $\frac{\varepsilon_3}{\|\theta\|_T}$ *— the portion of the total wealth invested into the risky assets under constrained VaR. Thus, under constrained VaR, a higher value of risk implies a lower investment into risky assets. ii) In the same way, simply increasing the time horizon* T *leads to increasing* $\|\theta\|_T$*, again further decreasing the ratio* $\frac{\varepsilon_3}{\|\theta\|_T}$*. Thus, choosing VaR as a risk measure leads to investing less wealth into the risky assets over a longer time horizon, which runs counter to the idea that stocks are more likely to outperform bonds in the long run.*

Finally, the asymptotic behaviour of mean–*RVaR* efficient portfolios follows immediately from Theorem 3.4, and we state it in the following corollary.

COROLLARY 4.1. *If* ε_4 *is the optimal wealth coefficient for problem (39), then, for fixed* D*,* ε_4 *is constant, and* $\lim_{T\to\infty} \frac{\varepsilon_4}{\|\theta\|_T} = 0$*.*

REMARK 4.3. *The above corollary has the same undesirable implications for the wealth distribution as in case of VaR, i.e., under constrained RVaR, we invest less wealth into the risky assets in better markets and over a longer time horizon. However, this drawback is less severe in this case since the ratio* $\frac{\varepsilon_4}{\|\theta\|_T}$ *decreases less rapidly than the ratio* $\frac{\varepsilon_3}{\|\theta\|_T}$*, as the time horizon increases.*

We will illustrate the results of Propositions 4.1 and 4.2, and Corollary 4.1 in the following section.

5. Numerical Results

In this section we consider several examples which illustrate the results from the previous sections. We recall that the stock returns' variance-covariance matrix, which we denote by $\Gamma(t)\,dt$, is equal to $\sigma(t)\,\sigma(t)'\,dt$. We also recall that $\Gamma(t)$ can be decomposed as

$$\Gamma(t) = \nu(t)\,\rho(t)\,\nu(t)\,,$$

where $\rho(t)$ is the stock returns' correlation matrix, and $\nu(t)\sqrt{dt}$ is a diagonal matrix with the entries equal to the stock returns' standard deviations. Therefore, we have that

$$\Gamma(t) = \nu(t)\,\rho(t)\,\nu(t) = \sigma(t)\,\sigma(t)'\,. \tag{47}$$

Although, theoretically, we assume that the vector of independent Brownian motions $W(t)$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$ adapted, i.e. known at time $t \in [0,T]$, in practice we only observe $\Gamma(t)$ or, equivalently, $\nu(t)$ and $\rho(t)$, but not $\sigma(t)$. From (47) we see that this leads to a nonunique decomposition of $\Gamma(t)$ into the product $\sigma(t)\sigma(t)'$. Despite that fact, the Euclidean norm—and consequently the L^2 norm—of the market price of risk are uniquely determined by

$$\|\theta(t)\|^2 = \|\sigma(t)^{-1}B(t)\|^2 = B(t)'(\sigma(t)\sigma(t))^{-1}B(t),$$

or, in the terms of the standard deviation and the correlation matrix, as

$$\|\theta(t)\|^2 = B(t)'\nu(t)^{-1}\rho(t)^{-1}\nu(t)^{-1}B(t). \tag{48}$$

We now turn our attention to three special cases with the following characteristics:

i) The interest rate is $r(t) = 0.05$, and the number of stocks is $m = 3$.

ii) In order to capture cycles in the economy or in the dynamics of the stocks we model the drifts as

$$b_i(t) = \mu_i + \beta_i \cos(\varphi_i t), \ i = 1, 2, 3. \tag{49}$$

We note that a similar model was used in [20].

iii) We let $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$, with $\varphi = 0.75$, that is, the economic cycles of all three stocks are the same, approximately equal to 8.4 years, and we consider $\beta_1 = 0.75\beta$, $\beta_2 = 0.5\beta$, $\beta_3 = 0.25\beta$, with $\beta = 0.015$, which corresponds to a 1.5% deviation around the constant values μ_i.

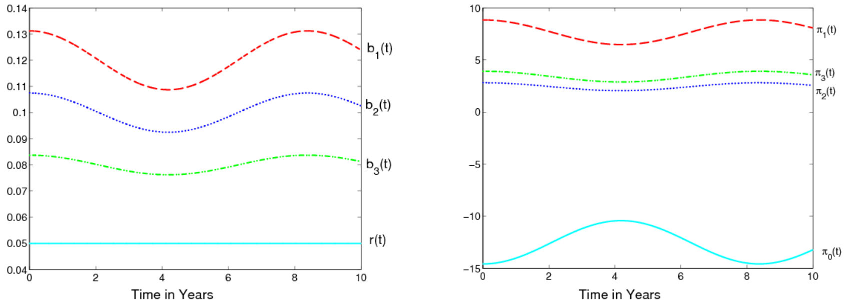
iv) We assume that the stocks' returns have constant standard deviations given below

$$\nu_1(t) = 20\%, \ \nu_2(t) = 25\%, \ \nu_3(t) = 30\%.$$

In all numerical computations and the corresponding plots we use an annual time scale for the drifts, standard deviations and the correlation matrix, and look at the time horizon of 10 years.

EXAMPLE 5.1. *In addition to assumptions i), ii) and iii) we assume that the average return rates are* $\mu_1 = 0.12$, $\mu_2 = 0.10$ *and* $\mu_3 = 0.08$. *We also assume that the correlation matrix is as given below*

$$\rho_1 = \begin{bmatrix} 1.0 & -0.6 & -0.8 \\ -0.6 & 1.0 & 0.5 \\ -0.8 & 0.5 & 1 \end{bmatrix}.$$



a. b.

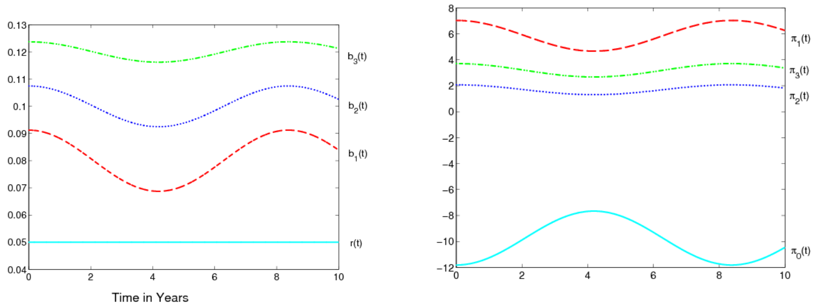
Figure 1. Example 5.1: a) Stock drifts given by $b_1(t) = 0.12 + 0.01125 \cos(0.75t)$, $b_2(t) = 0.1 + 0.0075 \cos(0.75t)$ and $b_3(t) = 0.08 + 0.00375 \cos(0.75t)$, with $T = 10$ years, and $r(t) = 0.05$. b) Portfolio weights of Merton's portfolio, with $T = 10$ years.

The actual time evolutions of the drifts of the three stocks and the interest rate are plotted in Figure 1.a, while the portfolio weights of Merton's portfolio for Example 5.1 are given in Figure 1.b.

In this example, we see the intuitively justified result, i.e., stock 1, which has the largest constant part in the drift and the smallest volatility, is present in the optimal portfolio in the highest percentage.

EXAMPLE 5.2. *We keep the same correlation matrix* $\rho = \rho_1$, φ_i *and* β_i, *as in Example 5.1, while changing the constant parts of the drifts into* $\mu_1 = 0.08$, $\mu_2 = 0.10$, $\mu_3 = 0.12$.

The stocks' drifts movements are given in Figure 2a. The portfolio weights of the corresponding Merton's portfolio are given in Figure 2b.



a. b.

Figure 2. Example 5.2: a) Stock drifts given by $b_1(t) = 0.08 + 0.01125 \cos(0.75t)$, $b_2(t) = 0.1 + 0.0075 \cos(0.75t)$ and $b_3(t) = 0.12 + 0.00375 \cos(0.75t)$, with $T = 10$ years, and $r(t) = 0.05$. b) Portfolio weights of Merton's portfolio, with $T = 10$ years.

In this example, we see that stock 1, which now has the smallest constant part in the drift, is still present in the optimal portfolio in the highest percentage, due to its high negative correlations with both stocks 2 and 3.

EXAMPLE 5.3. *In this example we assume again that the stocks have the same constant parts of the drifts μ_i, φ_i and β_i as in Examples 5.2, while the correlation matrix is*

$$\rho_3 = \begin{bmatrix} 1.0 & 0.2 & -0.3 \\ 0.2 & 1.0 & 0.1 \\ -0.3 & 0.1 & 1 \end{bmatrix}.$$

The portfolio weights of the corresponding Merton's portfolio are given in Figure 3.

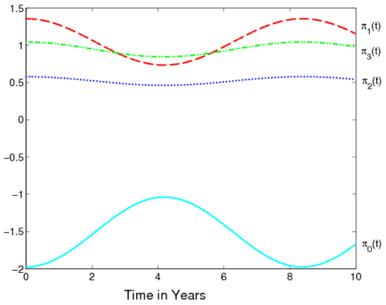


Figure 3. Portfolio weights of Merton's portfolio for Example 5.3, with $T = 10$ years.

We note that in the case of low correlations, such as in Example 5.3, the percentages of total wealth invested into the risky assets are much lower than in the previous examples.

We now look at the actual proportions of wealth, given by $\frac{\varepsilon}{\|\theta\|_T}$ invested into risky assets, under constrained *CaR*, *VaR* and *RVaR*.

Capital at Risk

We assume that the initial wealth is $X_0 = 1000$, for all three examples, and give the tables with the L^2 norms of the market price of risk $\|\theta\|_T$, values of the wealth coefficient ε_1, *CaR* and the expected wealth. We consider the risk level $\alpha = 0.05$.

Table I illustrates the results of Theorem 3.1 and Proposition 4.1. We see that in cases where $\|\theta\|_T > |z_\alpha|$ minimal *CaR* is negative, and that the wealth coefficient ε_1 increases when $\|\theta\|_T$ increases, so that under minimal *CaR* we invest more wealth into the risky assets in markets with higher market price of risk, and over a longer time horizon.

Table I. Optimal values of ε_1, $E\left[X^\pi(T)\right]$ and CaR for $C = C_{\min}$ under constrained CaR.

	Example 5.1	Example 5.2	Example 5.3
$\|\theta\|_T$	2.8268	2.2711	1.1420
ε_1	1.1656	0.6058	0
CaR	−1603.3	−332.1	0
$E\left[X^\pi(T)\right]$	43639	6447.2	1648.7

Table II. Optimal values of ε_3, $E\left[X^\pi(T)\right]$ and VaR, with VaR constrained to be less than $C = 0.9C_{\max}$.

	Example 5.1	Example 5.2	Example 5.3
$\|\theta\|_T$	2.8268	2.2711	1.1420
ε_3	0.286	0.318	0.43
VaR	1484	1484	1484
$E\left[X^\pi(T)\right]$	3701	3395	2694

Value at Risk

We recall that VaR is bounded below by 0. Since the assumption $\pi \in \mathcal{Q}$ guarantees that VaR is bounded above too, we can choose any real number for the upper bound for VaR. In order to be able to compare the optimal wealth and the risks as measured by VaR and $RVaR$, in the following examples we will look at the values for which $VaR\left(X_0, \pi, T\right) \leq 0.9C_{\max}$ where $C_{\max} = X_0 R_0(T)$. Applying Theorem 3.3 yields Table II.

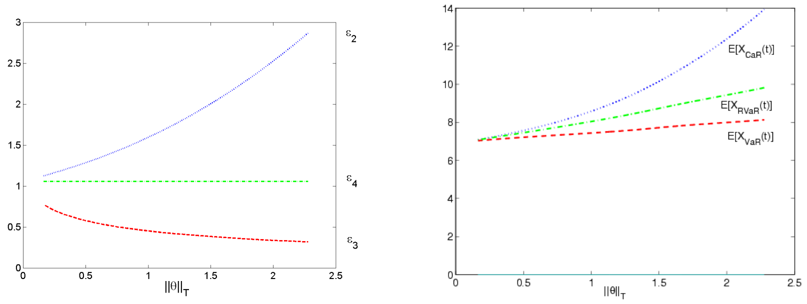
Table II illustrates Proposition 4.2 and Remark 4.2 ii), and shows the pathological behaviour of VaR, in the sense that a minimal VaR strategy allocates more wealth into markets with smaller L^2 norm of the market price of risk $\|\theta\|_T$.

Relative Value at Risk

As value at risk, relative value at risk is bounded below by 0, while above by 1. In order to compare the optimal results obtained under constrained $RVaR$ with other results, we choose for the upper bound of $RVaR$ to be 0.9. Applying Theorem 3.4, and choosing $C = 0.9$ we get the results in Table III.

Table III. Optimal values of ε_4, $E\left[X^{\pi}\left(T\right)\right]$ and *RVaR* for *RVaR* constrained to be less than $D = 0.9$.

	Example 5.1	Example 5.2	Example 5.3
$\Vert\theta\Vert_T$	2.8268	2.2711	1.1420
ε_4	1.059	1.059	1.059
RVaR	0.9	0.9	0.9
$E\left[X^{\pi}\left(T\right)\right]$	32896	18264	5525



a. b.

Figure 4. Optimal values of a) ε_2, ε_3, ε_4 and b) optimal wealth under constrained *CaR*, *VaR* and *RVaR*, on a logarithmic scale, for Example 5.1 as T changes from 0 to 10 years.

Table III is given for the sake of completenes, and illustrates the results of Theorem 3.4, showing that under constrained *RVaR* optimal wealth coefficient is constant.

Figure 4 illustrates the asymptotic behaviour of the wealth coefficients, and the expected wealth corresponding to constrained *CaR*, *VaR*, and *RVaR* as the time horizon increases. Also shown are the corresponding values of the optimal wealth. Note how the optimal wealth coefficients ε_2 and ε_3 move in opposite directions as the time horizon increases, while ε_4 remains constant, and observe how the magnitude of the optimal wealth coefficient has a significant impact on the magnitude of the expected wealth.

6. Conclusion

In this work we investigated continuous time portfolio selection under the constrained capital at risk, value at risk and relative value at risk, within the Black-Scholes asset pricing paradigm, with time dependent coefficients. Based on an idea from [9], generalized in [4] and [5], we employed a dimension reduction procedure which transforms $m-$dimensional optimization problems into one-dimensional problems, within the class of admissible portfolios, which are Borel measurable, bounded and deterministic. It is important to emphasize that we considered time dependent portfolios, in which case the methods developed in [9] no longer work. We illustrated the two-fund separation theorem, by showing that all optimal strategies are the weighted averages of Merton's portfolio and the bond, and the weights depend on the investor's risk tolerance only. Finding an optimal strategy under constrained *VaR* requires solving a nonlinear equation whose solution, i.e. the wealth coefficient ε_3, can be only found numerically. We proved the uniqueness of the optimal solution, and gave the bounds for it. We proved that the optimal wealth coefficient ε_1 and ε_2, corresponding to *CaR*, are increasing functions of the L^2 norm of the market price of risk $\|\theta\|_T$, so that optimal strategies under minimal or constrained *CaR* favour investing into stocks, over a longer time horizon. This indicates that *CaR* could be used as a risk measure in portfolio optimization problems over a longer time horizon. As opposed to ε_1 and ε_2, the wealth coefficients ε_3, corresponding to constrained *VaR*, is a decreasing function of $\|\theta\|_T$, which is counterintuitive and counterproductive. For this reason we suggest that *VaR* be used as a short term risk measure, but not in portfolio optimization problems over a longer time horizon. Finally, the wealth coefficient ε_4, corresponding to constrained *RVaR*, is a constant, depending only on the risk level α, and the risk constant $D \in (0,1)$, so that *RVaR* can be used for comparing risks of portfolios from different markets. Finally we provided several numerical examples which illustrate the importance of diversification, given by the correlation matrix. We also illustratde the asymptotic behaviour of the wealth coefficients corresponding to constrained *CaR*, *VaR*, and *RVaR* as the time horizon increases.

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