# A CLASSICAL PARTICLE WITH SPIN REALIZED BY REDUCTION OF A NONLINEAR NONHOLONOMIC CONSTRAINT

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In this paper we describe the motion of a nonlinear nonholonomically constrained system which after reduction realizes a nonrelativistic classical particle with spin.

## 1. Introduction

The usual physical interpretation of spin is that of internal angular momentum. The main difference between classical spin and angular momentum of a rigid body is that the length of the spin vector is fixed a priori, while the length of the angular momentum vector $J$ is a dynamical variable. This suggests that the dynamics of a spinning particle should be related to that of a rigid body by restricting the phase space to those points where the length of the angular momentum vector is fixed and reducing the rigid body degrees of freedom. In the present paper we carry out this program and obtain Souriau's formulation of the dynamics of a spinning particle, see Souriau [5, p. 195–6].

Since the constraint given by fixing the length of the angular momentum vector is nonlinear in velocities we are lead into the field of dynamics of systems with nonlinear nonholonomic constraints, see Arnold [1], Naimark and Fufaev [4]. The main problem of deciding what the dynamics of such a system should be is avoided here, because we know the dynamics of the rigid body and that the postulated constraint is preserved. Thus we are left with the problem of decribing the reduction of a system with nonlinear nonholonomic constraints. To solve this problem we use the procedure of Bates and Śniatycki [2]. We have two checks that this approach is correct: (i) our procedure gives the well established Souriau model and (ii) it is equivalent to Marsden–Weinstein reduction.

## 2. The unconstrained system

Consider a uniformly charged spherically symmetric rigid body with stationary center of mass. Impose a constant magnetic field $b = (b_1, b_2, b_3)$. Mathematically this unconstrained system is described as follows. Its configuration space is the three-dimensional rotation group SO(3) and its phase space (after trivialization by left translation) is SO(3) $\times$ so(3), where so(3) is the Lie algebra of $3 \times 3$ skew symmetric matrices. On phase space we have the symplectic form

$$\omega(A,X)((AY_1,Z_1),(AY_2,Z_2)) = Ik(Y_1,Z_2) - Ik(Y_2,Z_1) + Ik(X,[Y_1,Y_2]), \qquad (1)$$

where $I$ is the common value of the principal moments of inertia, $(A,X) \in \mathrm{SO}(3) \times \mathrm{so}(3)$, $(AY_i, Z_i) \in T_{(A,X)}(\mathrm{SO}(3) \times \mathrm{so}(3))$ for $i = 1,2$, and $k : \mathrm{so}(3) \times \mathrm{so}(3) \to \mathbb{R} : (X,Y) \to -\frac{1}{2}\,\mathrm{tr}\,XY$ is the Killing metric.[1] The Hamiltonian of the unconstrained system is

$$h : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathbb{R} : (A,X) \to \tfrac{1}{2} I\, k(X,X) - gI\, k(\mathrm{Ad}_A X, B), \qquad (2)$$

where

$$B = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} = i^{-1}(b_1, b_2, b_3) \qquad (3)$$

and $g$ is the gyromagnetic ratio. The second term in (2) represents the interaction of the charged rigid body with the magnetic field. A straightforward calculation shows that the Hamiltonian vector field $X_h$ associated with the Hamiltonian $h$ has integral curves which satisfy

$$\begin{cases} \dot{A} = A(X - g\mathrm{Ad}_{A^{-1}}B), \\ \dot{X} = 0. \end{cases} \qquad (4)$$

There are two Hamiltonian actions of SO(3) on our unconstrained system. First, the "body" action, corresponding to a change of frame fixed in the body which is given by right multiplication on the tangent bundle TSO(3) of SO(3). After trivialization by left translation the body action becomes

$$\begin{aligned} \Psi &: \mathrm{SO}(3) \times (\mathrm{SO}(3) \times \mathrm{so}(3)) \to \mathrm{SO}(3) \times \mathrm{so}(3) : \\ &(C,(A,X)) \to (AC^{-1}, \mathrm{Ad}_C X). \end{aligned} \qquad (5)$$

Second, the "space" action, corresponding to rotation in physical space, given by left multiplication on TSO(3). After trivialization by left translation the space action becomes

$$\begin{aligned} \Phi &: \mathrm{SO}(3) \times (\mathrm{SO}(3) \times \mathrm{so}(3)) \to \mathrm{SO}(3) \times \mathrm{so}(3) : \\ &(C,(A,X)) \to (CA, X). \end{aligned} \qquad (6)$$

[1] See Cushman and Bates [3] for more details about this Lie group model for the rigid body.

The right action (5) is a symmetry of the unconstrained system whereas the left action (6) is not, due to the presence of the magnetic field. The momentum of the left action

$$S : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathrm{so}(3) : (A, X) \to I\mathrm{Ad}_A X \tag{7}$$

has the physical interpretation of angular momentum of the body (spin angular momentum). In (7) we have identified so(3) with so(3)* using the Killing metric $k$. Calculating the Lie derivative of $S$ along the integral curves (4) of the unconstrained vector field $X_h$ we obtain

$$\begin{aligned} \dot{S}(A, X) &= I\Big(\mathrm{Ad}_A(\mathrm{ad}_{X - g\mathrm{Ad}_{A^{-1}}B} X)\Big) \\ &= -gI\,[B, \mathrm{Ad}_A X] \;=\; g\,[S(A, X), B]. \end{aligned} \tag{8}$$

Since the space action is not a symmetry, spin angular momentum $S$ is not conserved. However, from (8) it follows that its magnitude is conserved because

$$\begin{aligned} L_{X_h}\Big(k(S(A, X), S(A, X))\Big) &= 2\,k(\dot{S}(A, X), S(A, X)) \\ &= -2g\,k([B, S(A, X)]\,, S(A, X)) \\ &= 0. \end{aligned}$$

## 3. The constrained system

For a fixed $\sigma > 0$, we constrain the Hamiltonian system $(h, \mathrm{SO}(3) \times \mathrm{so}(3), \omega)$ to the submanifold $M$ consisting of those points $(A, X)$ for which the length of the angular momentum vector $S(A, X)$ equals $\sigma$. In other words,

$$M = \{(A, X) \in \mathrm{SO}(3) \times \mathrm{so}(3) \mid k(S(A, X), S(A, X)) = \sigma^2\}. \tag{9}$$

Using (7) we can write

$$M = \{(A, X) \in \mathrm{SO}(3) \times \mathrm{so}(3) \mid k(X, X) = \sigma^2 I^{-2}\}. \tag{10}$$

Because the magnitude of the angular momentum is conserved by the body action of SO(3), the unconstrained vector field $X_h$ is tangent to $M$.

By its very definition $M$ is a *nonlinear nonholonomic constraint*. $M$ determines a constraint 1-form $\varphi$ on $\mathrm{SO}(3) \times \mathrm{so}(3)$ given by

$$\varphi(A, X)(AY, Z) = k(X, Y). \tag{11}$$

Because $M$ is not a subbundle of $\mathrm{T}M$, the constrained system $(M, X_h|M, \varphi)$ is not linear nonholonomic and therefore falls outside the theory of Bates and Śniatycki [2]. The 1-form $\varphi$ and the tangent bundle $\mathrm{T}M$ of $M$ determine a constraint distribution $H$ on $\mathrm{SO}(3) \times \mathrm{so}(3)$ defined by

$$\begin{aligned} H_{(A,X)} &= \ker \varphi(A, X) \cap \mathrm{T}_{(A,X)}M \\ &= \{(AY, Z) \in \mathrm{T}_{(A,X)}(\mathrm{SO}(3) \times \mathrm{so}(3)) \mid k(X, Y) = k(X, Z) = 0\}. \end{aligned} \tag{12}$$

Observe that the constrained vector field $X_h|M$ does *not* lie in the constraint distribution $H$. To remedy this, we note that $H_{(A,X)}$ is a symplectic subspace of $T_{(A,X)}(SO(3) \times so(3)), \omega(A,X)$. Therefore we may write

$$T_{(A,X)}(SO(3) \times so(3)) = H_{(A,X)} \oplus H^{\perp}_{(A,X)}$$

for every $(A,X) \in (SO(3) \times so(3))$. Here $H^{\perp}_{(A,X)}$ is the symplectic perpendicular of $H_{(A,X)}$.

We can decompose $X_h|M$ into its components on $H_{(A,X)}$ and $H^{\perp}_{(A,X)}$:

$$X_h(A,X) = X_h^H(A,X) + X_h^{H^\perp}(A,X).$$

A calculation shows that for every $(A,X) \in M$,

$$X_h^H(A,X) = (A(-g\,\mathrm{Ad}_{A^{-1}}B + gI^2\sigma^{-2}\,k(X,\mathrm{Ad}_{A^{-1}}B)X), 0) \tag{13}$$

and

$$X_h^{H^\perp}(A,X) = (A(X - gI^2\sigma^{-2}\,k(X,\mathrm{Ad}_{A^{-1}}B)X), 0). \tag{14}$$

Looking forward to the next section, we observe that $X_h^{H^\perp}$ is killed by the reduction process (see the proof of (20)).

## 4. Symmetry and its reduction

We now show that the nonholonomic system $(M, X_h^H|M, \varphi)$ has a symmetry. Consider the action $\Psi$ given by (5). Since the constraint manifold $M$ is invariant under $\Psi$, the induced action $\widetilde{\Psi} = \Psi|(SO(3) \times M)$ is defined. Because the 1-form $\varphi$ is $\widetilde{\Psi}$-invariant, it follows that the constraint distribution $H$ is also $\widetilde{\Psi}$-invariant. Consequently, $\widetilde{\Psi}$ is a symmetry of the nonholonomic system $(M, X_h^H|M, \varphi)$, that is,

$$T_{(A,X)}\Psi_C(X_h^H(A,X)) = X_h^H(\Psi_C(A,X))$$

for every $C \in SO(3)$ and every $(A,X) \in M$.

To gain some insight into the dynamical meaning of $(M, X_h^H|M, \varphi)$ we remove its SO(3) symmetry following the procedure of Bates and Śniatycki [2]. (We also use their notation.) The infinitesimal generator of $\widetilde{\Psi}$ in the direction $Y \in so(3)$ is given by the vector field $X^Y(A,X) = (-AY, -[X,Y])$. Thus the SO(3) symmetry distribution $V$ on $M$ is

$$V_{(A,X)} = \{X^Y(A,X) \in T_{(A,X)}M \mid Y \in so(3)\}.$$

A calculation shows that the distribution $V \cap H$ on $M$ is given by

$$(V \cap H)_{(A,X)} = \{(-AY, -[X,Y]) \in T_{(A,X)}M \mid k(X,Y) = 0\}.$$

Consequently, the distribution

$$U = \{u \in H \,|\, \omega_H(u,v) = 0 \quad \text{for every } v \in V \cap H\}$$

on $M$ is

$$U_{(A,X)} = \{(AY, 0) \in \mathrm{T}_{(A,X)}M \mid k(X,Y) = 0\}. \tag{15}$$

Note that $X_h^H(A,X) \in U_{(A,X)}$ for every $(A,X) \in M$ since

$$k(-g\,\mathrm{Ad}_{A^{-1}}B + gI^2\sigma^{-2}\,k(X, \mathrm{Ad}_{A^{-1}}B)X, X) = 0.$$

Now consider the map

$$\rho : M \to \mathrm{so}(3) : (A,X) \to I(\mathrm{Ad}_A X) = S. \tag{16}$$

Because $\rho$ is constant on $\widetilde{\Psi}$ orbits, it induces a map

$$\widetilde{\rho} : M/\mathrm{SO}(3) \to \mathrm{so}(3)$$

on the orbit space $M/\mathrm{SO}(3)$. Since the adjoint action of SO(3) is transitive on

$$\mathcal{S} = \{X \in \mathrm{so}(3) \mid k(X,X) = \sigma^2 I^{-2}\}, \tag{17}$$

the range of $\rho$ (and hence of $\widetilde{\rho}$) is the SO(3)-adjoint orbit

$$\mathcal{O}_{S_0} = \{\mathrm{Ad}_A S_0 \mid A \in \mathrm{SO}(3)\},$$

where $S_0 = IX_0$ for some $X_0 \in \mathcal{S}$. From the fact that the fiber $\rho^{-1}(S_0)$ is a single $\widetilde{\Psi}$ orbit, it follows that the map $\widetilde{\rho}$ is a diffeomorphism of $M/\mathrm{SO}(3)$ onto $\mathcal{O}_{S_0}$. Therefore $\rho$ is the reduction map of the SO(3) symmetry of $(M, X_h^H|M, \varphi)$.

We know that the distribution $U$ pushes down under $\rho$ to the reduced distribution $\overline{H}$ on $\mathcal{O}_{S_0}$. A calculation shows that

$$\begin{aligned}\overline{H}_S &= \{\mathrm{ad}_{\mathrm{Ad}_A Y} S \mid k(X,Y) = 0,\ Y \in \mathrm{so}(3)\}\\ &= \{\mathrm{ad}_{\mathrm{Ad}_A Y} S \mid Y \in \mathrm{so}(3)\} = \mathrm{T}_S\mathcal{O}_{S_0}\}.\end{aligned}$$

We also know that the 2-form $\omega_H$ on $H$ pushes down under $\rho$ to a symplectic form $\omega_{\overline{H}}$ on $\mathcal{O}_{S_0}$. Again a calculation shows that

$$\omega_{\overline{H}}(S)(\mathrm{ad}_{\mathrm{Ad}_A Y_1} S, \mathrm{ad}_{\mathrm{Ad}_A Y_2} S) = k\Big(S, \mathrm{Ad}_A[Y_1, Y_2]\Big).$$

Since

$$\mathrm{T}_{(A,X)}\rho : \mathrm{T}_{(A,X)}M \to \mathrm{T}_S\mathcal{O}_{S_0} : (AY, Z) \to \mathrm{ad}_{\mathrm{Ad}_A Y} S + I\,\mathrm{Ad}_A Z, \tag{18}$$

the reduced vector field is

$$\begin{aligned}X_{\overline{H}}(S) &= \mathrm{T}_{(A,X)}\rho(X_h^H(A,X))\\ &= g\,\mathrm{ad}_{\overline{B}}S,\\ &= -g\,\mathrm{ad}_B S = g\,[S,B],\end{aligned} \tag{19}$$

where $\overline{B} = -B + I^2\sigma^{-2}\,k(B, \mathrm{Ad}_A X)\mathrm{Ad}_A X$, and $S$ is given by (16).

We now show that we obtain the same result using Marsden–Weinstein reduction. Below we prove

$$X_h^{H^\perp}(A,X) \in \ker \mathrm{T}_{(A,X)}\rho \tag{20}$$

for every $(A,X) \in M$. For the moment let us assume (20). Applying Marsden–Weinstein reduction on $M$ in order to remove the body SO(3) symmetry from the vector field $X_h|M$, we find that the reduced vector field is

$$\mathrm{T}_{(A,X)}\rho(X_h(A,X)) = \mathrm{T}_{(A,X)}\rho(X_h^H(A,X)) = X_{\overline{H}}(\rho(A,X)),$$

which is the same as (19). We now prove (20). Using (18) and formula (14) for $X_h^{H^\perp}$, we obtain

$$\begin{aligned}\mathrm{T}_{(A,X)}\rho(X_h^{H^\perp}(A,X)) &= (1 - gI^2\sigma^{-2}\,k(X,\mathrm{Ad}_{A^{-1}}B))\mathrm{T}_{(A,X)}\rho(AX,0)\\ &= (1 - gI^2\sigma^{-2}\,k(X,\mathrm{Ad}_{A^{-1}}B))\mathrm{ad}_{\mathrm{Ad}_A X}S.\end{aligned}$$

But

$$\mathrm{ad}_{\mathrm{Ad}_A X}(I\mathrm{Ad}_A X) = I[\mathrm{Ad}_A X, \mathrm{Ad}_A X] = 0.$$

Thus (20) follows.

The reduced vector field $X_{\overline{H}}$ (19) is Hamiltonian on $(\mathcal{O}_{S_0}, \omega_{\overline{H}})$ with the Hamiltonian

$$\overline{h} : \mathcal{O}_{S_0} \subseteq \mathrm{so}(3) \to \mathbb{R} : S \to \tfrac{1}{2}\,k(S_0,S_0) - g\,k(S,B). \tag{21}$$

If $X \in \mathcal{S}$ (17), then the values of function $S$ (16) range over all of $\mathcal{O}_{S_0}$. Consequently, (19) reads

$$\dot{S} = g\,[S,B]. \tag{22}$$

Physically, the reduced system $(\overline{h}, \mathcal{O}_{S_0}, \omega_{\overline{H}})$ is a stationary classical nonrelativistic particle with spin.

## 5. A moving nonrelativistic particle with spin

Suppose a spherically symmetric rigid body with charge $q$ and gyromagnetic ratio $g$ is allowed to move freely in a spatially varying magnetic field $b$. Its configuration space is $\mathbb{R}^3 \times \mathrm{SO}(3)$ and its phase space is $\mathrm{T}\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3))$, where $\mathrm{T}\mathbb{R}^3$ is identified with $\mathbb{R}^3 \times \mathbb{R}^3$. Using the same notation as in Section 2, the symplectic form on phase space is

$$\begin{aligned}&\omega(x,v,A,X)((\dot{x}_1,\dot{v}_1,AY_1,Z_1),(\dot{x}_2,\dot{v}_2,AY_2,Z_2))\\ &\quad = m\,\langle \dot{x}_1,\dot{v}_2\rangle - m\,\langle \dot{x}_2,\dot{v}_1\rangle - q\,\langle b,\dot{x}_1\times\dot{x}_2\rangle \\ &\quad\quad + Ik(Y_1,Z_2) - Ik(Y_2,Z_1) + Ik(X,[Y_1,Y_2]).\end{aligned} \tag{23}$$

Here $\langle\, ,\, \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is the usual Euclidean inner product on $\mathbb{R}^3$ and $(\dot{x}_i, \dot{v}_i, AY_i, Z_i) \in T_{(x,v,A,X)}(\mathrm{T}\mathbb{R}^3 \times (\mathrm{SO}(3)\times\mathrm{so}(3)))$ for $i = 1,2$. The Hamiltonian for the unconstrained system is

$$
\begin{aligned}
&h : \mathrm{T}\,\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3)) \to \mathbb{R} : \\
&\qquad (x, v, A, X) \to \tfrac{1}{2}\, m\, \langle v, v\rangle + \tfrac{1}{2}\, I\, k(X, X) - gI\, k\big(\mathrm{Ad}_A X, B(x)\big).
\end{aligned} \tag{24}
$$

The Hamiltonian vector field $X_h$ associated with the Hamiltonian $h$ has integral curves which satisfy

$$
\begin{cases}
\dot{x} = v, \\
\dot{v} = \dfrac{q}{m}(v \times b) + \dfrac{gI}{m} DB^*(x)(k^\sharp(\mathrm{Ad}_A X)), \\
\dot{A} = A(X - g\,\mathrm{Ad}_{A^{-1}} B(x)), \\
\dot{X} = 0,
\end{cases} \tag{25}
$$

where $DB^*(x) = j^\flat(DB(x)^t)$ and $j^\sharp(x)y = \langle x, y\rangle$.

The "body" action $\varPsi$ (5) can be modified to an SO(3)-action on $\mathrm{T}\,\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3))$ by defining

$$
\begin{aligned}
&\varPsi : \mathrm{SO}(3) \times \Big(\mathrm{T}\,\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3))\Big) \to \mathrm{T}\,\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3)) : \\
&\qquad \Big(C, (x, v, A, X)\Big) \to (x, v, AC^{-1}, \mathrm{Ad}_C X).
\end{aligned} \tag{26}
$$

Similarly, we have the "space" action, corresponding to rotation in physical space. In terms of the trivialization by left translation it is given by

$$
\begin{aligned}
&\varPhi : \mathrm{SO}(3) \times (\mathrm{T}\,\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3))) \to \mathrm{T}\,\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3)) : \\
&(C, (x, v, A, X)) \to (Cx, Cv, CA, X).
\end{aligned} \tag{27}
$$

As before, the right action $\varPsi$ is a symmetry of the unconstrained system, whereas the left action $\varPhi$ is not, because of the presence of the magnetic field. The momentum of the left action is

$$
J = L + S : \mathrm{T}\,\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3)) \to \mathrm{so}(3).
$$

$L$ is the orbital angular momentum given by

$$
L(x, v, A, X) = i^{-1}(x \times v),
$$

where the map $i$ is implicitly defined by (3), and $S$ is the spin angular momentum given by

$$
S(x, v, A, X) = I(\mathrm{Ad}_A X).
$$

Taking the Lie derivative of $S$ with respect to the unconstrained vector field $X_h$ gives

$$
\begin{aligned}
\dot{S}(x, v, A, X) &= I\mathrm{Ad}_A(\mathrm{ad}_{X - g\mathrm{Ad}_{A^{-1}}B} X) \\
&= -gI\,\mathrm{ad}_B(\mathrm{Ad}_A X) \;=\; g\,[S(x, v, A, X), B].
\end{aligned}
$$

Since the space action is not a symmetry, the angular momentum $J$ is not conserved. Neither is the spin angular momentum. However, the magnitude of the spin angular momentum is conserved since

$$L_{X_h}\Big(k(S(x,v,A,X),S(x,v,A,X))\Big) = 2\,k(\dot{S}(x,v,A,X),S(x,v,A,X))$$
$$= -2g\,k([B,S(x,v,A,X)],S(x,v,A,X)) = 0.$$

Now restrict the Hamiltonian system $(h, \mathrm{T}\mathbb{R}^3 \times (\mathrm{SO}(3)\times \mathrm{so}(3)), \omega)$ to the submanifold

$$M = \{(x,v,A,X) \in \mathrm{T}\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3)) \mid k(S,S) = \sigma^2\}, \tag{28}$$

where $S = S(x,v,A,X) = I(\mathrm{Ad}_A X)$. Again $M$ is a nonlinear nonholonomic constraint and determines a constraint 1-form $\varphi$ on $\mathrm{T}\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3))$ given by

$$\varphi(x,v,A,X)(\dot{x},\dot{v},AY,Z) = k(X,Y). \tag{29}$$

The 1-form $\varphi$ and the tangent bundle $\mathrm{T}M$ of $M$ determine a constraint distribution $H$ on $\mathrm{T}\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3))$ defined by

$$\begin{aligned} H_{(x,v,A,X)} &= \ker\varphi(x,v,A,X) \cap \mathrm{T}_{(x,v,A,X)}M \\ &= \{(\dot{x},\dot{v},AY,Z) \in \mathrm{T}_{(x,v,A,X)}(\mathrm{T}\mathbb{R}^3 \times (\mathrm{SO}(3) \times \mathrm{so}(3))) \mid \\ &\qquad k(X,Y) = k(X,Z) = 0\}. \end{aligned} \tag{30}$$

Repeating what was done in Section 4, we can define the dynamical system $(M, X_h^H|M, \varphi)$, where the induced action $\widetilde{\Psi} = \Psi|(\mathrm{SO}(3) \times M)$ is a symmetry. The infinitesimal generator of $\widetilde{\Psi}$ in the direction $Y \in \mathrm{so}(3)$ is given by the vector field $X^Y(x,v,A,X) = (0,0,-AY,-[X,Y])$. Thus the SO(3) symmetry distribution $V$ on $M$ is

$$V_{(x,v,A,X)} = \{X^Y(x,v,A,X) \in \mathrm{T}_{(x,v,A,X)}M \mid Y \in \mathrm{so}(3)\}.$$

A calculation shows that the distribution $V \cap H$ on $M$ is given by

$$(V \cap H)_{(x,v,A,X)} = \{(0,0,-AY,-[X,Y]) \in \mathrm{T}_{(x,v,A,X)}M \mid k(X,Y) = 0\}.$$

Therefore the distribution $U$ on $M$ is

$$U_{(x,v,A,X)} = \{(\dot{x},\dot{v},AY,0) \in \mathrm{T}_{(x,v,A,X)}M \mid k(X,Y) = 0\}. \tag{31}$$

Again $X_h^H(x,v,A,X) \in U_{(x,v,A,X)}$ for every $(x,v,A,X) \in M$. Now consider the map

$$\rho : M \to \mathrm{T}\mathbb{R}^3 \times \mathrm{so}(3) : (x,v,A,X) \to (x,v,S). \tag{32}$$

$\rho$ is the reduction map for the SO(3) symmetry of $(M, X_h^H|M, \varphi)$. Under $\rho$ the distribution $U$ pushes down to the reduced distribution $\overline{H}$ on $\mathrm{T}\mathbb{R}^3 \times \mathcal{O}_{S_0}$ given by

$$\begin{aligned} \overline{H}_{(x,v,S)} &= \{(\dot{x},\dot{v},\mathrm{ad}_{\mathrm{Ad}_A Y}S) \mid k(X,Y) = 0,\ Y \in \mathrm{so}(3)\} \\ &= \{(\dot{x},\dot{v},\mathrm{ad}_{\mathrm{Ad}_A Y}S) \mid Y \in \mathrm{so}(3)\} = \mathrm{T}_{(x,v)}(\mathrm{T}\mathbb{R}^3) \times \mathrm{T}_S\mathcal{O}_{S_0}. \end{aligned}$$

Here $S_0 = I X_0$ for some $X_0$ with $k(X_0, X_0) = k(I^{-1}S, I^{-1}S) = \sigma^2 I^{-2}$. The 2-form $\omega_H$ on $H$ pushes down under $\rho$ to a symplectic form $\omega_{\overline{H}}$ on $\mathrm{T}\,\mathbb{R}^3 \times \mathcal{O}_{S_0}$ defined by

$$\omega_{\overline{H}}(x, v, S)((\dot{x}_1, \dot{v}_1, \mathrm{ad}_{\mathrm{Ad}_A Y_1} S), (\dot{x}_2, \dot{v}_2, \mathrm{ad}_{\mathrm{Ad}_A Y_2} S)) \\ = m\,\langle \dot{x}_1, \dot{v}_2\rangle - m\,\langle \dot{x}_2, \dot{v}_1\rangle - q\,\langle b, \dot{x}_1 \times \dot{x}_2\rangle + k\Big(S, \mathrm{Ad}_A[Y_1, Y_2]\Big). \tag{33}$$

Thus the reduced vector field $X_{\overline{H}}(x, v, S)$ has integral curves which satisfy

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= \frac{q}{m}(v \times b) + \frac{g}{m} DB^*(x)(S), \\ \dot{S} &= g\,\mathrm{ad}_S B = g\,[S, B]. \end{aligned} \tag{34}$$

The reduced Hamltonian is

$$\overline{h} : \mathrm{T}\,\mathbb{R}^3 \times \mathcal{O}_{S_0} \to \mathbb{R} : (x, v, S) \to \tfrac{1}{2} m\,\langle v, v\rangle + \tfrac{1}{2} I^{-1} k(S_0, S_0) - g\,k(S, B). \tag{35}$$

Physically, the reduced system $(\overline{h}, \mathcal{O}_{S_0}, \omega_{\overline{H}})$ is a moving classical nonrelativistic particle with spin as defined by Souriąu [5, p. 195–6]. This can be seen as follows. Because $k(X, X) = k(I^{-1}S, I^{-1}S) = \sigma^2 I^{-2}$, we may choose $O \in \mathrm{SO}(3)$ so that

$$IX = \mathrm{Ad}_{O^{-1}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sigma \\ 0 & \sigma & 0 \end{pmatrix}.$$

Restricting the map $i$ (3) to the adjoint orbit $\mathcal{O}_{S_0}$, we obtain the diffeomorphism

$$\mathcal{O}_{S_0} \subseteq \mathrm{so}(3) \to S^2_\sigma \subseteq \mathbb{R}^3 : S = \mathrm{Ad}_A IX \to s = AO^{-1} \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix}. \tag{36}$$

Using (36), the reduced equations (34) become

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= \frac{q}{m} v \times b + \frac{g}{m} \nabla(b \cdot s), \\ \dot{s} &= g\, s \times b, \end{aligned} \tag{37}$$

and the reduced Hamiltonian (35) becomes

$$\overline{h} : \mathrm{T}\,\mathbb{R}^3 \times S^2_\sigma \to \mathbb{R} : (x, v, s) \to \tfrac{1}{2} m\,\langle v, v\rangle - g\,\langle s, b\rangle, \tag{38}$$

up to an additive constant.

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