

THE ODD SYMPLECTIC GROUP IN GEOMETRY

Larry Bates¹, Richard Cushman²§

¹Department of Mathematics
University of Calgary
Calgary, Alberta, T2N 1N4, CANADA
e-mail: bates@math.ucalgary.ca

²Mathematics Institute
University of Utrecht
Utrecht, 3508TA, THE NETHERLANDS
e-mail: cushman@math.uu.nl

Abstract: We describe the relation of the odd symplectic group to contact geometry and its use in the study of periodic geodesics in Riemannian geometry.

AMS Subject Classification: 53C50

Key Words: odd symplectic group, symplectic geometry, contact geometry, geodesic flow

1. The Odd Symplectic Group

Let ω be the symplectic form on \mathbb{R}^{2n+2} with matrix

$$\tilde{J} = \begin{pmatrix} & & & 1 \\ & & I_n & \\ & -I_n & & \\ -1 & & & \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. The odd symplectic group $\text{Sp}(2n+$

Received: October 28, 2002

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§Correspondence author

$\mathbb{1}, \mathbb{R}$) is defined to be the subgroup of the the symplectic group $\mathrm{Sp}(2n + 2, \omega)$ of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ v & A & 0 \\ w & (JA^{-1}v)^t & 1 \end{pmatrix}, \quad (1)$$

where $v \in \mathbb{R}^{2n}$, $w \in \mathbb{R}$, and $A \in \mathrm{Sp}(2n, \mathbb{R})$. Here J is the almost-complex matrix on \mathbb{R}^{2n} given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We have chosen to present the odd symplectic group in this fashion as it more clearly shows that $\mathrm{Sp}(2n + 1, \mathbb{R})$ is a semidirect product of an ordinary symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ and a Heisenberg group. One also sees that one may think of the odd symplectic group as a group of symplectic matrices that fix a given nonzero vector. Thus there is an interpolation

$$\cdots \subset \mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{Sp}(2n + 1, \mathbb{R}) \subset \mathrm{Sp}(2n + 2, \mathbb{R}) \cdots$$

Differentiating (1) at the identity shows that the Lie algebra of the odd symplectic group may be realized as the algebra of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ v & B & 0 \\ w & (Jv)^t & 0 \end{pmatrix},$$

with $v \in \mathbb{R}^{2n}$, $w \in \mathbb{R}$, and $B \in \mathfrak{sp}(2n, \mathbb{R})$.

Let G be the affine symplectic group, and let X, Y belong to its Lie algebra \mathfrak{g} . Define a bracket in the Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$ of the central extension \widehat{G} by

$$[(X, s), (Y, t)] = ([X, Y], \Sigma(X, Y)),$$

where Σ is the cocycle of the affine symplectic action on \mathbb{R}^{2n} . Then \widehat{G} is isomorphic to the odd symplectic group $\mathrm{Sp}(2n + 1, \mathbb{R})$.

2. Contact Geometry

We explain how the odd symplectic group arises naturally in contact geometry.

To start with, consider a $2n + 1$ dimensional contact manifold (M, ϑ) and its symplectification of $(N, \omega) = (\mathbb{R}_{>0} \times M, d(t\vartheta))$, [6, p. 113].

Here t is the coordinate on $\mathbb{R}_{>0}$. Denote by ϑ the pull back to N by the projection map of the one-form ϑ on M . Consider a symplectic diffeomorphism F of N that leaves invariant each leaf of the foliation of N given by the level sets of t . In other words, F is of the form

$$F(t, m) = (t, G(t, m)). \quad (2)$$

A theorem of Darboux [4] asserts that there exist local coordinates (x^a, y_a, z) on M so that the contact form $\vartheta = dz - y_a dx^a$. In addition, in the induced local coordinates (t, x^a, y_a, z) on N the symplectic form

$$\omega = dt \wedge dz + t dx^a \wedge dy_a - y_a dt \wedge dx^a.$$

The $2(n+1) \times 2(n+1)$ matrix of ω is

$$\Omega = \begin{pmatrix} 0 & -y^T & 0 & 1 \\ y & 0 & tI_n & 0 \\ 0 & -tI_n & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Because F is symplectic, $(DF)^T \Omega DF = \Omega$, whose infinitesimalization is

$$u^T \Omega + \Omega u = 0. \quad (3)$$

Here u a $2(n+1) \times 2(n+1)$ matrix having zero entries in its first row. Specializing to the case $(t, y) = (1, 0)$ and writing (3) out shows that u is of the form

$$u = \begin{pmatrix} 0 & 0 & 0 \\ v & B & 0 \\ a & (Jv)^T & 0 \end{pmatrix},$$

where $v \in \mathbb{R}^{2n}$, $a \in \mathbb{R}$, $B \in \mathfrak{sp}(2n, \mathbb{R})$ with symplectic form J . Since $u \in \mathfrak{sp}(2n+2, \mathbb{R})$ with symplectic form \tilde{J} by construction, it follows that u is in the Lie algebra of the odd symplectic group $\mathfrak{sp}(2n+1, \mathbb{R})$, see [3].

If X is a vector field on N such that the Lie derivative

$$L_X(d(t\vartheta)) = 0 \quad \text{and} \quad Xt = 0, \quad (4)$$

then the flow of X is a one parameter group F_t of diffeomorphisms of N of the form (2). Moreover, from

$$L_X(d(t\vartheta)) = d(tL_X\vartheta), \quad (5)$$

it follows that group of diffeomorphisms of the the form (2) contains the subgroup \mathcal{C} of contactomorphisms of (M, ϑ) that preserve the contact form. To see this, fix t in (5). Note that in the usual theory of contact structures, as described in [1], [5], or [6], one is interested in the hyperplane field which is the kernel of the contact form, and not in the contact form per se.

A vector field X on M is a contact vector field if and only if

$$L_X \vartheta = g \vartheta, \tag{6}$$

for some function g , see [6, p. 110]. When (6) holds, there is a Hamiltonian function h which satisfies

$$X \lrcorner \vartheta = -h \text{ and } X \lrcorner d\vartheta = dh - (Y \lrcorner dh)\vartheta.$$

The vector field Y is the Reeb field of the one-form ϑ , that is,

$$Y \lrcorner \vartheta = 1 \text{ and } Y \lrcorner d\vartheta = 0.$$

Furthermore, $g = -Y \lrcorner dh$. In our case, the tangent at zero of a one parameter subgroup of \mathcal{C} is a contact vector field X on M such that $L_X \vartheta = 0$. Thus g is identically 0. So we are interested only in those functions h for which $Y \lrcorner dh = 0$. In other words, h is constant on the integral curves of the characteristic distribution of the contact one-form ϑ . In a Darboux chart this implies that h is a function only of x and y .

If we consider inhomogeneous quadratic polynomials of x and y under the Jacobi (contact) bracket [5, p. 320], then we get a Lie algebra that is isomorphic to the Lie algebra of the odd symplectic group. The isomorphism is obtained by using the map

$$C^\infty(M) \hookrightarrow C^\infty(N) : f \rightarrow tf,$$

from the symplectification of M . Since functions under the Jacobi bracket do not form a Poisson algebra, because the constant function 1 generates a nontrivial vector field equal to the Reeb field, we may think of the map $f \rightarrow tf$ as embedding a Lie algebra into a Poisson algebra.

Alternatively, if one starts with the affine symplectic transformations on a symplectic vector space with a global Darboux chart and considers the algebra of Hamiltonian functions that it generates under the Poisson

bracket, one may view the resulting algebra as the algebra of inhomogeneous quadratic polynomials on a symplectic vector space. Thought of in this manner, it is immediate that the odd symplectic algebra is a semidirect product of a symplectic algebra with a Heisenberg algebra. This representation of the odd symplectic algebra is not faithful since the Poisson bracket of two linear functions may be a nonzero constant. Passing to the contact manifold provides a faithful representation in the space of contact vector fields. Adding a dimension once more giving the symplectification provides a faithful representation of the odd symplectic algebra within the algebra of Hamiltonian vector fields.

3. Periodic Geodesics

We now show how the odd symplectic group arises in the study of periodic geodesics.

As is well known, the geodesics on a $n + 1$ -dimensional Riemannian manifold (M, g) arise from the flow of a Hamiltonian vector field corresponding to the Hamiltonian function $h = \frac{1}{2} g_{ab} v^a v^b$ on (TM, ω) . Here (q, v) are natural coordinates on TM , $\omega = -d\vartheta$ the symplectic form and $\vartheta = g_{ab} v^a dq^b$ is the fundamental one-form. In other words, the geodesic vector field X_h satisfies $X_h \lrcorner \omega = dh$. Define the Liouville vector field L on TM as $L = v^a \frac{\partial}{\partial v^a}$. Since $[X_h, L] = -X_h$, the distribution $\Delta = \text{span}\{X_h, L\}$ on TM is integrable. However, its symplectic perpendicular Δ^ω is not. This follows from the fact that the distribution Δ^ω is the kernel of the fundamental one-form ϑ restricted to an energy surface, which is well known to be a contact one-form.

The Riemannian metric g on M induces a metric g_1 on TM called the Sasaki metric, see [7]. For each $(q, v) \in TM$ the metric g_1 gives a splitting of $T_{(q,v)}(TM)$ into orthogonal horizontal and vertical subspaces H and V , respectively, which are Lagrangian planes. Observe that $X_h \in H$ and $L \in V$. Next pick orthonormal vectors Y^a and Z_a in $T_{(q,v)}(TM)$ which span the distribution Δ^ω so that $Y^a \in H$ and $Z^a \in V$. This can be done because if J_1 is a Jacobi field along a geodesic γ with $\gamma(0) = q$, $\dot{\gamma}(0) = v$, and $(J_1(0), \dot{J}_1(0)) = (w_1, 0)$, then Y is the horizontal lift of w_1 . Similarly, if J_2 is a Jacobi field with $(J_2(0), \dot{J}_2(0)) = (0, w_2)$, then Z is the vertical lift of w_2 . Furthermore, w_1 and w_2 are chosen to be orthogonal to v , namely, $g(v, w_1) = 0$, $g(v, w_2) = 0$.

At a point (q, v) on $h^{-1}(\frac{1}{2})$ a basis for $T_{(q,v)}(TM)$ is given by

$\{L, Z_a, Y^a, X_h\}$. With respect to this basis, the $2n \times 2n$ matrix of the symplectic form ω is

$$\Omega = \begin{pmatrix} & & & 1 \\ & & I_n & \\ & -I_n & & \\ -1 & & & \end{pmatrix}.$$

Let ϕ_t be the geodesic flow, that is, the flow of X_h . Then integrating $[X_h, L] = -X_h$ gives $\phi_t^* L = L - tX_h$. Consequently, $(\phi_t)_* L = L + tX_h$. If γ is a periodic geodesic of period τ , then the matrix of $(\phi_\tau)_*$ with respect to the frame $\{L, Y, Z, X_h\}$ is

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ \tau & 0 & 1 \end{pmatrix} \quad \text{with } A \in \text{Sp}(2n, \mathbb{R}). \quad (7)$$

Thus Φ is an element of the odd symplectic group $\text{Sp}(2n+1, \mathbb{R})$.

What is interesting here is that the conjugacy class of Φ in the real symplectic group $\text{Sp}(2(n+1), \mathbb{R})$ does not involve the period τ . To see this consider the $2(n+1) \times 2(n+1)$ matrix $B = \begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 & 0 \\ 0 & I_{2n} & 0 \\ 0 & 0 & \sqrt{\tau} \end{pmatrix}$. Then $B \in \text{Sp}(2(n+1), \mathbb{R})$, that is, $B^T \tilde{J} B = \tilde{J}$ and

$$B^{-1} \Phi B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

which does *not* depend on τ . Thus the period τ is *not* a symplectic invariant. It is an invariant of the odd symplectic group, because every element of $\text{Sp}(2n+1, \mathbb{R})$ which preserves the block structure (7) of Φ is of the form $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P & 0 \\ a & 0 & 1 \end{pmatrix}$, where $P \in \text{Sp}(2n, \mathbb{R})$ and $a \in \mathbb{R}$. Now

$$C^{-1} \Phi C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P^{-1} A P & 0 \\ \tau & 0 & 1 \end{pmatrix}.$$

Thus the period τ , which is a modulus of the conjugacy class of Φ in the odd symplectic group $\text{Sp}(2(n+1), \mathbb{R})$, is invariant, see [3]. Hence the *period* of a periodic geodesic can be detected by the conjugacy class of the linear Poincaré map in the odd symplectic group.

4. Completely Integrable Systems

Consider a completely integrable system with Hamiltonian h and action-angle variables $j_1, \dots, j_n, \phi^1, \dots, \phi^n$. Suppose there is a periodic orbit through the point p . We can always arrange things so that the Hamiltonian vector field

$$X_h = \partial_1 h(j_1, \dots, j_n) \frac{\partial}{\partial \phi^1},$$

at p . Hence the period τ is

$$\tau = \frac{2\pi}{\partial_1 h(p)}.$$

Let ψ_t be the flow of the integrable system and consider the matrix of $(\psi_\tau)_*$ in the basis $\{j_1, \dots, j_n, \phi^2, \dots, \phi^n, \phi^1\}$ (note the order). We have

$$\begin{pmatrix} I & 0 \\ \tau D(\partial_1 h) & I \end{pmatrix}.$$

Since we write elements of $\mathrm{Sp}(2n-1, \mathbb{R})$ in the form

$$\begin{pmatrix} 1 & 0 & 0 \\ v & A & 0 \\ w & (JA^{-1}v)^T & 1 \end{pmatrix},$$

we may identify $v \in \mathbb{R}^{2n-2}$ and $w \in \mathbb{R}$ as $v^T = (0, \dots, 0, \tau D\partial_1 h)$, and $w = \tau \partial_{11} h$. The symplectic matrix A is

$$\begin{pmatrix} I_{n-1} & 0 \\ \tau \partial_{kl} h & I_{n-1} \end{pmatrix}_{k,l=2,\dots,n}.$$

The interest here is that presumably the zero components of the vector v are related to the fact that the map $(\psi_\tau)_*$ preserves more than a single vector, namely it preserves an entire Lagrange plane. We say presumably because it is an open problem to list the conjugacy classes of the group $\mathrm{Sp}(2n+2, k, \mathbb{R})$, which is the subgroup of the symplectic transformations that preserve an isotropic subspace of dimension k .

Acknowledgements

The first author is partially supported by NSERC.

The first author is partially supported by European Community Funding for the Research and Training Network MASIE (HPRN-CT-2000-00113).

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