Journal of Differential Equations **172**, 42–58 (2001)
doi:10.1006/jdeq.2000.3852, available online at http://www.idealibrary.com on IDEAL®

Non-Hamiltonian Monodromy

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Received July 9, 1999; revised March 21, 2000

It is well known that the energy momentum map of a two degree of freedom integrable Hamiltonian system with a focus-focus singularity has monodromy. In this paper we generalize this result to systems which are not necessarily Hamiltonian. © 2001 Academic Press

1. INTRODUCTION

Let us recall the Hamiltonian version of the monodromy theorem. For simplicity suppose that phase space is the cotangent bundle $T^*\mathbf{R}^2$ of $\mathbf{R}^2$ with coordinates $z=(q,p)=(q_1,q_2,p_1,p_2)$ and standard symplectic form $\omega=\sum_{i=1}^2 dq_i \wedge dp_i$. Suppose that we are given two smooth functions H and K on $T^*\mathbf{R}^2$ whose Poisson bracket $\{H,K\}(z)=\omega(z)(X_H(z), X_K(z)) =0$ for every $z\in T^*\mathbf{R}^2$. We say that z_0 is a focus-focus singularity of the above integrable Hamiltonian system if $X_H(z_0)=X_K(z_0)=0$ and the span of the linearized vector fields $DX_H(z_0)$, $DX_K(z_0)$ is conjugate by a linear symplectic mapping to the Cartan subalgebra of the Lie algebra $\mathrm{sp}(4,\mathbf{R})$ of all linear Hamiltonian vector fields spanned by the linear Hamiltonian vector fields $-q_2(\partial/\partial q_1)+q_1(\partial/\partial q_2)-p_2(\partial/\partial p_1)+p_1(\partial/\partial p_2)$ and $q_1(\partial/\partial q_1) +q_2(\partial/\partial q_2)-p_1(\partial/\partial p_1)-p_2(\partial/\partial p_2)$. Without loss of generality we may assume that $z_0=0$ and $H(z_0)=K(z_0)=0$. Now consider the energy momentum map of this integrable system, namely

$$\mathscr{E}\mathscr{M}\colon T^*\mathbf{R}^2 \to \mathbf{R}^2 : z \to (H(z), K(z)).$$

Assume that $\mathscr{E}\mathscr{M}$ has the following properties: 1) there is a open ball $U\subseteq \mathbf{R}^2$ about 0 such that 0 is its only critical value; 2) for every $u\in U\setminus\{0\}$ the fiber $\mathscr{E}\mathscr{M}^{-1}(u)$ is connected and compact. (Hence by the Arnol'd-Liouville theorem $\mathscr{E}\mathscr{M}^{-1}(u)$ is diffeomorphic to a smooth 2-dimensional torus); 3) the singular fiber $\mathscr{E}\mathscr{M}^{-1}(0)$ is connected and compact and the rank of the derivative of $\mathscr{E}\mathscr{M}$ on $\mathscr{E}\mathscr{M}^{-1}(0)\setminus\{0\}$ is 2. The Hamiltonian monodromy theorem says that under the above hypotheses, the smooth

0022-0396/01 $35.00

2-torus bundle $\mathcal{EM}^{-1}(\Gamma)$ over a smooth circle Γ in $U\backslash\{0\}$ is *nontrivial.* In particular, using a suitable basis for the first homology group of the 2-torus $\mathcal{EM}^{-1}(\Gamma_0)$ where $\Gamma_0 \in \Gamma$, the classifying (or monodromy) map of the bundle $\mathcal{EM}^{-1}(\Gamma)$ is $\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)$, having identified the fiber $\mathcal{EM}^{-1}(\Gamma_0)$ with the lattice $\mathbf{R}^2/\mathbf{Z}^2$.

Mondromy has been found in a number of concrete 2 degree of freedom integrable Hamiltonian systems. We mention the spherical pendulum, (Duistermaat [14], Cushman [6], Cushman and Bates [9, IV.5]), the Hamiltonian Hopf bifurcation, (van der Meer [16, Cor. 4.14, p. 83], Duistermaat [15]), the Lagrange top (Cushman and Knörrer [11], Cushman and Bates [9, V, 7.3], Cushman and van der Meer [12]), the champagne bottle (Bates [2]), the magnetic spherical pendulum (Cushman and Bates [8]), and the hydrogen atom in orthogonal electric and magnetic fields (Cushman and Sadovskii [13]). In Duistermaat [14] it is shown that nontrivial monodromy is the coarsest obstruction to the existence of global action-angle variables. A case of confluence of two focus-focus singularities has been studied by Bates and Zou [3].

Now let us state the assumptions for our generalization of the Hamiltonian monodromy theorem. The set up is the following. Let v and w be two smooth vector fields on a smooth 4-dimensional manifold M. Let $p \in M$ and suppose that $f\colon M \to \mathbf{R}^2$ is a map which is smooth on $M\backslash\{p\}$, continuous at p, and for which $f(p)=0$. For every $c \in \mathbf{R}^2$ we shall write $F_c = \{x \in M \mid f(x)=c\}$ for the fiber of f over c. We make the following assumptions.

Assumptions

(a) The vector fields v and w commute.

(b) $v(p)=0$. There is a suitable real linear combination of v and w (which we again denote by v and w) so that one complex conjugate pair of eigenvalues of $\mathrm{D}v(p)$ has negative real part while the other has positive real part.

(c) The derivative of f in the direction of both vector fields v and w is equal to 0.

(d) At each point $x \in F_0\backslash\{p\}$, the vectors $v(x)$ and $w(x)$ are linearly independent and the rank of $\mathrm{D}f(x)$ is equal to 2.

(e) The subset $F_0 = f^{-1}(\{0\})$ of M is both compact and connected.

Remark 1.1. If F_0 is not connected, but is equal to the union of two disjoint closed subsets K and L, where $p \in K$ and K is compact and connected, then one can replace M by an open neighborhood $\widetilde{M}$ of K such that $\widetilde{M} \cap F_0 = K$. One then requires that assumptions (a)–(e) hold with M replaced by $\widetilde{M}$.

One says that v has an equilibrium point of *focus-focus type* at p if (b) holds.

Informally, the main conclusion which can be drawn form the above assumptions is that the integral map f has monodromy.

Before we state the non-Hamiltonian monodromy theorem precisely, one would like to know if there are any interesting physical systems which satisfy the above assumptions. The answer is yes, namely Routh's sphere. Routh's sphere is a sphere whose center of mass is not at is geometric center and which is dynamically symmetric about the line joining its geometric center with its center of mass. In other words, two of its principal moments of inertia about this axis are equal. Routh's sphere rolls without slipping on a horizontal plane under the influence of a constant vertical gravitational force. The constraint that the sphere rolls without slipping on the plane is nonholonomic. This destroys the possibility of the system being Hamiltonian. Why would one even guess that Routh's sphere has monodromy? Gyroscopic stabilization seems to be the right physical intuition. In more detail, recall that the Lagrange top undergoes gyroscopic stablization when its figure axis is vertical, namely, when the spin about this axis is small the motion is unstable; whereas when it is sufficiently large its motion is stable. Routh's sphere also undergoes gyroscopic stabilization, namely when its center of mass is vertically above its geometric center, the motion of the sphere is unstable; whereas when it spins sufficiently fast about the axis joining the center of mass with its geometric center and this axis is vertical, its motion is stable. For more details showing that Routh's sphere satisfies the assumptions (a)–(e) see Cushman [7]. Using equations found by Chaplygin [5] for describing the motion of a smooth convex solid of revolution rolling without slipping on a horizontal plane under the influence of a constant vertical gravitational field, one can show that this non-Hamiltonian system has monodromy, see [10]. The authors know of no integrable system which undergoes gyroscopic stabilization and does not have monodromy.

We now state the non-Hamiltonian monodromy theorem.

THEOREM 1.2. *The assumptions* (a)–(e) *lead to the following conclusions.*

(a) *$F_0 \setminus \{p\}$ is diffeomorphic to the cylinder $(\mathbf{R}/2\pi\,\mathbf{Z}) \times \mathbf{R}$ and F_0 is homeomorphic to the one point compactification of this cylinder. Near p, F_0 is equal to the union of two 2-dimensional submanifolds of M which intersect transversally at the point p. Let ϕ^t be the flow of v. If $S_\pm$ denotes the set of $x \in M$ such that $\phi^t(x) \to p$ as $t \to \pm\infty$, then $F_0 = S_+ = S_-$.*

(b) *There is an open neighborhood $\widetilde{M}$ of F_0 in M and a simply connected open neighborhood U of 0 in $\mathbf{R}^2$ such that $f \mid (\widetilde{M} \setminus F_0) : \widetilde{M} \setminus F_0 \to U \setminus \{0\}$ defines a locally trivial 2-torus fibration over $U \setminus \{0\}$.*

(c) *Let* $\alpha\colon [0, 1] \to U \backslash \{0\}$ *be a smooth closed curve in* $U \backslash \{0\}$ *which winds once around the origin in the positive direction. The* 2*-torus bundle* $\coprod_{\varphi \in [0, 1]} F_{\alpha(\varphi)} \to \alpha$ *over the loop* α *has monodromy* $\mathscr{M} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ *with respect to a suitable basis of generators of the* 2*-dimensional lattice* $\mathrm{H}_1 = \mathrm{H}_1(F_{\alpha(0)} \cap \tilde{M}, \mathbf{Z})$.

Remark 1.3. Because the cylinder is diffeomorphic to the 2-dimensional sphere minus two points $p_{\pm}$, part (a) of the theorem says that F_0 can be viewed as a 2-sphere S^2 with p_+ and p_- identified with the point p and embedded in M in such a way that the tangent spaces $T_p S^2$ of the 2-sphere S^2 at the points $p_{\pm}$ are identified with complementary subspaces $T_{\pm}$ of $T_p M$. Alternatively, F_0 can be described as a 2-dimensional torus in which a generating circle is pinched to the point p.

The set S_+ (and S_-) is called the *stable* (and (*unstable*)) *manifold* of p for the vector field v. Because $S_+ = S_- = F_0$, the description of F_0 in assumption (b) implies that $S_{\pm}$ can be viewed as an immersed smooth submanifold of M, with $\mathrm{T}_p S_{\pm}$ equal to the tangent space at p of one of the two manifolds which intersect at p. Actually, $\mathrm{T}_p S_+$ ($\mathrm{T}_p S_-$) is equal to the eigenspace of $Dv(p)$ on which the real part of the eigenvalues of $Dv(p)$ are negative (positive).

Remark 1.4. In the Hamiltonian case conclusion (c) of Theorem 1.2 has been obtained by Zou [20, Theorem 1.1], who ascribes to Flaschka the idea that the monodromy is determined by the local behaviour near the point p. However, Zou's proof of Lemma 1.3 (which is needed in the proof of Theorem 1.1) is not correct, since one can pinch off a small sphere from the torus whose pinching circle is homotopically trivial. Independently, Tien Zung [19] obtained conclusion (c) of Theorem 1.2 in the Hamiltonian case.

To the conclusions of Theorem 1.2 we may add the following three propositions and a corollary.

Proposition 1.5. *Let* ϕ^t *and* ψ^s *denote the flow of the vector fields* v *and* w, *respectively. Then* $(s, t) \mapsto \psi^s \circ \phi^t$ *defines a smooth action of* $\mathbf{R}^2$ *on* $\tilde{M}$. *For each* $c \in U \backslash \{0\}$, $\tilde{F}_c = F_c \cap \tilde{M}$ *is equal to an orbit of the* $\mathbf{R}^2$*-action. The stabilizer subgroup of the* $\mathbf{R}^2$*-action on* $\tilde{F}_c$ *is a* 2*-dimensional lattice* Γ_c *in* $\mathbf{R}^2$ *which depends smoothly on* $c \in U \backslash \{0\}$. *Under the canonical isomorphism of* Γ_c *with* $\mathrm{H}_1(\tilde{F}_c, \mathbf{Z})$, *the action of the monodromy operator* $\mathscr{M}$ *on* H_1 *corresponds to the automorphism of* $\Gamma_{\alpha(0)}$ *which one obtains by following the elements of* $\Gamma_{\alpha(\varphi)}$ *continuously as* φ *runs from* 0 *to* 1. *Finally,* $F_0 \backslash \{p\}$ *is an* $\mathbf{R}^2$*-orbit and its stabilizer subgroup is isomorphic to* $\mathbf{Z}$.

Proposition 1.6. *There exist unique smooth functions* σ *and* τ *on* U *such that* $\sigma(0) > 0$ *and the flow of the vector field* $u = (\sigma \circ f)\, w + (\tau \circ f)\, v$ *defines a free action of the circle group* $\mathbf{R}/2\pi\mathbf{Z}$ *on* $\tilde{M} \backslash \{p\}$.

Remark 1.7. In [19] Tien Zung obtained a Hamiltonain version of proposition 1.6.

Let $c \in U \setminus \{0\}$, then a u-circle in F_c defines a generator $\delta_1 = \delta_1(c)$ of the subgroup of elements of $\mathrm{H}_1(F_c, \mathbf{Z})$ which are fixed by the monodromy operator $\mathscr{M} = \mathscr{M}_c$. For the second generator $\delta_2 = \delta_2(c)$ we can take a v-solution curve, starting and ending on a u-circle in F_c, followed by a part of the u-circle in order to close it up.

The linear transformation $\mathrm{D}u(p)$ on $\mathrm{T}_p M$ defines a complex structure on $\mathrm{T}_p M$, which in turn defines an orientation on $\mathrm{T}_p S_-$, the eigenspace of $Dv(p)$ on which the real part of its eigenvalues are positive. This orientation extends in a continuous fashion to an orientation on $\mathrm{T}_x M/\mathrm{T}_x S_+$, for $x \in S_+$, $x \neq p$.

PROPOSITION 1.8. *If, for* $x \in S_+$, $x \neq p$, *the orientation on* $\mathrm{T}_x M/\mathrm{T}_x S_+$ *defined by the complex structure on* $\mathrm{T}_p M$ *agrees with the pull back of the orientation of* $\mathbf{R}^2$ *by means of* $\mathrm{T}_x f$, *then we can take* (δ_1, δ_2) *as the ordered basis of* H_1. *We obtain the monodromy matrix* $\mathscr{M}$ *as given in part* (c) *of Theorem* 1.2. *Otherwise, we get the inverse of* $\mathscr{M}$ *as the monodromy matrix.*

COROLLARY 1.8. *In the Hamiltonian case, the orientations in Proposition* 1.8 *agree.*

Remark 1.9. In [19] Tien Zung gives a generalization of the Hamiltonian version of Theorem 1.2 to the situation where there is a cycle of n focus-focus singularities. His proofs make essential use of the Hamiltonian character of the system. His main conclusion is that the monodromy of a cycle of n focus-focus singularities is equal to the nth power of the monodromy with 1 focus-focus singularity. This is consequence of our Corollary 1.8.

In the Hamiltonian systems given by the spherical pendulum [14] and the Hamiltonian Hopf bifurcation [15], the monodromy matrix indeed has the minus sign in the upper right corner if one goes around the origin in U in the positive, counter-clockwise direction. In [14] a plus sign occured in the upper right corner because the monodromy was taken over a clockwise loop around the origin in U.

Remark 1.10. In the non-Hamiltonian case, the orientations in Proposition 1.8 need no longer agree, because the integral mapping f need no longer be determined by the vector fields v and w as in the Hamiltonian case. For example, consider an prolate ellipsoid of revolution rolling without slipping on a horizontal plane under the influence of a constant vertical gravitational force. One can have a cycle of two focus-focus points where the signs in the monodromy matrix are *different* (and thus cancel when going around the cycle). Such a cycle occurs when both equilibrium positions of the body with the symmetry axis in the vertical direction are unstable, the height of the center of mass being equal to or larger than the height of the center of mass at all other positions. The heteroclinic orbits consist of the body rolling over a half turn along a meridian. Because the prolate ellipsoid of revolution is symmetric with respect to the reflection about the center of mass, the orientations of the unstable manifolds near the two equilibria can be compared by using the corresponding reflection in phase space as a time reversing map. Because the monodromy going around this heteroclinic cycle is the identity, the rolling prolate ellipsoid of revolution can not be made into a Hamiltonian system, even though it is time reversible and energy conserving. This is an example where a global invariant (namely, monodromy) has been used to show that a 4-dimensional conservative time reversible system is not Hamiltonian.

2. THE SINGULAR FIBER

2.1. We use the notation ϕ^t and ψ^s for the flow of the vector field v and w, respectively. We begin with collecting some facts about the stable and unstable manifold S_+ and S_- of p, which are defined as the set of $x \in M$ such that $\phi^t(x)$ converges to p when $t \to \pm\infty$.

Let us write $V = \mathrm{D}v(p)$, which is regarded as a linear transformation on $T = \mathrm{T}_p M$. Let T_+ and T_- be the V-invariant 2-dimensional linear subspace of T such that the eigenvalues of the restriction V_+ and V_- of V to T_+ and T_- has negative and postive real part, respectively. If B is an open neighborhood of p in M, we denote by $S_\pm^B$ the set of $x \in M$ such that $\phi^t(x) \in B$ when $\pm t \geqslant 0$. The hyperbolicity of v at p implies that there exists an open neighborhood B of p in M such that $S_\pm^B$ is a smooth connected submanifold of M, with tangent space at p equal to $T_\pm$. In particular, $S_\pm^B$ is 2-dimensional and $S_\pm^B \backslash \{p\}$ is connected as well. It follows from the definition that $S_\pm$ is equal to union over all $t \in \mathbf{R}$ of the 2-dimensional manifolds $\phi^t(S_\pm^B)$. In particular $S_\pm$ and $S'_\pm := S_\pm \backslash \{p\}$ are connected.

Because $f(x) = f(\phi^t(x))$ converges to zero when $x \in S_\pm$, $t \to \pm\infty$, we find that $S_\pm \subset F_0$.

Since F_0 is a compact subset of M, which is locally invariant under the v-flow, it is globally invariant in the sense that for every $t \in \mathbf{R}$ and $x \in F_0$, the flow $\phi^t \mid F_0$ is well-defined and $\phi^t(x) \in F_0$.

In view of the rank condition for f in assumption (d), the set $F'_0 := F_0 \backslash \{p\}$ is a smooth 2-dimensional submanifold of M. It follows that $\phi^t(S_\pm^B) \backslash \{p\}$ is an open subset of F'_0 and therefore $S_\pm \backslash \{p\}$, the union of these sets over all $t \in \mathbf{R}$, is an open subset of F'_0 as well.

2.2. The compact set F_0 is also globally invariant under the w-flow. Because $[v, w]=0$, $\phi^t \circ \psi^s = \psi^s \circ \phi^t$, which implies that $(s, t) \mapsto \psi^s \circ \phi^t$ defines an action of the additive group $\mathbf{R}^2$ on F_0. Thus the fixed point set Z of the flow ϕ^t is invariant under the flow ψ^s. Because p is an isolated point of Z, it is a fixed point for the ψ^s. Thus $w(p)=0$. Moreover, if $x \in S_{\pm}$ then $\phi^t(\psi^s(x)) = \psi^s(\phi^t(x))$ converges to $\psi^s(p)=p$ when $t \to \pm\infty$, so $S_{\pm}$ is not only invariant under the v-flow, but also under the w-flow. Because $\{p\}$ is a fixed point for both flows, $S'_{\pm}$ is invariant under the $\mathbf{R}^2$-action as well.

Since the vector fields v and w are linearly independent at each point of the 2-dimensional smooth manifold F'_0, each orbit O of the $\mathbf{R}^2$-action in F'_0 is open in F'_0. From the fact that the complement of O in F'_0 (which is equal to the union of other $\mathbf{R}^2$-orbits in F'_0) is also open, we deduce that O is a connected component of F'_0. In the same way the invariant open subset $S'_{\pm}$ of F'_0 is a union of orbits. Because $S'_{\pm}$ is connected, we conclude that $S'_{\pm}$ is equal to an $\mathbf{R}^2$-orbit in F'_0.

Let O be an orbit in F'_0. It is open and closed in F'_0, which implies that it is open in $F_0 = F'_0 \cup \{p\}$. If p does not belong to the closure of O in M, then O is also closed in F_0 and hence is equal to F_0 because F_0 is connected. This is in contradiction with the fact that $p \in F_0$. Thus p belongs to the closure of O. On the other hand, $O \cup \{p\}$ is equal to the complement of the other orbits in F'_0, which are open in F_0. Hence $O \cup \{p\}$ is closed in F_0 and therefore closed in M. In other words, p is the unique limit point in $M \backslash O$ of the orbit O. We conclude that for any orbit O in F'_0, the closure of O in M is equal to $O \cup \{p\}$. This holds in particular for $O = S_{\pm}$.

2.3. In general, if O is an orbit of an $\mathbf{R}^2$-action and $x \in O$, then the mapping $(s, t) \mapsto \psi^s \circ \phi^t(x)$ induces a diffeomorphism from $\mathbf{R}^2/\Gamma_x$ onto O. Here $\Gamma_x := \{(s, t) \mid \psi^s \circ \phi^t(x) = x\}$ denotes the stabilizer subgroup of the point x. Γ_x is a closed additive subgroup of $\mathbf{R}^2$ and does not depend on the choice of $x \in O$. For this reason we write Γ_O instead of Γ_x.

Suppose now that O is an orbit in F'_0 and that $\Gamma_O \cap (\{0\} \times \mathbf{R}) \neq \varnothing$. Then the flow of v in O would be periodic, with a fixed common period $t_O > 0$. Because periodic solutions in O, which start near p and leave a fixed neighborhood of p, need arbitrarily long time for this, we deduce that the v-solutions in O which start close to p remain close to p. However, the hyperbolicity of v at p implies that v does not have periodic solutions which remain close to p other than p. Thus we arrive at a contradiction.

Because $\Gamma_O \cap (\{0\} \times \mathbf{R}) = \varnothing$, for every $x \in O$ the integral curve $\phi^t(x)$ runs out of every compact subset of O when $|t| \to \infty$. Combined with the fact that p is the only limit point in $M \backslash O$ of O (and that O is contained in the compact subset F_0 of M), it follows that $\phi^t(x)$ actually converges to p as $t \to \infty$ and also as $t \to -\infty$. In other words, $x \in S_+$ and $x \in S_-$.

Recalling that S'_+ and S'_- are orbits in F'_0, if follows that $O = S'_+ = S'_-$. Because this holds for every orbit O in F'_0, we deduce that $F'_0 = S'_+ = S'_-$ and $F_0 = S_+ = S_-$. We have proved the last statement in part (a) of Theorem 1.2.

2.4. In order to obtain a better understanding of the structure of $S_\pm$ as an $\mathbf{R}^2$-orbit and also as a first step in the proof of Proposition 1.5, we take a closer look at the linear transformation $W = \mathrm{D}w(p)$ of $T = \mathrm{T}_p M$. Because $[v, w] = 0$ we have $[V, W] = 0$, which implies that $T_\pm$ is W-invariant. We write $W_\pm$ for the restriction of W to $T_\pm$. Note that $W_\pm$ commutes with $V_\pm$.

Let $a, b \in \mathbf{R}$. We claim that under the assumptions (a), (b), the following statements (i)–(iii) are equivalent.

(i) $a\, W_\pm + b\, V_\pm = 0$.

(ii) For each $x \in S_\pm \backslash \{p\}$ we have $a\, w(x) + b\, v(x) = 0$.

(iii) There exists an $x \in S_\pm \backslash \{p\}$ such that $a\, w(x) + b\, v(x) = 0$.

In particular assumption (d) implies that $V_\pm$ and $W_\pm$ are linearly independent over $\mathbf{R}$.

Proof. Because there are no resonances for $V_\pm$ in $T_\pm$, the theorem of Poincaré–Sternberg [18] yields the existence of a smooth coordinate system in $S^B_\pm$, where B is a sufficiently small neighborhood of p, in which the vector field v is linear. Because w commutes with v, we have

$$w(x) = \mathrm{e}^{-t\, V_\pm}(w(\mathrm{e}^{t\, V_\pm} x)), \qquad x \in S^B_\pm.$$

The differentiability of w at the origin implies that the right hand side converges to $W_\pm\, x$ as $t \to \pm\infty$, so $w(x) = W_\pm\, x$, which means that w is linear in these coordinates as well. Using the fact that $S_\pm$ is equal to the union of the $\phi^t(S^B_\pm)$, $t \in \mathbf{R}$, we obtain the equivalence of (i)–(iii).

On $T_\pm$ there is a complex structure in terms of which $V_\pm$ is equal to multiplication by a complex number, which we will denote by the symbol $\widetilde{V}_\pm$. The fact that $W_\pm$ commutes with $V_\pm$ then implies that $W_\pm$ is also equal to multiplication by a complex number, denoted by $\widetilde{W}_\pm$. The linear independence over $\mathbf{R}$ of $V_\pm$ and $W_\pm$ implies that every complex number c can be written as $c = a\, \widetilde{W}_\pm + b\, \widetilde{V}_\pm$ for some $a, b \in \mathbf{R}$. In particular we can find $a, b \in \mathbf{R}$ such that the linear map $U = a\, W_\pm + b\, V_\pm$ has the property that U^2 is equal to minus the identity on $T_\pm$. Clearly $a \neq 0$ in this case and $a = a_\pm$ and $b = b_\pm$ are unique if we require that $a > 0$. The linear transformation $U_\pm = a_\pm\, W_\pm + b_\pm\, V_\pm$ is our choice of the aforementioned complex structure on $T_\pm$.

2.5. Consider the vector field $u_\pm = a_\pm w + b_\pm v$ on $S_\pm$. From the linearization argument in the proof of the equivalence of (i)–(iii) given above, we see that the $u_\pm$-solution curves are periodic with primitive period equal to 2π. In other words, if Γ denotes the stabilizer subgroup for the orbit $F'_0 = S'_+ = S'_-$, then both $(2\pi\, a_+, 2\pi\, b_+)$ and $(2\pi\, a_-, 2\pi\, b_-)$ belong to Γ and are not equal to $n\,T$ for some $T \in \Gamma$ and some integer $n \neq \pm 1$. Because $F'_0 \simeq \mathbf{R}^2/\Gamma$ is not compact, the rank of the lattice Γ cannot be equal to two and because $\Gamma \neq \{0\}$ we conclude that $\Gamma \simeq \mathbf{Z}$. Since a_+ and a_- have the same sign, this implies that that $a_+ = a_-$ and $b_+ = b_-$; we will write $a = a_\pm$ and $b = b_\pm$ in the sequel. We have $\Gamma = \mathbf{Z}\, T$, where $T = 2\pi\,(a, b)$.

If $\chi^s = \psi^{a\,s} \circ \phi^{b\,s}$ denotes the u-flow after time s, then, for any $x \in F'_0$, the mapping $(s, t) \mapsto \chi^s \circ \phi^t(x)$ induces the diffeomorphism from the cylinder $(\mathbf{R}/2\pi\,\mathbf{Z}) \times \mathbf{R}$ onto F'_0 which is mentioned in part (a) of Theorem 1.2. We have proved all the statements in part (a) of Theorem 1.2 including the one concerning $F'_0 = F_0 \backslash \{p\}$ in Proposition 1.5.

3. THE NEARBY REGULAR FIBERS

3.1. Let B be an open ball around p, which is chosen small enough so that its boundary ∂B intersects F_0 in two circles: $S_+ \cap \partial B$ and $S_- \cap \partial B$. Using our knowledge about $V_\pm$ and $W_\pm$, we can also arrange that except at p the v, w-orbits in a neighborhood of the closure $\bar{B}$ of B in M are 2-dimensional and that the orbits through points close to p intersect ∂B in two circles close to $F_0 \cap \partial B$.

The set $F_0 \backslash B$ is a compact smooth 2-dimensional submanifold of M having the two circles $S_\pm \cap \partial B$ as its boundary. Thus $F_0 \backslash B$ is diffeomorphic to a compact cylinder. Using a suitable 2-dimensional fibration transversal to $F_0 \backslash B$ and the facts that the rank of $Df(x)$ is equal to 2 for each $x \in F_0 \backslash B$ and that $v(x)$, $w(x)$ are linearly independent, we obtain an open neighborhood N of $F_0 \backslash B$ in $M \backslash B$, an open neighborhood U of 0 in $\mathbf{R}^2$, which we can take to be simply connected, and a diffeomorphism Φ from N onto $(F_0 \backslash B) \times U$, such that

(a) The restriction f_N of f to N is equal to $\pi_2 \circ \Phi$ where π_2 is the projection from $(F_0 \backslash B) \times U$ onto the second factor U.

(b) For each $x \in N$, $v(x)$ and $w(x)$ are linearly independent.

(c) The fibers of f_N intersect ∂B in two circles, which are close to the two circles where the fiber $F_0 \backslash B$ over 0 intersects ∂B.

3.2. On the other hand, the orbits of the local $\mathbf{R}^2$-action in $\bar{B}$ through points in $(N \cap \partial B) \backslash F_0$ also intersect ∂B in two circles. These two circles coincide with the intersection of the fibers of f_N with ∂B, because f is constant on the orbits. We now define $\tilde{M}$ as the union of N, the orbits in $\bar{B}$ which intersect $N \cap \partial B$, and $\{p\}$. The union of the orbits through $N \cap \partial B$ form an open subset of $\bar{B}$, which contains $V \backslash \{p\}$, where V is a sufficiently small open neighborhood of p in M. This follows because the v-solutions through points at a distance δ to p enter and leave B at points in ∂B which are at a distance $\mathrm{O}(\delta)$ to $\partial B \cap S_+$ and $\partial B \cap S_-$, respectively.

Consequently $\tilde{M}$ is an open neighborhood of F_0 in M such that $F_c \cap \tilde{M}$ is equal to the union of $F_c \cap N$ and the orbit in $\bar{B}$ through $F_c \cap \partial B$ for each $c \in U \backslash \{0\}$. From this description it is clear that the restriction of f to $\tilde{M}$ is a proper mapping from $\tilde{M}$ onto U and that its fibers are connected. The invariance of f under the local $\mathbf{R}^2$-action, combined with the fact that for every $x \in N \cap \partial B$ the rank of $Df(x)$ is equal to 2, implies that the rank of $Df(x)$ is also equal to 2 for x on each orbit which intersects $N \cap \partial B$. Consequently, $Df(x)$ is surjective for every $x \in \tilde{M} \backslash \{p\}$. Thus the restriction of f to $\tilde{M} \backslash F_0$ is a locally trivial fibration over $U \backslash \{0\}$ with compact and connected fibers.

Since we have replaced M by $\tilde{M}$, we will simplify our notation by writing F_c instead of $F_c \cap \tilde{M}$ in the sequel. Because the flows of v and w leave the fibers F_c invariant, these flows are complete and define an action of $\mathbf{R}^2$ on F_c. Now let $c \in U \backslash \{0\}$. Using (ii) above, combined with the fact that the vectors $v(x)$ and $w(x)$ are linearly independent for each $x \in B \backslash \{p\}$, we deduce that each orbit O in F_c is open. Because F_c is connected, we conclude that $O = F_c$. Since O is diffeomorphic to $\mathbf{R}^2/\Gamma_O$ and $O = F_c$ is compact, Γ_O is a 2-dimensional lattice in $\mathbf{R}^2$, and $O = F_c$ is diffeomorphic to a 2-dimensional torus. In the sequel we will write Γ_c instead of Γ_O. We have now proved all the statements part (b) of Theorem 1.2.

3.3. Recall that there exists $a, b \in \mathbf{R}$, with $a > 0$, such that all solution curves of the vector field $a\,w + b\,v$ on $F'_0 = S'_- = S'_+$ are periodic with primitive period equal to 2π. Let Y be a local smooth section of f at a point $y \in F'_0$. Applying the implicit function theorem to the equation $\phi^t \circ \psi^s(x) = x$, with unkowns (s, t) near $2\pi \cdot (a, b)$ and treating $x \in Y$ as a parameter, we obtain unique smooth functions σ and τ on an open neighborhood U' of 0 in U such that $\sigma(0) = a$, $\tau(0) = b$, and $\phi^t \circ \psi^s(x) = x$ when $x \in Y \cap f^{-1}(U')$, $s = 2\pi\,\sigma(f(x))$, $t = 2\pi\,\tau(f(x))$. In other words, all solutions of the vector field $u := (\sigma \circ f)\,w + (\tau \circ f)\,v$ which start in $Y \cap f^{-1}(U')$ are periodic with period equal to 2π. Because 2π is a primitive period for the solution starting at y, a continuity argument shows 2π is a primitive period for all solutions starting in $Y \cap f^{-1}(U')$ if we take U' sufficiently small. In other words, for each $c \in U'$, the element $T(c) = 2\pi \cdot (\sigma(c), \tau(c))$ is a primitive element of the lattice Γ_c.

We can take U' to be simply connected. We will write U and $\tilde{M}$ instead of U' and $f^{-1}(U') \cap \tilde{M}$. Using this notation, all the previous statements about $\tilde{M}$ and U remain valid. In particular we have proved Proposition 1.6.

4. MONODROMY

The trivialization outside B is used in order to prove that the monodromy is determined by what happens inside B, and this leads to the description of the monodromy matrix as in part (c) of Theorem 1.2 and Proposition 1.8.

4.1. The homology class of any u-circle in F_c can be taken as the first element $\delta_1(c)$ of a basis of $H_1 = H_1(F_c, \mathbf{Z})$. $\delta_1(c)$ depends smoothly and in a single-valued way on $c \in U \backslash \{0\}$. Thus when c encircles the origin in $U \backslash \{0\}$, we return to the same element of H_1. In other words, $\delta_1(c)$ is a fixed element for the monodromy operator in H_1.

If χ^s denotes the flow of the vector field u, then it will be convenient to work with the action $(s, t) \mapsto \phi^t \circ \chi^s$ of $\mathbf{R}^2$ on $\tilde{M} \backslash F_0$, instead of the action $(s, t) \mapsto \phi^t \circ \psi^s$. Consider the mapping which assigns to each $(s, t) \in \Gamma_c$ the homology class of the solution curve of the vector field $s\,u + t\,v$ defined on a time interval of unit length, which is a closed curve. This mapping is an isomorphism from Γ_c onto $H_1(F_c, \mathbf{Z})$. We know that $(2\pi, 0) \in \Gamma_c$ corresponds to the previously defined fixed element $\delta_1(c)$ of the monodromy in $H_1(F_c, \mathbf{Z})$.

4.2. In order to find the second element $\delta_2(c)$ of a basis of $H_1(F_c, \mathbf{Z})$, we observe that the projection $(s, t) \mapsto t$ maps Γ_c onto a subset of the form $\mathbf{Z}\, t_c$, where t_c is a uniquely determined positive number. Any $(s, t_c) \in \Gamma_c$ corresponds to an element $\delta_2(c)$ such that $\delta_1(c)$ and $\delta_2(c)$ form a $\mathbf{Z}$-basis of $H_1(F_c, \mathbf{Z})$.

In terms of integral curves of u and v this means the following. Let C_c be any orbit in F_c of the circle action defined by the flow of u. The vector field v is transversal to C_c. Let γ_c be an integral curve of v in F_c such that $\gamma_c(0) \in C$. Then t_c is the smallest positive time t such that $\gamma_c(t_c) \in C$. Moreover, there exists an $s \in \mathbf{R}$ such that $\chi^s(\gamma_c(t_c)) = \gamma_c(0)$. This corresponds to the condition that $(s, t_c) \in \Gamma_c$.

We have proved the first part of Proposition 1.8.

4.3. In order to understand how the second generator $\delta_2(c)$ moves as a function of $c \in U \backslash \{0\}$, it is be convenient to use local coordinates around p in which the circle action of u is a one-parameter group of linear transformations. (It is a theorem of Bochner [4, Theorem 1] that every compact group of smooth transformations can be linearized around a fixed point).

Furthermore, because the local stable and unstable manifold are u-invariant, one can perform a u-equivariant smooth coordinate transformation which maps the local stable and unstable manifold to open subsets of linear subspaces. In the new coordinates u is still linear.

In terms of the complex structures defined in paragraph 2.4, we can identify the neighborhood B of p in M with an open ball of small positive radius r around the origin in $\mathbf{C}^2$, such that $u(z) = i\,z$ and the local stable and unstable manifold correspond to an open neighborhood of the origin in $\mathbf{C} \times \{0\}$ and $\{0\} \times \mathbf{C}$. We write $z = (z_+, z_-) \in \mathbf{C}^2$.

Next we choose a fixed $0 < \varepsilon < r$ and for every c close to 0 consider the u-circles $C_\pm(c)$ in F_c determined by the condition that $|z_\pm| = \varepsilon$. As the initial point of the solution curve γ_c in F_c of the vector field v we take the point $z = z(c) \in C_-(c)$ such that $z_- = \varepsilon$. Here we use that $z_+ \mapsto f(z_+, \varepsilon)$ is a diffeomorphism from an open neighborhood of the origin in $\mathbf{C} = \mathbf{R}^2$ onto an open neighborhood of the origin in $\mathbf{R}^2$. Therefore $z_+ = z_+(c)$ is uniquely determined and depends smoothly on c when c is sufficiently close to the origin in $\mathbf{R}^2$. Because $z_+(0) = 0$, $z(c)$ is close to the local unstable manifold when c is small. Also note that $z_+(c)$ winds around the origin once if c does.

The solution curve γ_0 will leave $C_-(0)$ with growing $|\gamma_0(t)_-|$, while $\gamma_0(t)_+ = 0$. If B is sufficiently small, then γ_0 leaves B and reenters B with $\gamma_0(t)_- = 0$ and $|\gamma_0(t)_+| > \varepsilon$, and lands on the circle $C_+(0)$ after some positive time. It follows that for small c there is a first positive time $t = t_1(c)$ for which $|\gamma_c(t)_+| = \varepsilon$, and both $t_1(c)$ and $y(c) := \gamma_c(t_1(c))$ depend smoothly on c.

4.5. The following observation is crucial. For small c the point $y(c)$ remains close to the point $y(0) = (y(0)_+, 0)$ on the local stable manifold. In particular, $y(c)_+$ does not wind around the circle of radius ε in $\mathbf{C}$ when c winds around the origin in U. On the other hand, $y(c)_-$ winds around the origin once when c does.

For the proof, we write $y(c)_+ = \mathrm{e}^{i\alpha_c}\, y(0)_+$, in which $\alpha_c \in \mathbf{R}$ depends smoothly on c, $\alpha_0 = 0$. From the equation

$$c = f(y(c)_+, y(c)_-) = f(y(0)_+, \mathrm{e}^{-i\alpha_c}\, y(c)_-)$$

$$\sim \frac{\partial f(y(0)_+, z_-)}{\partial z_-}\bigg|_{z_- = 0} \mathrm{e}^{-i\alpha_c}\, y(c)_-, \quad |c| << 1,$$

we find that $y(c)_-$ winds around the origin once when c does so. Writing $x = (y(0)_+, 0)$, we identify the z_--space with $\mathrm{T}_x M / \mathrm{T}_x S_+$. The winding number of $y(c)_-$, when c winds around the origin in $\mathbf{R}^2$ in the positive direction, is equal to $+1$ or -1 if the orientation of $\mathrm{T}_x M / \mathrm{T}_x S_+$ defined by the complex structure in $\mathrm{T}_p M$ agrees or does not agree with the pull back of the orientation of $\mathbf{R}^2$ by means of $\mathrm{T}_x f$, respectively.

4.6. We now have to find the first positive time $t = t_2(c)$ such that

$$\gamma_c(t_1(c) + t) = \phi^t(y(c)) \in C_-(c).$$

Then $t_c = t_1(c) + t_2(c)$ and $\gamma_c(t_c) = \phi^{t_2(c)}(y(c))$.

We can write

$$\gamma_c(t_c)_- = \phi^{t_2(c)}(y(c))_- = \mathrm{e}^{i\theta_c}\,\varepsilon,$$

where $\theta_c \in \mathbf{R}$ depends smoothly on c when c runs on a curve in $U \backslash \{p\}$. Therefore, if $\gamma_c(t)$, with t running from 0 to t_c, is followed by the curve $\mathrm{e}^{i\tau}$, with τ running from 0 to $-\theta_c$, we obtain a closed curve on F_c which represents $\delta_2(c)$, and $(-\theta_c, t_c) \in \Gamma_c$. From this we find that if k is the winding number of $c \mapsto \phi^{t_2(c)}(y(c))_-$ on the circle of radius ε as c winds once around the origin in $U \backslash \{0\}$, then $\mathscr{M}\,\delta_2 = \delta_2 - k\,\delta_1$. Note that the winding number k, and therefore the monodromy matrix $\mathscr{M}$, is completely determined by the behaviour of the solution curves of the vector field v in the ball B around p.

4.7. Write $v = V + R$ in which V is the linear part of v, $V(z) = (V_+ z_+, V_- z_-)$, and R is the remainder term, which vanishes to at least second order at the origin. The condition that $z_- = 0$ and $z_+ = 0$ are the stable and unstable manifold of v, invariant under the flow of v, corresponds to the condition that $R(z_+, 0)_- \equiv 0$ and $R(0, z_-)_+ \equiv 0$. If we replace v by $v_\delta = V + \delta\,R$ with $0 \leqslant \delta \leqslant 1$ then the winding number of $\phi_\delta^{t_2(c,\,\delta)}(y(c))_-$ is well defined and depends continuously on δ, and therefore does not depend on δ because it is an integer. Now we have

$$\phi_0^t(z) = (\mathrm{e}^{V_+ t} z_+, \mathrm{e}^{V_- t} z_-),$$

from which we see that $t = t_2(c, 0)$ is determined by the equation

$$|\mathrm{e}^{V_- t}\, y(c)_-| = \varepsilon.$$

Writing $\rho_- = \operatorname{Re} V_- > 0$ and substituting the solution

$$t = (\rho_-)^{-1} \log(\varepsilon / |y(c)_-|),$$

we obtain

$$\phi_0^t(y(c))_- = \mathrm{e}^{(V_-/\rho_-)\log(\varepsilon/|y(c)_-|)}\, y(c)_-,$$

the argument of which is equal to the sum of the single-valued function

$$(\mathrm{Im}\, V_-/\mathrm{Re}\, V_-)\log(\varepsilon/|y(c)_-|)$$

of c and the argument of $y(c)_-$. It follows that the winding number of $\gamma_c(t_c)_-$ on the circle of radius ε, as c winds once around the origin in $U\backslash\{0\}$, is equal to the winding number of $y(c)_-$. Using the fact that $y(c)_+$ remains close to $y(0)_+$ on the circle of radius ε and the fact that f is rotationally invariant, the latter winding number is equal to $+1$ or -1, according to whether the diffeomorphism $z_- \mapsto f(\varepsilon, z_-)$ is orientation preserving or orientation reversing. The proof of Proposition 1.8 and part (c) of Theorem 1.2 is complete.

Remark 4.8. In passing we have seen that t_c is asymptotically proportional to $-\log|c| \to \infty$ as $c \to 0$. This is the way in which the 2-dimensional lattice Γ_c converges to the 1-dimensional lattice which is the stabilizer subgroup of the action of $\mathbf{R}^2$ on the manifold $F_0\backslash\{p\} = S_+\backslash\{p\} = S_-\backslash\{p\}$, which is diffeomorphic to a cylinder.

5. THE HAMILTONIAN CASE

5.1. For the proof of corollary 1.8 we may replace w by u, which we may assume to be linear in our coordinate system around p. Furthermore, we may replace v by the vector field $z \mapsto (-z_+, z_-)$ and the Hamiltonian function h and g of u and v by the quadratic part of its Taylor expansion at the origin. We will give a basis which is adapted to the symplectic form Ω in T_pM and then determine h and g in terms of the coordinates with respect to this basis.

5.2. Because $V = \mathrm{D}v(p)$ is an infinitesimally symplectic transformation in T_pM, we find for each $a, b \in \mathrm{T}_pM$ that

$$\Omega(a, b) = \Omega(\mathrm{e}^{t\,V}a, \mathrm{e}^{t\,V}b).$$

If $a, b \in T_\pm$ then, if we let t run to $\pm\infty$ in the right hand side, we get $\Omega(a, b) = 0$. In other words, T_+ and T_- are isotropic subspaces of T_pM with respect to the symplectic form Ω.

Because T is equal to the direct sum of T_+ and T_- and Ω is non-degenerate, we also obtain that

$$z_- \mapsto (z_+ \mapsto \Omega(z_+, z_-))$$

is a bijective linear mapping from T_- onto $(T_+)^*$, the space of linear forms on T_+.

5.3. Choose any nonzero vector $e_1 \in T_+$, and write $e_2 = U e_1$. Because U is our complex structure, $U^2 = -1$, we have $U e_2 = -e_1$.

The last statement in Paragraph 5.2 implies that there are unique $\varepsilon_1, \varepsilon_2 \in T_-$, such that

$$\Omega(e_i, \varepsilon_j) = \delta_{ij}, \quad i, j = 1, 2.$$

Because U is a linear combination of the infinitesimally symplectic transformations $\mathrm{D}v(p)$ and $\mathrm{D}w(p)$ in $\mathrm{T}_p M$, it is infinitesimally symplectic as well. This implies that $\Omega(e_i, U\varepsilon_j) = -\Omega(U e_i, \varepsilon_j)$, which is equal to $-\Omega(e_2, U\varepsilon_j)$ and $\Omega(e_1, U\varepsilon_j)$ when $i = 1$ and $i = 2$, respectively. From this we see that $U\varepsilon_1 = \varepsilon_2$ and $U\varepsilon_2 = -\varepsilon_1$.

Using $\Omega(e_1, e_2) = 0$ and $\Omega(\varepsilon_1, \varepsilon_2) = 0$, we find that if

$$a = x_1 e_1 + x_2 e_2 + \xi_1 \varepsilon_1 + \xi_2 \varepsilon_2, \quad b = y_1 e_1 + y_2 e_2 + \eta_1 \varepsilon_1 + \eta_2 \varepsilon_2,$$

then

$$\Omega(a, b) = x_1 \eta_1 + x_2 \eta_2 - \xi_1 y_1 - \xi_2 y_2.$$

5.4. The Hamiltonian function g of the linear Hamiltonian vector field U is determined by the relation

$$\Omega(U(a), b) + \mathrm{d}g(a)(b) = 0,$$

which in terms of the notation in Paragraph 5.3 is equivalent to

$$\frac{\partial g(a)}{\partial x_1} = -\xi_2, \quad \frac{\partial g(a)}{\partial x_2} = \xi_1, \quad \frac{\partial g(a)}{\partial \xi_1} = x_2, \quad \frac{\partial g(a)}{\partial \xi_2} = -x_1.$$

Using the condition that $g(0) = 0$, this gives $g(a) = -x_1 \xi_2 + x_2 \xi_1$.

A similar calculation shows that the Hamiltonian function h of the linear Hamiltonian vector field

$$V\colon (x_1, x_2, \xi_1, \xi_2) \mapsto (-x_1, -x_2, \xi_1, \xi_2)$$

is given by $h(a) = x_1 \xi_1 + x_2 \xi_2$.

5.5. The matrix of the derivative of the mapping $a \mapsto (g(a), h(a))$ is equal to

$$\begin{pmatrix} -\xi_2 & \xi_1 & x_2 & -x_1 \\ \xi_1 & \xi_2 & x_1 & x_2 \end{pmatrix}.$$

In particular $\det \frac{\partial f(a)}{\partial \xi} = x_2^2 + x_1^2 > 0$ when $a = (x, 0) \in S_+$, $a \neq 0$. This completes the proof of corollary 1.8.

ACKNOWLEDGMENT

We thank San Vu Ngoc for stimulating discussions.

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