LOCAL INTEGRABILITY OF THE MIXMASTER MODEL

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In this paper we study the mixmaster universe whose evolution is described by a polynomial Hamiltonian. We show that this model is locally integrable. The Taub solutions form an integrable subsystem. We show that there are no solutions of the mixmaster universe asymptotic to a partially isotropic gravitational collapse, that is, a collapse where two of the metric coefficients are always equal, other than the Taub solutions.

1. Hamilton equations

The Bianchi IX or mixmaster model of the universe is given by the 4-metric

$$^4g = dt^2 - \sum_{k=1}^{3} q_k(t)\sigma_k^2 \tag{1}$$

on $\mathbf{R} \times SU(2)$. Here σ_k are left invariant 1-forms on $SU(2)$ such that

$$d\sigma_i = \tfrac{1}{2}\sum_{j,k} \varepsilon_{ijk}\, \sigma_j \wedge \sigma_k. \tag{2}$$

For example, we can take

$$\begin{aligned} \sigma_1 &= \sin\psi\, d\theta - \cos\psi\, \sin\theta\, d\varphi, \\ \sigma_2 &= \cos\psi\, d\theta + \sin\psi\, \sin\theta\, d\varphi, \\ \sigma_3 &= -d\psi - \cos\theta\, d\varphi, \end{aligned} \tag{3}$$

where (θ, φ, ψ) are spherical coordinates for the 3-sphere which is identified with $SU(2)$. The positive coefficients q_k of ^4g are the dynamical variables to be determined from the Einstein equations. Thus the minisuperspace of the mixmaster model is $Q = \mathbf{R}^3_>$.

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The autonomous Hamiltonian formulation for the evolution of the mixmaster universe in $T^*Q = \mathbf{R}^3_> \times \mathbf{R}^3$ was given first by C. W. Misner [1] in terms of coordinates Ω, β_+, and β_- such that

$$\begin{aligned} q_1 &= \exp\{2(-\Omega + \beta_+ + \sqrt{3}\,\beta_-)\}, \\ q_2 &= \exp\{2(-\Omega + \beta_+ - \sqrt{3}\,\beta_-)\}, \\ q_3 &= \exp\{2(-\Omega - 2\beta_+)\}. \end{aligned} \tag{4}$$

The Hamiltonian is

$$H = \tfrac{1}{2}(-p_\Omega^2 + p_+^2 + p_-^2) + \tfrac{1}{2}e^{-4\Omega}V, \tag{5}$$

where p_Ω, p_+ and p_- are the momenta canonically conjugate to Ω, β_+ and β_-. Here the potential energy is

$$\begin{aligned} V = {} & \tfrac{1}{3}\exp(-8\beta_+) - \tfrac{4}{3}\exp(-2\beta_+)\cosh(2\sqrt{3}\beta_-) + \\ & + \tfrac{2}{3}\exp(4\beta_+)[\cosh(4\sqrt{3}\beta_-) - 1]. \end{aligned} \tag{6}$$

Our aim is to study the Hamiltonian (5) in the variables q_i and canonically conjugate momenta p_i. In terms of q_i and p_i the momenta p_Ω, p_+, and p_- are given by

$$\begin{aligned} p_\Omega &= \sum_i p_i \frac{\partial q_i}{\partial \Omega} = -2\sum_i p_i q_i, \\ p_+ &= \sum_i p_i \frac{\partial q_i}{\partial \beta_+} = 2p_1q_1 + 2p_2q_2 - 4p_3q_3, \\ p_- &= \sum_i p_i \frac{\partial q_i}{\partial \beta_-} = 2\sqrt{3}\,p_1q_1 - 2\sqrt{3}\,p_2q_2, \end{aligned} \tag{7}$$

since the change of variables (4) is a point transformation. Using (4) and (7) we see that the kinetic energy term in (5) is

$$\begin{aligned} \tfrac{1}{2}(-p_\Omega^2 + p_+^2 + p_-^2) = {} & 6(p_1q_1)^2 + 6(p_2q_2)^2 + 6(p_3q_3)^2 - \\ & -12(p_1q_1)(p_2q_2) - 12(p_2q_2)(p_3q_3) - 12(p_1q_1)(p_3q_3) \end{aligned}$$

and the potential energy term is

$$\tfrac{1}{2}\exp(-4\Omega)V = \tfrac{1}{6}[q_1^2 + q_2^2 + q_3^2 - 2(q_1q_2 + q_2q_3 + q_1q_3)].$$

Hence the Hamiltonian is

$$\begin{aligned} H = {} & 6(p_1q_1)^2 + 6(p_2q_2)^2 + 6(p_3q_3)^2 - \\ & -12(p_1q_1)(p_2q_2) - 12(p_2q_2)(p_3q_3) - 12(p_1q_1)(p_3q_3) + \\ & + \tfrac{1}{6}[q_1^2 + q_2^2 + q_3^2 - 2(q_1q_2 + q_2q_3 + q_1q_3)]. \end{aligned} \tag{8}$$

The integral curves of the Hamiltonian vector field X_H of H satisfy the Hamilton equations:

$$\begin{aligned}
\dot{q}_1 &= 12q_1[p_1q_1 - p_2q_2 - p_3q_3],\\
\dot{q}_2 &= 12q_2[-p_1q_1 + p_2q_2 - p_3q_3],\\
\dot{q}_3 &= 12q_3[-p_1q_1 - p_2q_2 + p_3q_3],\\
\dot{p}_1 &= -12p_1[p_1q_1 - p_2q_2 - p_3q_3] - \tfrac{1}{3}[q_1 - q_2 - q_3],\\
\dot{p}_2 &= -12p_2[-p_1q_1 + p_2q_2 - p_3q_3] - \tfrac{1}{3}[-q_1 + q_2 - q_3],\\
\dot{p}_3 &= -12p_3[-p_1q_1 - p_2q_2 + p_3q_3] - \tfrac{1}{3}[-q_1 - q_2 + q_3],
\end{aligned} \tag{9}$$

where the dot denotes differentiation with respect to supertime s (denoted by λ in [2]).

The solutions of (9), which lie in the zero level set of the Hamiltonian H, describe the evolution of the mixmaster universe.

In order to relate (9) to the BKL equations [3], we first check that

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\dot{q}_1}{q_1}\right) &= 4\{(q_2 - q_3)^2 - q_1^2\},\\
\frac{d}{ds}\left(\frac{\dot{q}_2}{q_2}\right) &= 4\{(q_1 - q_3)^2 - q_2^2\},\\
\frac{d}{ds}\left(\frac{\dot{q}_3}{q_3}\right) &= 4\{(q_2 - q_1)^2 - q_3^2\}.
\end{aligned} \tag{10}$$

Substituting

$$q_1 = e^{2\alpha}, \qquad q_2 = e^{2b}, \qquad q_3 = e^{2\gamma},$$

we obtain

$$\begin{aligned}
2\ddot{\alpha} &= 4\{(e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha}\},\\
2\ddot{\beta} &= 4\{(e^{2\alpha} - e^{2\gamma})^2 - e^{4\beta}\},\\
2\ddot{\gamma} &= 4\{(e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma}\}.
\end{aligned} \tag{11}$$

Comparing (11) with BKL equations in [4] we see that our independent variable s is twice the BKL time coordinate τ, that is,

$$ds = d\tau = 2\frac{dt}{\sqrt{q_1q_2q_3}}. \tag{12}$$

2. Local integrability

In this section we show that the mixmaster model is locally integrable.

The function $v^2 = q_1q_2q_3$ is a measure of the Riemannian volume of space-like sections of the mixmaster universe. Let σ be a function on $(H|T^*Q)^{-1}(0)$ given by

$$\begin{aligned}\sigma &= \frac{d}{ds}(v^{-2}|(H|T^*Q)^{-1}(0)) \\ &= [12(q_1q_2q_3)^{-1}(p_1q_1 + p_2q_2 + p_3q_3)]|(H|T^*Q)^{-1}(0). \end{aligned} \tag{13}$$

It is well known that σ is a strictly increasing function of s. This assertion follows by straightforward calculation. In particular, we obtain

$$\frac{d\sigma}{ds} = 288(q_1q_2q_3)^{-1}[(p_1q_1)^2 + (p_2q_2)^2 + (p_3q_3)^2]|(H|T^*Q)^{-1}(0) \geq 0, \tag{14}$$

which vanishes only when $(p_1q_1)^2+(p_2q_2)^2+(p_3q_3)^2 = 0$. The second derivative of σ at its critical points vanishes identically, while the third derivative is positive on $(H|T^*Q)^{-1}(0)$.

The existence of the function σ on $(H|T^*Q)^{-1}(0)$, which is strictly increasing along the integral curves of X_H, leads to the following

THEOREM 1. *Every point $m = (q,p) \in (H|T^*Q)^{-1}(0)$ lies in the domain U of a chart of $(H|T^*Q)^{-1}(0)$ in which all the integral curves of X_H are straight lines. In this chart distinct lines correspond to distinct integral curves of X_H.*

Proof: The vector field X_H has no equilibrium points in $(H|T^*Q)^{-1}(0)$. Hence, for any $m \in (H|T^*Q)^{-1}(0)$, we have $X_H(m) \neq 0$. Thus there is a codimension 1 submanifold P of $(H|T^*Q)^{-1}(0)$ containing m such that X_H is transverse to P. Since $X_H(m)$ spans the kernel of the pull back of ω to $(H|T^*Q)^{-1}(0)$, we see that T_mP is a symplectic subspace of $(T_m(T^*Q), \omega_m)$. Therefore by Darboux theorem, P is a symplectic submanifold of T^*Q near m and we can choose canonical coordinates (x,y) for P near m.

Consider the map Φ: $\mathbf{R} \times \mathbf{R}^4 \to (H|T^*Q)^{-1}(0)$: $(s,x,y) \mapsto \varphi_s^H(x,y)$, where φ_s^H is the flow of X_H. Then $\Phi(0,0,0) = m$. Because X_H is transverse to P at m, it follows that the map Φ is a local diffeomorphism. Its inverse Φ^{-1} is a diffeomorphism of a neighbourhood U of $\Phi(m)$ in $(H|T^*Q)^{-1}(0)$ onto its image $\Phi^{-1}(U) \subseteq \mathbf{R} \times \mathbf{R}^4$. It is called a straightening chart for X_H, because in the coordinates defined by Φ the vector field X_H is given by $\frac{\partial}{\partial s}$, whose integral curves are straight lines. Straightening charts exist near nonequilibrium points of an arbitrary vector field on a manifold, see Abraham and Marsden [5, p. 67].

However, in our case there is a function σ which is strictly increasing along all integral curves of X_H. Thus the image under the map Φ of distinct straight lines in $\Phi^{-1}(U)$ are distinct integral curves of X_H. ∎

The above result is independent of the choice of parametrization of the trajectories of X_H. It excludes all forms of recurrence for the integral curves of X_H on $(H|T^*Q)^{-1}(0)$. In particular it excludes chaotic behaviour.

There are several notions of integrability for Hamiltonian systems. We say that a Hamiltonian system (M, H, ω) is *integrable* if it has $n = \frac{1}{2}\dim M$ Poisson commuting constants of motion which are functionally independent on a dense subset of M. If the Hamiltonian vector fields of the commuting constants of motion are complete, the system is called *completely integrable*. The Hamiltonian system (M, H, ω) will be called *locally integrable* if for every $m \in M$ there exists an open neighbourhood U of m

invariant under the flow of the Hamiltonian vector field of H, such that the induced Hamiltonian system $(U, H|_U, \omega_U)$ is integrable.

To illustrate the notion of local integrability we discuss the extended Hamiltonian system $(\widetilde{M}, \widetilde{H}, \widetilde{\omega})$ associated to a given Hamiltonian system (M, H, ω). Here evolution space is $\widetilde{M} = M \times \mathbf{R}^2$, the extended Hamiltonian is $\widetilde{H} = H - e$, and the symplectic form on the evolution space is $\widetilde{\omega} = \omega + de \wedge dt$. Moreover, t is the parameter along the integral curves of the Hamiltonian vector field of H and e is the energy variable canonically conjugate to t. The extended Hamiltonian system describes the *evolution* of the given Hamiltonian system. Writing $e = H$ gives rise to a canonical embedding of (M, H, ω) as the zero level set of the Hamiltonian $\widetilde{H}$ of the extended Hamiltonian system. Restricting the flow of the Hamiltonian vector field of $\widetilde{H}$ to $\widetilde{H}^{-1}(0)$ gives the flow of the Hamiltonian vector field X_H. We have

THEOREM 2. (i) *For every Hamiltonian system (M, H, ω) the corresponding extended Hamiltonian system is locally integrable.*

(ii) *If $M = \mathbf{R}^{2n}$ and $\omega = \sum_i dx_i \wedge dy^i$, where (x, y) are Cartesian coordinates on $\mathbf{R}^{2n}$, and the Hamiltonian vector field X_H is complete, then the corresponding extended Hamiltonian system is completely integrable.*

Proof: (i) Given a point $(m_0, e_0, t_0) \in M \times \mathbf{R}^2$, consider the set

$$M_0 = \{(m, e, t) \in \widetilde{H}^{-1}(H(m_0) - e_0) | \{t = t_0\}\}.$$

M_0 is a symplectic submanifold of $\widetilde{M} = M \times \mathbf{R}^2$, which contains the point (m_0, e_0, t_0). Let V be the domain of a Darboux chart on $M_0 \subseteq \widetilde{M}$ which contains (m_0, e_0, t_0), that is, for some smooth functions f_i and g^i on V we have $\omega|V = \sum_i df_i \wedge dg^i$. Let W be the subset of $\widetilde{H}^{-1}(H(m_0) - e_0)$ which is obtained by flowing out from V along the integral curves of the Hamiltonian vector field of $X_{\widetilde{H}}$. Finally, let $\widetilde{U}$ be the set obtained by flowing out from W along the integral curves of the Hamiltonian vector field $X_t = \frac{\partial}{\partial e}$ of the time variable t. Since $X_{\widetilde{H}}$ and X_t commute, it follows that $\widetilde{U}$ is $X_{\widetilde{H}}$-invariant. We can extend the functions f_i on V to functions $\widetilde{f}_i$ on $\widetilde{U}$, which are $X_{\widetilde{H}}$ and X_t invariant. Clearly, the $\widetilde{f}_i$ are Poisson commuting, functionally independent constants of motion of the Hamiltonian system $(\widetilde{U}, \widetilde{H}_{\widetilde{U}}, \widetilde{\omega}_{\widetilde{U}})$.

(ii) If $M = \mathbf{R}^{2n}$ and $\omega = \sum_i dx_i \wedge dy^i$, then we can take $M_0 = \mathbf{R}^{2n} \times \{(e_0, t_0)\}$. Since the Hamiltonain vector field X_H is complete, it follows that the Hamiltonian vector field $X_{\widetilde{H}}$ is also complete. Also every integral curve of $X_{\widetilde{H}}$ intersects M_0. Hence $\widetilde{U} = \widetilde{M}$. Moreover, the functions x_i on $M_0 \simeq \mathbf{R}^{2n}$ have complete Hamiltonian vector fields. Therefore, the Hamiltonian vector fields of the extended functions $\widetilde{x}_i$ are also complete. Consequently, the extended Hamiltonian system $(\widetilde{M}, \widetilde{H}, \widetilde{\omega})$ is completely integrable. ∎

Let us return to the mixmaster model. The zero energy level $(H|T^*Q)^{-1}(0)$ can be expressed as the union of the expansion phase $M^e = \{(q, p) \in (H|T^*Q)^{-1}(0) | \sigma > 0\}$, the contraction phase $M^c = \{(q, p) \in (H|T^*Q)^{-1}(0) | \sigma < 0\}$, and the submanifold

$M^0 = \{(q,p) \in (H|T^*Q)^{-1}(0)|\ p_1q_1 + p_2q_2 + p_3q_3 = 0\}$, which separates these two phases and corresponds to the maximum expansion.

THEOREM 3. *About any point $(q,p) \in M^e$ (or M^c) there exists an open neighbourhood in M^e (or M^c) invariant under the flow of X_H restricted to M^e (or M^c) which can be embedded as the zero level of an integrable Hamiltonian system.*

Proof: We give the proof for the expansion phase since the argument for the contraction phase is the same. Given a point $(q,p) \in M^e$, let U be a neighbourhood of $(q,p) \in M^e$ constructed in the proof of Theorem 1. Denote by V the invariant neighbourhood of (q,p) obtained by flow out from U along the integral curves of X_H restricted to M^e. The functions (x_a, y^a) in U defined in the proof of Theorem 1 can be extended to X_H-invariant functions on V, which will also be denoted by (x_a, y^a). The functions (x_a, y^a, σ) give global coordinates on V. Since (x_a, y^a) are constants of motion, the equations of motion in V are

$$\frac{dx_a}{d\sigma} = 0 \quad \text{and} \quad \frac{dy^a}{d\sigma} = 0. \tag{15}$$

Let $\widetilde{M} = V \times \mathbf{R}$, and let $(\widetilde{x}_a, \widetilde{y}^a, \widetilde{\sigma})$ be functions on $\widetilde{M}$ obtained by pulling back the functions (x_a, y^a, σ) by the projection $\widetilde{M} = V \times \mathbf{R} \to V$ on the first factor. Let $\widetilde{H}$ be the projection $\widetilde{M} = V \times \mathbf{R} \to \mathbf{R}$ on the second factor and define $\widetilde{\omega} = \sum_a dx_a \wedge dy^a + d\widetilde{H} \wedge d\sigma$. The Hamiltonian vector field of the Hamiltonian system $(\widetilde{M}, \widetilde{H}, \widetilde{\omega})$ when restricted to $\widetilde{H}^{-1}(0)$ has integral curves which satisfy (15). The functions $\widetilde{x}_a$ are commuting constants of motion. Hence we have obtained an embedding of V as the zero level set of an integrable Hamiltonian system $(\widetilde{M}, \widetilde{H}, \widetilde{\omega})$. ∎

3. Taub solutions

In this section we show that the equilibrium points of X_H corresponding to partially isotropic gravitational collapse are contained in a union of integrable, X_H-invariant subsystems which are a union of Taub solutions.

The boundary ∂Q of minisuperspace $Q = \mathbf{R}^3_> \subseteq \mathbf{R}^3$ corresponds to the configuration space gravitational collapse set. In particular, ∂Q is the union of the first quadrants of the coordinate planes in $\mathbf{R}^3$. Let C be the set of points $(q,p) \in \mathbf{R}^6$ such that q lies in ∂Q, that is,

$$C = \{(q,p) \in \mathbf{R}^6 |\, q_1 \geq 0, q_2 \geq 0, q_3 \geq 0, \text{ and } q_1q_2q_3 = 0\}. \tag{16}$$

C will be called the phase space gravitational collapse set. It should be noted that gravitational collapse can also occur at infinity, for example when some $q_i \to \infty$ but $q_1q_2q_3 \to 0$. Such solutions require separate investigation, which we will not carry out.

The principal advantage of expressing the Hamiltonian H (8) in our variables (q,p) is that H is defined and analytic on *all* of $\mathbf{R}^6$. Thus the integral curves of X_H can be used to study the evolution of the mixmaster universe in a neighbourhood of the phase space gravitational collapse set C.

Setting one of the q_i equal to zero in the equations of motion (9), we see that $\dot{q}_i$ equals zero. Thus the phase space gravitational collapse set C is invariant under the flow of X_H. Hence, an integral curve of X_H can approach C only if it approaches an equilibrium point of X_H in C. This implies that motions which lead to gravitational collapse, that is, which converge to points in C, must do so in infinite s-time.

We now look at the relevant set of equilibrium points more closely. The motions of physical interest lie in $(H|T^*Q)^{-1}(0)$. We consider

$$(H|T^*Q)^{-1}(0) \cap C$$
$$= \{(q,p) \in \mathbf{R}^6 |\, q_i = 0, q_j = q_k, \text{ and } p_j = p_k \text{ for } \{i,j,k\} = \{1,2,3\}\}.$$

Let E be the set of equilibrium points of X_H contained in $(H|T^*Q)^{-1}(0) \cap C$. We can write

$$E = E_0 \cup E_1 \cup E_2 \cup E_3,$$

where

$$E_0 = \{(q,p) \in \mathbf{R}^6 |\, q = 0\} \tag{17}$$

is the total gravitational collapse set and

$$E_i = \{(q,p) \in \mathbf{R}^6 - E_0 |\, q_i = 0, q_j = q_k, \text{ and } 36p_ip_j = 36p_ip_k = -1\} \tag{18}$$

is a set of partial gravitational collapse. For $i = 1,2,3$ each of the sets E_i has two connected components: one where $p_i > 0$, and the other where $p_i < 0$.

We now find a symplectic submanifold $(M_i, \Omega_i = \omega|M_i)$ of the phase space $(\mathbf{R}^6, \omega = \sum_i dq_i \wedge dp_i)$, which is invariant under the flow of X_H (9) and contains E_i. Moreover, the restricted Hamiltonian system $(H_i = H|M_i, M_i, \Omega_i)$ has an integral A_i which is functionally independent of H_i on a dense open subset of M_i. In other words, the Hamiltonian subsystem (H_i, M_i, Ω_i) is integrable.

For each $i = 1,2,3$, with j and k in $\{1,2,3\} - \{i\}$ and not equal to each other, let

$$M_i = \{(q,p) \in \mathbf{R}^6 |\, q_j = q_k, p_j = p_k\}. \tag{19}$$

From the definition of E_i (18) we see that $E_i \subseteq M_i$.

PROPOSITION 4. *Each M_i is an X_H-invariant symplectic manifold.*

Proof: We treat only the case $i = 1$. The other cases are handled in a similar manner. Introduce new variables

$$\begin{aligned} q^+ &= q_2 + q_3, & p^+ &= \tfrac{1}{2}(p_2 + p_3), \\ q^- &= q_2 - q_3, & p^- &= \tfrac{1}{2}(p_2 - p_3). \end{aligned} \tag{20}$$

Substituting (20) into the equations of motion (9) gives

$$\begin{aligned} \dot{q}_1 &= 12q_1[p_1q_1 - p^+q^+ - p^-q^-], \\ \dot{q}^+ &= -12(p_1q_1)q^+ + 12q^-(p^+q^- + p^-q^+), \\ \dot{q}^- &= -12(p_1q_1)q^- + 12q^+(p^+q^- + p^-q^+), \end{aligned} \tag{21}$$

$$\begin{aligned} \dot{p}_1 &= -12p_1[p_1q_1 - p^+q^+ - p^-q^-] - \tfrac{1}{3}[q_1 - q^+], \\ \dot{p}^+ &= 12(p_1q_1)p^+ - 12p^-(p^+q^- + p^-q^+) + \tfrac{1}{3}q_1, \\ \dot{p}^- &= 12(p_1q_1)p^- - 12p^+[p^+q^- + p^-q^+] - \tfrac{1}{3}q^-. \end{aligned} \tag{22}$$

Since the right-hand sides of (21) and (22) vanish when $q^- = p^- = 0$, it follows that M_1 is an invariant manifold of X_H. In fact M_1 is parametrized by (q_1, q^+, p_1, p^+). Since q^- and p^- are canonically conjugate, M_1 is symplectic with symplectic form $\Omega_1 = dq_1 \wedge dp_1 + dq^+ \wedge dp^+ = \omega|M_1$. ∎

COROLLARY. *The manifold*

$$M_0 = M_1 \cap M_2 \cap M_3$$

is an invariant symplectic manifold of X_H *containing* E_0.

Proof: Clearly M_0 is an invariant symplectic manifold of X_H. Since M_0 can be parametrized by (q_1, p_1), it follows that the restriction of the symplectic form ω to M_0 is $3\,dq_1 \wedge dp_1$. Hence M_0 is symplectic. Clearly, $E_0 \subseteq M_0$. ∎

For $(q,p) \in M_0$ the spatial part of the metric ${}^4g$ is isotropic since $q_1 = q_2 = q_3$. Hence ${}^4g$ is a Robertson–Walker metric. However, if $p_i = p$ and $q_i = q$ for $i = 1, 2, 3$, then the mixmaster Hamiltonian (9) becomes

$$H_0 = -18(pq)^2 - \tfrac{1}{2}q^2 \leq 0.$$

Hence $H_0^{-1}(0) = \{(0,p) \in \mathbf{R}^6 | p \in \mathbf{R}^3\} = E_0$, which does not belong to the phase space T^*Q of the mixmaster model.

For $(q,p) \in M_1 \cup M_2 \cup M_3$ two of the three metric coefficients q_1, q_2, q_3 are equal. Hence we have partial isotropy of the 3-dimensional metric on $SU(2)$. This is the subsystem we investigate here.

Restricted to the invariant symplectic manifold (M_1, Ω_1) the mixmaster Hamiltonian H (8) is the Hamiltonian

$$H_1 = 6(p_1q_1)^2 - 12(p_1q_1)(p^+q^+) + \tfrac{1}{6}q_1^2 - \tfrac{1}{3}q_1q^+. \tag{23}$$

The integral curves of the Hamiltonian vector field X_{H_1} on $M_1 = \{(q,p) \in \mathbf{R}^6 | q^- = p^- = 0\}$ satisfy the Hamilton equations:

$$\begin{aligned} \dot{q}^+ &= -12(p_1q_1)q^+, \\ \dot{q}_1 &= 12q_1(p_1q_1 - p^+q^+), \\ \dot{p}^+ &= 12(p_1q_1)p^+ + \tfrac{1}{3}q_1, \\ \dot{p}_1 &= -12p_1(p_1q_1 - p^+q^+) - \tfrac{1}{3}(q_1 - q^+). \end{aligned} \tag{24}$$

We now prove

PROPOSITION 5. *The function*

$$A_1 = \tfrac{1}{2}[36(p_1q_1 - p^+q^+)^2 + q_1^2] \tag{25}$$

is an integral of X_{H_1}.

Proof: Using (24) we compute

$$\begin{aligned}\dot{A}_1 &= 36(\dot{p}_1 q_1 + p_1\dot{q}_1 - \dot{p}^+ q^+ - p^+\dot{q}^+)(p_1q_1 - p^+q^+) + q_1\dot{q}_1\\ &= 36(p_1q_1 - p^+q^+)[-12(p_1q_1)(p_1q_1 - p^+q^+) - \tfrac{1}{3}q_1^2 + \tfrac{1}{3}q_1q^+ +\\ &\quad +12(p_1q_1)(p_1q_1 - p^+q^+) - 12(p_1q_1)(p^+q^+) - \tfrac{1}{3}q_1q^+ +\\ &\quad +12(p_1q_1)(p^+q^+)] + 12q_1^2(p_1q_1 - p^+q^+) = 0.\end{aligned}$$

■

Because H_1 and A_1 are polynomials, they are functionally dependent when $0 = dH_1 \wedge dA_1$. Since this condition defines an algebraic subset of M_1, the functions H_1 and A_1 are functionally independent on an open dense subset of M_1. Thus the Hamiltonian system (H_1, M_1, Ω_1) is integrable.

Exact solutions $s \rightarrow \gamma(s) = (q_1(s), q^+(s), p_1(s), p^+(s))$ of the Hamiltonian system (H_1, M_1, ω_1) which lie on $(H|T^*Q)^{-1}(0) \cap M_1 \cap T^*Q = H_1^{-1}(0) \cap T^*Q$ were discovered by Taub [6]. They are given by

$$q^+(s) = \sqrt{2a_1}\left[\frac{\cosh 2\sqrt{2a_1}(s - s^\dagger)}{1 + \cosh 2\sqrt{2a_1}(s - s^*)}\right], \tag{26}$$

$$q_1(s) = \frac{\sqrt{2a_1}}{\cosh 2\sqrt{2a_1}(s - s^\dagger)}, \tag{27}$$

$$p^+(s) = \frac{1}{6}\frac{\sinh 2\sqrt{2a_1}(s - s^*)}{\cosh 2\sqrt{2a_1}(s - s^\dagger)}, \tag{28}$$

$$\begin{aligned}p_1(s) = \tfrac{1}{6}\Bigg[&- \sinh 2\sqrt{2a_1}(s - s^\dagger) +\\ &+ \frac{\cosh 2\sqrt{2a_1}(s - s^\dagger)\sinh 2\sqrt{2a_1}(s - s^*)}{1 + \cosh 2\sqrt{2a_1}(s - s^*)}\Bigg],\end{aligned} \tag{29}$$

where $s^\dagger$ and s^* are integration constants. Specifically, $s^\dagger$ is the time at which q_1 assumes its maximum value and s^* is the time at which q_1q^+ reaches its maximum.

4. Geometry of asymptotic solutions

In this section we show that the set of solutions of X_H on $(H|T^*Q)^{-1}(0)$, which are asymptotic to equilibrium points in $E_1 \cup E_2 \cup E_3$ (18), consists of Taub solutions.

The analysis of the set of solutions asymptotic to the equilibrium points E_0 requires separate investigation. It should be noted that no solution asymptotic to E_0 can have two of the metric coefficients equal for all the time. Hence, the solutions asymptotic to E_0 describe an anisotropic gravitational collapse.

We begin by finding the set of all integral curves of the integrable vector field X_{H_1} on M_1 which are asymptotic to the manifold of equilibrium points:

$$E_1 = \{q^- = p^- = 0 \text{ and } \; q_1 = 0,\; q^+ \neq 0,\; 36p_1p^+ + 1 = 0\}.$$

Parametrize E_1 by u and v, where

$$q^+ = v, \;\; v \neq 0, \;\; p_1 = -\frac{1}{36u} \text{ and } p^+ = u, \;\; u \neq 0.$$

In the following we denote by (α, β) the equilibrium point $(0, v, -\frac{1}{36u}, u)$ of X_{H_1}. On M_1 we introduce new coordinates $(\xi_1, \xi_2, \eta_1, \eta_2)$ which bring (α, β) to the origin; namely,

$$\xi_1 = q_1, \qquad \eta_1 = p_1 + \frac{1}{36u},$$
$$\xi_2 = q_2, \qquad \eta_2 = p^+ - u.$$

The Hamiltonian H_1 (23) in the new coordinates is

$$H_1 = \frac{1}{6}\left(1 + \frac{1}{36u^2}\right)\xi_1^2 - 12uv\xi_1\eta_1 + \frac{v}{3u}\xi_1\eta_2 - \frac{1}{3u}\xi_1^2\eta_1 - \\ -12\,u\,\xi_1\xi_2\eta_1 - 12v\xi_1\eta_1\eta_2 + \frac{1}{3u}\xi_1^2\xi_2^2\eta_2 + 6\xi_1^2\eta_1^2. \tag{30}$$

Linearizing X_{H_1} at the origin gives

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -12uv & 0 & 0 & 0 \\ \frac{v}{3u} & 0 & 0 & 0 \\ -\frac{1}{3}\left(1 + \frac{1}{36u^2}\right) & 0 & 12uv & -\frac{v}{3u} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \mathcal{A}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{31}$$

where $\mathcal{A} = DX_{H_1}(0)$. Using the linear canonical change of variables

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{1}{72uv}\left(1 + \frac{1}{36u^2}\right) & 0 & 1 & -\frac{1}{36u^2} \\ \frac{1}{36u^2} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{P}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{32}$$

equation (31) becomes

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12uv & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -12uv & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{B}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{33}$$

The eigenvectors $w_1 = \mathcal{P}e_2 = (0,1,0,0)$ and $w_2 = \mathcal{P}e_4 = (0,0,\frac{1}{36u^2},1)$ of $\mathcal{A}$ corresponding to the eigenvalue 0 span the tangent space to E_1 at (α,β). On E_1 the parameters u and v are nonzero. Hence for $(\alpha,\beta) \in E_1$ the eigenvectors $w_3 = \mathcal{P}e_1$ and $w_4 = \mathcal{P}e_3$ of $\mathcal{A}$ corresponding to the nonzero real eigenvalues $\lambda = \pm 12uv$ span a space which is normal to E_1 at (α,β). Since the expanding eigenvalue $|\lambda|$ and contracting eigenvalue $-|\lambda|$ of $DX_{H_1}(\alpha,\beta)$ in the normal direction to E_1 are larger in absolute value than any eigenvalue of $DX_{H_1}(\alpha',\beta')$ corresponding to eigenvectors tangential to E_1 at (α',β'), it follows that the manifold E_1 is normally hyperbolic [7]. Thus, according to [8], the set $W^s(E_1)$ of all points of M_1 which are asymptotic to E_1 under the flow $\varphi_s^{H_1}$ of X_{H_1} as $s \to \infty$ is a smooth manifold called the stable manifold of E_1. Similarly, the set $W^u(E_1)$ of all points of M_1 which are asymptotic to E_1 under the flow $\varphi_s^{H_1}$ of X_{H_1} as $s \to -\infty$ is a smooth manifold called the unstable manifold of E_1. By definition, the stable manifold of E_1

$$W^s(E_1) = \{P \in M_1 - E_1 \mid \lim_{s\to\infty} \varphi_s^{H_1}(P) \in E_1\} \tag{34}$$

and the unstable manifold of E_1

$$W^u(E_1) = \{P \in M_1 - E_1 \mid \lim_{s\to-\infty} \varphi_s^{H_1}(P) \in E_1\}$$

are invariant manifolds of X_{H_1}. In fact, the stable manifold $W^s(E_1)$ of E_1 is the *unique* invariant manifold of X_{H_1} whose tangent space at the point $(\alpha,\beta) \in E_1$ is the sum of the tangent space to E_1 at (α,β) and the eigenspace $V_{-|\lambda|}$ corresponding to the eigenvalue $-|\lambda|$ of $\mathcal{A}$ at (α,β). Since $\dim E_1 = 2$ and $\dim V_{-|\lambda|} = 1$, it follows that $\dim W^s(E_1) = 3$. Similarly, the unstable manifold $W^u(E_1)$ of E_1 is the unique invariant manifold of X_{H_1} whose tangent space at the point $(\alpha,\beta) \in E_1$ is the sum of the tangent space to E_1 at (α,β) and the eigenspace $V_{|\lambda|}$ corresponding to the eigenvalue $|\lambda|$ of $\mathcal{A}$ at (α,β). Again $\dim W^u(E_1) = 3$. Another way to show that $W^s(E_1)$ is a smooth manifold goes as follows. For every $(\alpha,\beta) \in E_1$, the stable manifold $W^s(\alpha,\beta)$ of the point (α,β) is the unique X_{H_1}-invariant manifold whose tangent space at (α,β) is the eigenspace $V_{-|\lambda|}$ of $\mathcal{A}$ at (α,β). Since the Hamiltonian H_1 (30) and the change of variables $\mathcal{P}$ (33) depend smoothly on the point (α,β), the eigenspace $V_{-|\lambda|}$ depends smoothly on (α,β). Hence the stable manifold $W^s(\alpha,\beta)$ varies smoothly with (α,β). Consequently, the stable manifold of E_1, which is the disjoint union of $W^s(\alpha',\beta')$ as (α',β') ranges over E_1, is a smooth manifold. A similar argument shows that the unstable manifold $W^u(E_1)$ of E_1 is also a smooth manifold.

After these generalities we give an explicit description of the X_{H_1}-stable and unstable manifolds of $E_1 \cap T^*Q$. Because $E_1 \subseteq H_1^{-1}(0)$ and H_1 is an integral of X_{H_1}, from (34) it follows that $W^s(E_1) \cup W^u(E_1) \subseteq H_1^{-1}(0)$. To proceed further, we need the following lemma which describes the asymptotic behaviour of the Taub solutions of X_{H_1}.

LEMMA 6. *Let* $s \to \gamma(s) = (q_1(s), q^+(s), p_1(s), p^+(s))$ *be an integral curve of* X_{H_1} *in* T^*Q *given by a Taub solution. Then*

$$\begin{aligned}\gamma\ (\infty) &= \lim_{s\to\infty}\gamma(s)\\ &=(0, \sqrt{2a_1}\exp(2\sqrt{2a_1}(s^*-s^\dagger)), -\tfrac{1}{6}\exp(\sqrt{2a_1}(s^\dagger-s^*)), \tfrac{1}{6}\exp(\sqrt{2a_1}(s^*-s^\dagger))), \quad (35)\\ \gamma\ (-\infty) &= \lim_{s\to-\infty}\gamma(s)\\ &=(0, \sqrt{2a_1}\exp(-2\sqrt{2a_1}(s^*-s^\dagger)), \tfrac{1}{6}\exp(\sqrt{2a_1}(s^\dagger-s^*)), -\tfrac{1}{6}\exp(\sqrt{2a_1}(s^*-s^\dagger))). \quad (36)\end{aligned}$$

Proof: We only compute $q^+(\pm\infty)$. The other cases are similar and are omitted. From (26) we have

$$q^+(s) = \sqrt{2a_1}\frac{\exp(2\sqrt{2a_1}(s-s^\dagger)) + \exp(-2\sqrt{2a_1}(s-s^\dagger))}{2 + \exp(2\sqrt{2a_1}(s-s^*)) + \exp(-2\sqrt{2a_1}(s-s^*))}. \quad (37)$$

Dividing numerator and denominator of (37) by $\exp(2\sqrt{2a_1}s)$ gives

$$\begin{aligned}q^+(s) &= \sqrt{2a_1}\times\\ &\times\frac{\exp(-2\sqrt{2a_1}s^\dagger) + \exp(-4\sqrt{2a_1}s)\exp(2\sqrt{2a_1}s^\dagger)}{2\exp(-2\sqrt{2a_1}s) + \exp(-2\sqrt{2a_1}s^*) + \exp(-4\sqrt{2a_1}s)\exp(2\sqrt{2a_1}s^*)}.\end{aligned}$$

Taking the limit as $s\to\infty$ gives the second component of (35). Dividing the numerator and denominator of (37) by $\exp(-2\sqrt{2a_1}s)$ gives

$$q^+(s) = \sqrt{2a_1}\frac{\exp(2\sqrt{2a_1}s^\dagger) + \exp(4\sqrt{2a_1}s)\exp(-2\sqrt{2a_1}s^\dagger)}{2(\exp 2\sqrt{2a_1}s) + \exp(2\sqrt{2a_1}s^*) + \exp(4\sqrt{2a_1}s)\exp(-2\sqrt{2a_1}s^*)}.$$

Taking the limit as $s\to-\infty$ gives the second component of (36). ∎

Using (35) and (36) it is easy to see that

$$\gamma_2(\varepsilon\infty) > 0 \quad \text{and} \quad 36\gamma_3(\varepsilon\infty)\gamma_4(\varepsilon\infty) + 1 = 0,$$

where $\varepsilon^2 = 1$, that is, $\gamma(\pm\infty)$ is in

$$E_1^+ = \{P\in E_1\cap T^*Q \mid q^+(P) > 0\}.$$

Note that E_1^+ is the set of equilibrium points of X_{H_1} which lie in $H_1^{-1}(0)\cap T^*Q$. The manifold E_1^+ has two connected components:

$$E_1^{++} = \{P\in E_1^+ \mid p^+(P) > 0\}, \qquad E_1^{+-} = \{P\in E_1^+ \mid p^+(P) < 0\}.$$

Clearly $\gamma(+\infty)\in E_1^{++}$ and $\gamma(-\infty)\in E_1^{+-}$. From Lemma 6 we see that every integral curve of X_{H_1} in $H_1^{-1}(0)\cap T^*Q$ which is not an equilibrium point of X_{H_1} converges to a point in E_1^{++} as $s\to\infty$, and converges to a point in E^{+-} as $s\to-\infty$. Thus we have proved

PROPOSITION 7. *The manifold*

$$W^s(E_1^{++}) = W^u(E_1^{+-}) = (H_1^{-1}(0) - E_1^+)\cap T^*Q. \quad (38)$$

The fact that the stable manifold of E_1^{++} is equal to the unstable manifold of E_1^{+-} is not surprising since X_{H_1} is integrable on M_1. From (23) it follows that

$$H_1 = q_1[\tfrac{1}{6}(36p_1^2 + 1)q_1 - \tfrac{1}{3}q^+(36p_1p^+ + 1)].$$

Consequently, $H_1^{-1}(0) \cap T^*Q$ is the union of

$$\{q_1 = 0,\ q^- = p^- = 0 \text{ and } q^+ > 0\} \tag{39}$$

and

$$\left\{q_1 = \frac{2q^+(36p_1p^+ + 1)}{36p_1^2 + 1},\ q^- = p^- = 0, \text{ and } q^+ > 0\right\}, \tag{40}$$

whose intersection is E_1^+. Combining (38) with (39) and (40) we obtain an explicit description of the set of points in $H_1^{-1}(0)$ which are initial conditions of solutions of X_{H_1} asymptotic to an equilibrium point in T^*Q as $s \to \pm\infty$.

Now let us look at the dynamics of the full Hamiltonian vector field X_H on $H^{-1}(0) \cap T^*Q$. Using the equations of motion (21) for the integral curves of X_H in the coordinates $(q, q^+, q^-, p_1, p^+, p^-)$, it is easy to check that

$$E_1 = \{q^- = p^- = 0,\ q_1 = 0,\ q^+ \neq 0, \text{ and } 36p_1p^+ + 1 = 0\}$$

are equilibrium points of X_H which lie in $H^{-1}(0) \cap T^*Q$. Parametrize E_1 by nonzero u, v such that $q_1 = q^- = p^- = 0$, $q^+ = v$, $p_1 = -\frac{1}{36}$, and $p^+ = u$. Introducing canonical coordinates

$$\begin{aligned} \xi_1 &= q_1, & \eta_1 &= p_1 + \frac{1}{36u}, \\ \xi_2 &= q^+ - v, & \eta_2 &= p^+ - u, \\ \xi_3 &= q^-, & \eta_3 &= p^-, \end{aligned}$$

which brings the equilibrium point $(\alpha, \beta) = (0, v, 0, -\frac{1}{36u}, u, 0)$ to the origin, we find that the Hamiltonian H becomes

$$\begin{aligned} \widetilde{H} = \frac{1}{6}\Bigg(&\left(1 + \frac{1}{36u^2}\right)\xi_1^2 - 12uv\xi_1\eta_1 + \frac{v}{3u}\xi_1\eta_2 + \tfrac{1}{6}(1 + 36u^2)\xi_3^2 + \\ &+ 12uv\xi_3\eta_3 + 6v^2 + 12u\xi_3^2\eta_2 + 12\xi_3\eta_2\eta_3 + 12v\xi_2\eta_3^2 + \\ &- \frac{1}{3u}\xi_1^2\eta_1 + \frac{1}{3u}\xi_1\xi_2\eta_2 + \frac{1}{3u}\xi_1\xi_3\eta_3 + 6\,\xi^2\eta^2\Bigg). \end{aligned}$$

Hence the linearization $DX_{\widetilde{H}}(0)$ of the vector field $X_{\widetilde{H}}$ at the origin is

$$\frac{d}{ds}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \widetilde{A}\begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

where

$$\widetilde{\mathcal{A}} = \begin{pmatrix} -12uv & 0 & 0 & 0 & 0 & 0 \\ \frac{v}{3u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12uv & 0 & 0 & 12v^2 \\ -\frac{1}{3}\left(1 + \frac{1}{36u^2}\right) & 0 & 0 & -12uv & -\frac{v}{3u} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3}\left(1 + \frac{1}{36u^2}\right) & 0 & 0 & -12uv \end{pmatrix}.$$

Note that $\widetilde{\mathcal{A}}$ restricted to the space spanned by $\{e_1, e_2, e_4, e_5\}$ equals $\mathcal{A}$ (31). Moreover, the space spanned by $\{e_3, e_6\}$ is an $\widetilde{\mathcal{A}}$-invariant subspace on which its characteristic polynomial is $\lambda^2 + 4v^2$. Thus $\widetilde{\mathcal{A}}$ has eigenvalues $0, 0, 12uv, -12uv, 2iv, -2iv$. The real eigenvalues $\lambda = \pm 12uv$ are nonzero on $E_1^+ = E_1 \cap \{q^+ > 0\}$ and the corresponding eigenvectors, when projected onto the space spanned by $\{e_1, e_2, e_4, e_5\}$, are the same as those of $\mathcal{A}$. Hence $\widetilde{H}$, $X_{\widetilde{H}}$ and the eigenspaces of the real eigenvalues of $\widetilde{\mathcal{A}}$ depend smoothly on the parameters u and v such that (α, β) lies in E_1^+. This shows that the $X_{\widetilde{H}}$- stable manifold

$$\widetilde{W}^s(E_1^+) = \{P \in T^*Q - E_1^+ \mid \lim_{s \to \infty} \varphi_s^{\widetilde{H}}(P) \in E_1^+\}$$

is smooth. A similar argument shows that the $X_{\widetilde{H}}$-unstable manifold

$$\widetilde{W}^u(E_1^+) = \{P \in T^*Q - E_1^+ \mid \lim_{s \to -\infty} \varphi_s^{\widetilde{H}}(P) \in E_1^+\}$$

is smooth. Since the tangent space to $\widetilde{W}^s(E_1^+)$ at (α, β) is the sum of the tangent space to E_1^+ at (α, β) and the space spanned by the eigenvector of $\widetilde{\mathcal{A}}$ corresponding to the positive real eigenvalue, we see that the dimension of $\widetilde{W}^s(E_1^+)$ is three. Because the tangent space to $\widetilde{W}^s(E_1^+)$ at $(\alpha, \beta) \in E_1^+$ is equal to the tangent space to the X_{H_1}-stable manifold $W^s(E_1^+)$ considered as a subspace of $\mathbf{R}^6$, from *uniqueness* it follows that $\widetilde{W}^s(E_1^+) = W^s(E_1^+)$. A similar argument shows that $\widetilde{W}^u(E_1^+) = W^u(E_1^+)$. Thus we have proved

PROPOSITION 8. *The set of all initial conditions in $T^*Q \cap H^{-1}(0)$ of solutions of X_H which are asymptotic as $s \to \pm\infty$ to equilibrium points in E_i^+ corresponding to points of partial gravitational collapse is equal to the union of the stable manifold $W^s(E_i^+)$ and unstable manifold $W^u(E_i^+)$ of X_{H_i}.*

The conclusion of Proposition 8 is consistent with the local integrability of the mixmaster model. In fact it is an important global qualitative feature of this model.

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