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The rotation number and the herpolhode angle in Euler's top

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Abstract. We describe the relation between the rotation number and the herpolhode angle in the rotation of the force-free rigid body.

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In a numerical study of the Euler top [9] one of the authors (ES) observed that the angle swept out on the invariant plane by the point of contact of the moment of inertia ellipsoid during a period of the corresponding solution of Euler's equations did not always agree with the rotation number of the flow of the Euler-Arnol'd equations on a two-torus $T^2_{e,\mu}$ in phase space of constant energy and angular momentum. This is contrary to an assertion in [4, p.133]. In fact, the numerical answer differed by 2π from the rotation number. This paper explains this phenomenon by studying the geometry of the Poinsot map $\mathcal{R}$, which assigns to the body angular momentum the corresponding spatial angular velocity. We show that the image of a circle $\mathcal{S}$ on $T^2_{e,\mu}$, which is transverse to the rulings by the integral curves of the angular momentum vector field, is not always null homotopic in the annulus $\mathcal{A} = \mathcal{R}(T^2_{e,\mu})$ in which the point of contact of the inertia ellipsoid moves. This accounts for the missing 2π.

Related work on computing the amount of rotation of a free rigid body may be found in [6] and [7], where the rotation number is described in terms of an area integral in the reduced space, and is computed modulo 2π. Hence one may view our computation as providing the actual multiple of 2π that has been modded out.

Basic facts

We recall some known results about the Euler top, see [4, ch.III]. The motion of a force free rigid body in space may be modelled by the Hamiltonian system which

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is the geodesic flow of a left-invariant metric on the Lie group $G = \mathrm{SO}(3,\mathbb{R})$with Lie algebra $\mathfrak{g} = \mathrm{so}(3,\mathbb{R}) = \mathbb{R}^3$. In the left trivialization

$$G \times \mathfrak{g}^* \to T^*G : (A, p) \mapsto Ap$$

we have the attitude matrix A, which represents the orientation of the body in space and $p \in {\mathbb{R}^3}^*$ is the angular momentum in the body frame. $I = \mathrm{diag}(I_1, I_2, I_3)$ is the inertia tensor in a principal axis frame. We assume $I_1 < I_2 < I_3$, and set $a = I_1^{-1}$, $b = I_2^{-1}$ and $c = I_3^{-1}$. The angular velocity in the body frame is $\omega = I^{-1}p$, and the spatial angular velocity is $\Omega = A\omega$. $P = AIA^{-1}\Omega$ is the angular momentum in the space frame. The Hamiltonian of the Euler top is $h = \frac{1}{2}\langle p, I^{-1}p\rangle$, where $\langle\, ,\, \rangle$ is the Euclidean inner product. The equations of motion are Hamilton's equations. Since we do not have canonical variables, these equations are also called the Euler-Arnol'd equations. They are

$$\begin{aligned} \dot{A} &= A\widehat{I^{-1}p} \\ \dot{p} &= p \times I^{-1}p, \end{aligned}$$

where $\widehat{I^{-1}p} = \begin{pmatrix} 0 & -c\,p_3 & b\,p_2 \\ c\,p_3 & 0 & -a\,p_1 \\ -b\,p_2 & a\,p_1 & 0 \end{pmatrix}$. The equations $\dot{p} = p \times I^{-1}p$ are called the Euler equations.

Precis of known results

Proofs of the results in this section may be found in [4, ch.III]. It is standard and easily checked that P, h and $\langle \Omega, P\rangle$ are conserved quantities for the Euler-Arnol'd equations. The momentum map for the $\mathrm{SO}(3,\mathbb{R})$ action is

$$j : T^*G \to \mathfrak{g}^* : (A, p) \to P$$

and the energy-momentum mapping is

$$\mathcal{EM} : (A, p) \to (h(A, p), j(A, p)).$$

For every $\mu \neq 0$, $j^{-1}(\mu)$ is a smooth manifold diffeomorphic to $\mathbb{R}P^3$. A connected component of $\mathcal{EM}^{-1}(e, \mu)$ is a two-torus $T^2_{e,\mu}$ at all regular values of $h|j^{-1}(\mu)$. The reduced space S^2_μ is a sphere of radius $|\mu|$ and $j^{-1}(\mu)$ is an $\mathrm{SO}(3,\mathbb{R})_\mu = \{B \in \mathrm{SO}(3,\mathbb{R}) \,\big|\, B\mu = \mu\}$ principal bundle with base S^2_μ and bundle projection map π.

Let $t \to p(t)$ be a solution of Euler's equations. If $|\mu| = l$, then for $c \le 2e/l^2 \le b$,

$$p_1(t) = A\,\mathrm{cn}(nt; k),\ \ p_2(t) = B\,\mathrm{sn}(nt; k),\ \ p_3(t) = C\,\mathrm{dn}(nt; k),$$

where

$$A^2 = \frac{2e - cl^2}{a - c}, \quad B^2 = \frac{2e - cl^2}{b - c}, \quad C^2 = \frac{al^2 - 2e}{a - c},$$

and

$$n = \sqrt{(al^2 - 2e)(b - c)}, \qquad k = \sqrt{\frac{(a-b)(2e - cl^2)}{(b-c)(al^2 - 2e)}};$$

while for $b \leq 2e/l^2 \leq a$,

$$p_1(t) = A\,\mathrm{dn}(nt;k),\ p_2(t) = B\,\mathrm{sn}(nt;k),\ p_3(t) = C\,\mathrm{cn}(nt;k),$$

where

$$A^2 = \frac{2e - cl^2}{a - c}, \quad B^2 = \frac{al^2 - 2e}{a - b}, \quad C^2 = \frac{al^2 - 2e}{a - c},$$

and

$$n = \sqrt{(2e - cl^2)(a - b)}, \qquad k = \sqrt{\frac{(b-c)(al^2 - 2e)}{(a-b)(2e - cl^2)}}.$$

Consider Poinsot's map $\mathcal{R}$ defined by

$$\mathcal{R} : j^{-1}(\mu) \to \mathrm{so}(3,\mathbb{R}) : (A,p) \mapsto \Omega = AI^{-1}p. \tag{0.1}$$

In other words, to each point in phase space with spatial angular momentum μ we assign the corresponding spatial angular velocity $\Omega = AI^{-1}A^{-1}\mu$. Arnol'd [1] showed that $\mathcal{R}$ maps $T^2_{e,\mu}$ to an affine annulus $\mathcal{A}$. From a somewhat different point of view this was of course known to Poinsot [8]. If $\Gamma : t \to (A(t), p(t))$ is an integral curve of the Hamiltonian vector field on $T^2_{e,\mu}$, then the curve $t \mapsto \mathcal{R}\circ\Gamma(t)$ is called the herpolhode. The annulus lies in a plane with normal parallel to the vector μ and with centre collinear with μ. The torus $T^2_{e,\mu}$ is ruled by the fibers of the bundle projection π. To see this we recall some of the discussion of the Euler top in [4, p.128–29]. Suppose that Z is a unit vector in $\mathrm{so}(3,\mathbb{R})_\mu$, the Lie algebra of $\mathrm{SO}(3,\mathbb{R})_\mu$. Then

$$\phi^*_{-t} : T^*\,\mathrm{SO}(3,\mathbb{R}) \to \mathrm{SO}(3,\mathbb{R}) : (A,p) \mapsto ((\exp tZ)A, p)$$

is the flow of a Hamiltonian vector field corresponding to the Hamiltonian

$$j^Z : T^*\,\mathrm{SO}(3,\mathbb{R}) \to \mathbb{R} : (A,p) \mapsto P(Z) = \langle j(A,p), Z\rangle,$$

which is the Z-component of the $\mathrm{SO}(3,\mathbb{R})$-momentum mapping j. By construction $Z\mu = 0$. If $(A,p) \in j^{-1}(\mu)$, then

$$j(\phi^*_{-t}(A,p)) = j((\exp tZ)A, p) \ = \ \exp tZ(Ap) \ = \ (\exp tZ)\mu \ = \ \mu.$$

So ϕ^*_t is a diffeomorphism of $j^{-1}(\mu)$ into itself. Since ϕ^*_t clearly preserves $h^{-1}(e)$, it is a diffeomorphism of $T^2_{e,\mu}$ into itself. Because Z has unit length, the flow ϕ^*_t is periodic of period 2π. Thus the integral curves of Z on $T^2_{e,\mu}$ give a ruling by circles.

The fibers of the bundle projection π are mapped diffeomorphically by the Poinsot map $\mathcal{R}$ (0.1) onto concentric circles in $\mathcal{A}$. Furthermore, the inverse image under $\mathcal{R}$ of an interior point of $\mathcal{A}$ is four points; while the inverse image of a boundary point is two points.

Geometry of the Poinsot map $\mathcal{R}$

To get a more complete picture of the geometry of the map $\mathcal{R}$, we should understand how $\mathcal{R}$ maps a circle in $T^2_{e,\mu}$, transverse to the fibers of π, to the annulus $\mathcal{A}$. More formally, if $\mathcal{C}$ is the circle which is the fiber of π over the point p, we choose another circle $\mathcal{S}$ in $T^2_{e,\mu}$ so that the pair $\{\mathcal{C}, \mathcal{S}\}$ form a basis for the homology group $H_1(T^2_{e,\mu}, \mathbb{Z})$.

Let ξ be the angular variable in the plane that contains the annulus $\mathcal{A}$, which measures the amount of rotation about the center of $\mathcal{A}$. We would like to compute the integral

$$\int_{\mathcal{S}} \mathcal{R}^* \, \mathrm{d}\xi.$$

To do so it helps to first fix some values and introduce a little more notation. Let $\mu = (0, 0, l)$ with $l > 0$ be the angular momentum in the space frame, and suppose that the attitude matrix A is written as $A = \mathrm{col}(x, y, x \times y)$ with x and y being orthogonal unit vectors in $\mathbb{R}^3$. We consider two cases: 1) when $cl^2 < 2e < bl^2$ and 2) when $bl^2 < 2e < al^2$. We also need a model for the circle $\mathcal{S}$. Many choices are possible. One that facilitates the computation is obtained by the section

$$s : S^2_\mu \to j^{-1}(\mu),$$

which is defined by

$$\begin{aligned} x_1 &= \sqrt{1 - p_1^2/l^2}, & x_2 &= 0, & x_3 &= p_1/l, \\ y_1 &= -p_1 p_2 x_1/(l^2 - p_1^2), & y_2 &= l p_3 x_1/(l^2 - p_1^2), & y_3 &= p_2/l, \end{aligned}$$

see [4, p.124]. Since the bundle $\pi : j^{-1}(\mu) \to S^2_\mu$ is not trivial, s is not defined on all of S^2_μ. In fact, its domain is S^2_μ less the two points where $p_1 = \pm l$. The curve $\mathcal{S}$ is parametrized by applying the section s to a closed integral curve $\gamma : t \to p(t)$ of Euler's equations of positive period with energy $2e \neq al^2$, bl^2, or cl^2. Using $A = \mathrm{col}(x, y, x \times y)$, the definition of the Poinsot map $\mathcal{R}$ and the section s we obtain

$$\begin{aligned} (\mathcal{R} \circ s)^* \Omega_1 &= \frac{l p_1}{\sqrt{l^2 - p_1^2}} (a - 2h/l^2), \\ (\mathcal{R} \circ s)^* \Omega_2 &= \frac{p_2 p_3}{\sqrt{l^2 - p_1^2}} (b - c). \end{aligned}$$

With the orientation convention that counterclockwise is positive, pull back the angle form

$$\mathrm{d}\xi := \frac{\Omega_1 \, \mathrm{d}\Omega_2 - \Omega_2 \, \mathrm{d}\Omega_1}{\Omega_1^2 + \Omega_2^2}$$

on the annulus $\mathcal{A}$ via $\mathcal{R} \circ s$ and integrate over γ. If we are in case 1 with $cl^2 <$

$2e < bl^2$, then taking the limit as $2e \to cl^2$, gives

$$\int_{\mathcal{S}} \mathcal{R}^* \, \mathrm{d}\xi = \int_{\gamma = s^* \mathcal{S}} (\mathcal{R} \circ s)^* \, \mathrm{d}\xi = \int_0^{2\pi/n} \frac{l(a-c)(b-c)}{(a-c) - (a-b)\sin^2(nt)} \, \mathrm{d}t,$$

where $n = l\sqrt{(a-c)(b-c)}$. Note that taking the limit is justified because the above integral is a locally constant function of e. This integral is simplified by the substitutions $u = nt$ and $\sin\theta = \sqrt{(b-c)/(a-c)}$, which yield

$$\int_0^{2\pi} \frac{\sin\theta}{1 - \cos^2\theta \sin^2 u} \, \mathrm{d}u.$$

A routine calculation with residue calculus shows that the above integral is 2π whenever $0 < \theta < \pi/2$.

In case 2 when $bl^2 < 2e < al^2$, we take the limit as $2e \to al^2$. After the change of variable $u = nt$ we are left with the problem of evaluating

$$l^{-1}\sqrt{(a-c)(b-c)} \int_0^{2\pi} \frac{1 - 2\sin^2 u}{(\sin^2 u - \sec^2\theta)(\sin^2 u + \tan^2\theta)} \, \mathrm{d}u.$$

A residue calculation yields that the integral equals zero when $0 < \theta < \pi/2$.

The herpolhode angle and the rotation number

Recall (see [4, p.126]) that the rotation number of the flow of the Hamiltonian vector field X_h on $T^2_{e,\mu}$ is $\Delta\theta(\tau)/2\pi$, where τ is the period of a solution γ of Euler's equations, which is obtained by applying the bundle map π to an integral curve Γ of X_h on $T^2_{e,\mu}$. Moreover,

$$\Delta\theta(t) = \theta(t) - \theta(0) = \int_0^t \beta(p(u)) \, \mathrm{d}u$$

with

$$\beta(p) = \frac{l(b\,p_2^2 + c\,p_3^2)}{l^2 - p_1^2}.$$

Consider the affine frame on $T^2_{e,\mu}$ constructed as follows. First, lift the reduced Hamiltonian vector field $X_{\overline{h}}$ to the section s to get $s_* X_{\overline{h}}$. Push this vector field around by the action of $\mathrm{SO}(3,\mathbb{R})_\mu$ (whose integral manifolds are the fibres of π) to get an $\mathrm{SO}(3,\mathbb{R})_\mu$-invariant vector field Y on the entire torus $T^2_{e,\mu}$. Let Z denote the infinitesimal generator of the $\mathrm{SO}(3,\mathbb{R})_\mu$ action. The pair of vector fields $\{Y, Z\}$ define an affine framing of the torus $T^2_{e,\mu}$ because $\pi_* Z \equiv 0$ while $\pi_* Y = X_{\overline{h}} \neq 0$. Note that contrary to the case of action-angle variables, X_h is *not* a constant linear combination of Y and Z. In fact, by construction of the rotation number $X_h = Y + \beta Z$. We have $[Y, Z] = [Z, X_h] = 0$, but $[X_h, Y] \neq 0$.

We are now in a position to describe the relation between the herpolhode angle and the rotation number. Since the Poinsot map is $\mathrm{SO}(3,\mathbb{R})_\mu$ equivariant, the relation

$$\mathcal{R}_* Z = \partial_\xi \tag{0.2}$$

holds. To see this recall that $t \mapsto ((\exp tZ)A, p)$ is the flow of the Hamiltonian vector field X_{j^Z} whose integral curves on $T^2_{e,\mu}$ give its rulings. From the definition of the Poinsot map (0.1), for $(A, p) \in T^2_{e,\mu}$, we have

$$\mathcal{R}((\exp tZ)A, p) = (\exp tZ)AI^{-1}p \;=\; (\exp tZ)\mathcal{R}(A, p). \tag{0.3}$$

The infinitesimal version of this is (0.2). (See the appendix for a second proof of (0.2)).

The evolution of the herpolhode angle is

$$\dot{\xi} = \langle \mathrm{d}\xi, \mathcal{R}_* X_h \rangle = \langle \mathrm{d}\xi, \mathcal{R}_*(Y + \beta Z)\rangle = \langle \mathrm{d}\xi, \mathcal{R}_* Y + \beta \partial_\xi \rangle.$$

Hence after one period τ of the solution of Euler equations

$$\begin{aligned}
\Delta\xi &= \xi(\tau) - \xi(0), \\
&= \int_0^\tau \langle \mathrm{d}\xi, \mathcal{R}_* Y\rangle \,\mathrm{d}t + \int_0^\tau \beta(t)\,\mathrm{d}t, \\
&= \int_{s^*\mathcal{S}} (\mathcal{R}\circ s)^*\,\mathrm{d}\xi + \int_0^\tau \beta(t)\,\mathrm{d}t \qquad (0.4)\\
&= \begin{cases} 0, & \text{if } bl^2 < 2e < al^2 \\ 2\pi, & \text{if } cl^2 < 2e < bl^2 \end{cases} + \Delta\theta(\tau). \qquad (0.5)
\end{aligned}$$

where the integral curve of X_h on $T^2_{e,\mu}$ projects to a closed integral curve of Euler's equations on S^2_μ of energy e and $|\mu| = l$.

Relation with Levi-Montgomery formula

In this section we connect our result (0.5) with the formula of Levi-Montgomery [6, 7]. From the integration of the Euler-Arnol'd equations [4, p.123–126] we know that the function β in (0.4) is

$$\beta(t) = \frac{l(b\,p_2^2 + c\,p_3^2)}{l^2 - p_1^2} \;=\; \frac{2e}{l} + \frac{2e - al^2}{l}\,\frac{p_1^2}{l^2 - p_1^2},$$

since $2e = a\,p_1^2 + b\,p_2^2 + c\,p_3^2$. Hence

$$\Delta\theta = \theta(\tau) - \theta(0) \;=\; \int_0^\tau \beta(t)\,\mathrm{d}t \;=\; \frac{2e}{l}\,\tau - \frac{al^2 - 2e}{l}\int_0^\tau \frac{p_1^2}{l^2 - p_1^2}\,\mathrm{d}t.$$

This splits the rotation number $\Delta\theta$ into two parts: the "dynamic phase " and the "geometric phase".

We now identify the geometric phase as the area enclosed by an integral curve Γ of Euler's equation with energy e and magnitude of angular momentum l, which has been normalized to lie on the unit sphere and bounds a disk D. To compute this area we project the oriented area 2-form σ on the unit sphere to the p_2/l–p_3/l-plane (viewing the eastern hemisphere as a graph) and observe that

$$\sigma = \mathrm{d}\Big(-p_1 \left[\frac{p_2 \,\mathrm{d}p_3 - p_3 \,\mathrm{d}p_2}{p_3^2 + p_3^2} \right] \Big).$$

To determine the *unoriented area* $|A|$ we have to compute the integral

$$|A| = \int_D |\sigma| \;=\; \frac{1}{l} \int_0^{\tau} p_1 \left(\frac{p_2 \,\dot{p}_3 - p_3 \,\dot{p}_2}{p_2^2 + p_3^2} \right) \mathrm{d}t.$$

Substituting Euler's equations

$$\begin{aligned} \dot{p}_1 &= (b-c)p_2p_3 \\ \dot{p}_2 &= (c-a)p_1p_3 \\ \dot{p}_3 &= (a-b)p_1p_2, \end{aligned}$$

where $a > b > c$, yields

$$\begin{aligned} \frac{1}{l}\int_0^{\tau} p_1^2 \,\frac{[(a-b)p_2^2 + (a-c)p_3^2]}{p_2^2+p_3^2}\,\mathrm{d}t \;&=\; \frac{1}{l}\int_0^{\tau} a\,p_1^2 - \frac{1}{l}\int_0^{\tau} \frac{b\,p_1^2p_2^2 + c\,p_1^2p_3^2}{p_2^2+p_3^2}\,\mathrm{d}t \\ &= \frac{1}{l}\int_0^{\tau} \left[a\,p_1^2 + \frac{a\,p_1^4}{l^2 - p_1^2} \right] \mathrm{d}t - \frac{2e}{l}\int_0^{\tau} \frac{p_1^2}{l^2-p_1^2}\,\mathrm{d}t, \\ &\qquad \text{since } 2e = a\,p_1^2 + b\,p_2^2 + c\,p_3^2 \\ &= \frac{al^2 - 2e}{l}\int_0^{\tau} \frac{p_1^2}{l^2 - p_1^2}\,\mathrm{d}t. \end{aligned}$$

To determine the correct oriented area A note that $\sigma = -|\sigma|$. Moreover, if $al^2 > 2e > bl^2$ then the integral curve of Euler's equations in the eastern hemisphere which encloses the positive p_1 axis is traced out *clockwise* and hence has a *negative* orientation. Thus $A = |A|$. When $bl^2 > 2e > cl^2$ the integral curve of Euler's equation is traced out in a counterclockwise fashion. After a rotation which takes the positive p_3 axis to the positive p_1 axis, a calculation similar to that given above shows that $A = -|A|$. Hence we have proved

$$\Delta\theta = \frac{2e\tau}{l} - \begin{cases} A, & \text{if } al^2 > 2e > bl^2 \\ -A, & \text{if } bl^2 > 2e > cl^2, \end{cases}$$

which is exact; whereas the Levi-Montgomery formula is modulo 2π.

This sort of angle theorem has a long pedigree. Mac Cullagh gave a geometric interpretation [2] with a refinement in [3]. A splitting similar to the dynamic and geometric phase appears in Poinsot [8]. More recently, the angle theorem appears in Goodman and Robinson [5]. There is also an interesting account in Zhuravlev [10].

Appendix

We give a second proof of the equivariance of the Poinsot map $\mathcal{R}$. The proof follows from realizing that the Poinsot map may be thought of as the composition

$$(A,p) \in G \times \mathfrak{g}^* \xrightarrow{\lambda} T^*G \xrightarrow{\mathscr{L}^{-1}} TG \xrightarrow{\rho^{-1}} G \times \mathfrak{g} \to \mathfrak{g} \ni \Omega$$

where $\mathscr{L}^{-1}$ is the inverse of the Legendre transformation, ρ the right trivialization, and λ is the left trivialization. Since λ and ρ are G-equivariant by design, all that remains for us to do is establish the following lemma.

lemma. *Let the Lie group G act on a configuration space Q, with the induced actions ϕ on TQ and ϕ^* on T^*Q. Suppose that ℓ is a G-invariant Lagrangian on TQ. Then the Legendre transform*

$$\mathscr{L} : TQ \to T^*Q$$

is a G-equivariant map.

Proof. The Legendre transform is defined by

$$\left.\frac{d}{dt}\right|_{t=0} (\ell(v+tw)) = \langle \mathscr{L}(v), w \rangle$$

for all vertical vectors w. Hence if $g \in G$ we have

$$\left.\frac{d}{dt}\right|_{t=0} \ell(\phi_g(v+tw)) \quad = \quad \langle \mathscr{L}(\phi_g v), \phi_g w \rangle, = \langle \phi_g^* \mathscr{L}(\phi_g v), w \rangle,$$

and since the Lagrangian ℓ is G-invariant, it follows that $\phi_g^* \circ \mathscr{L} \circ \phi_g = \mathscr{L}$. Composing with $\phi_{g^{-1}}^*$ yields

$$\mathscr{L} \circ \phi_g = \phi_{g^{-1}}^* \circ \mathscr{L},$$

which is what we wanted to show. Note that in the case at hand there is no essential difference between considering the Lagrangian or Hamiltonian description as $\ell = \mathscr{L}^* h$.

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