GEOMETRY OF NONHOLONOMIC CONSTRAINTS

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This paper presents a Hamiltonian treatment of nonholonomically constrained mechanical systems. We assume that the total energy of the systems is a sum of kinetic and potential energies and that the constraints are linear in velocities. We prove a nonholonomic version of Noether's theorem and treat a special case of the nonholonomic reconstruction problem. The entire theory is illustrated by a disc, which rolls on a plane without slipping.

1. Phenomenological constraints

Constraints in dynamics are restrictions on the phase space of the system. They may appear for several different reasons. Phenomenological constraints are introduced instead of unknown forces to describe observed motions. For example, constraints forcing a body to move on a given surface or a no slip condition in the motion of a rolling disc are phenomenological constraints. The notion of phenomenological constraints also involves an assumption on the unknown forces, which will be discussed below.

We consider a dynamical system with configuration space Q. The constraints are given by k independent functions $f_1, \ldots, f_k$ on the tangent bundle TQ of configuration space. The motions $q(t)$ of our system are restricted by the constraint conditions

$$f_a(\dot{q}(t)) = 0 \qquad \forall\, a = 1, 2, \ldots, k\,. \tag{1}$$

For each $a = 1, \ldots, k$ let ϕ_a be a 1-form on TQ such that for every $w \in T_v(TQ)$,

$$\langle \phi_a(v) \,|\, w \rangle = \frac{d}{dt} f_a(v + t\, T\pi(w))\Big|_{t=0}\,, \tag{2}$$

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where $\pi : TQ \to Q$ is the tangent bundle projection, and $T\pi : TTQ \to TQ$ is its tangent map. The idea is that the forms only need to be compatible with the constraint functions in the base directions. We assume that the 1-forms ϕ_a are linearly independent, that is,

$$\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_k \neq 0 . \tag{3}$$

This assumption together with equation (2) seems to suggest that our considerations exclude holonomic constraints. This is not the case. Purely holonomic constraints, given by the functions $h_a : Q \to \mathbb{R}$, can also be discussed by identifying the set on which the constraint functions vanish with the set of $v \in TQ$ such that $\langle dh_a \,|\, v \rangle = 0$.

The kinetic energy of the system defines a Riemannian metric g on Q. The motion of our system can be determined if we know all the forces acting on it, assuming Newton's law. Differentiating (1) with respect to t and splitting the second derivative of $q(t)$ into its horizontal and vertical parts relative to the Levi-Civita connection corresponding to the metric g gives

$$\left\langle \phi_a(q(t)) \,\Big|\, \frac{D^2 q}{dt^2} \right\rangle = -\left\langle df_a(\dot{q}(t)) \,\Big|\, \mathrm{hor}\frac{d^2 q}{dt^2} \right\rangle = 0 \qquad \forall\, a = 1, \ldots , k , \tag{4}$$

where $\frac{D^2 q}{dt^2}$ is the acceleration of the motion. Hence, the components of the acceleration in the direction of the 1-forms ϕ_a are completely determined by the constraint equations. Thus, whatever the external forces acting on the system, the constraints provide a reaction force of the form $\lambda^a \, \phi_a(\dot{q}(t))$ (where λ^a are Lagrange multipliers and the summation convention is understood) such that the total force in the direction of the 1-forms ϕ_a gives rise to the acceleration determined from (4). The other components of the reaction force of the constraints cannot be determined from the constraint equations alone. Hence they have to be specified together with other external forces. Therefore, we assume that the total force acting of the system is of the form $F + \lambda^a \, \phi_a$, where F is a known external force. This gives rise to the equations of motion

$$\frac{D^2 q}{dt^2} = F + \lambda^a \, \phi_a , \tag{5}$$

where the Lagrange multipliers λ^a are determined from the constraint equation (2).

In all known examples in dynamics $f_1, \ldots , f_k$ are linear functions of the velocities (possibly inhomogeneous). Nonlinear constraints seem to appear in the theory of electrical circuits. Since we are studying constrained mechanical systems, we assume here that the constraint functions f_a are given by 1-forms χ_a on Q, namely

$$f_a(v) = \langle \chi_a \,|\, v \rangle \qquad \forall v \in TQ , \tag{6}$$

so we may as well assume

$$\phi_a = \pi^* \chi_a . \tag{7}$$

Moreover we assume that the known forces come from a potential, that is,

$$F = -\pi^* dV . \tag{8}$$

This allows us to rewrite the equations of motion (5) in Lagrangian form

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\dot{q}(t), q(t))\right] - \frac{\partial L}{\partial q}(\dot{q}(t), q(t)) = \lambda^a \, \phi_a(\dot{q}(t)), \tag{9}$$

where

$$L(v, q) = \tfrac{1}{2}\, g(v, v) - V(q). \tag{10}$$

2. Velocity space formulation

In [1] we use the Legendre transformation

$$\mathcal{L} : TQ \to T^*Q : (v, q) \to (p, q),$$

where

$$p = \frac{\partial L}{\partial v}(v, q), \tag{11}$$

to describe the theory of linear nonholonomic constraints in the cotangent bundle T^*Q to configuration space. In order to find the Hamiltonian of this theory we have to express (v, q) in terms of (p, q), that is, we need to find the inverse of the Legendre transformation. For the mechanical systems under consideration the Legendre tranformation is a linear map in velocities depending on parameters. For a nonholonomic system with a large number of degrees of freedom, the computation of its inverse can be quite cumbersome. To avoid this, one can work directly on TQ using the Legendre transformation to pull back the canonical symplectic form ω_0 on the cotangent bundle. In this way we obtain a Hamiltonian description of the nonholonomic system on $P = TQ$ with symplectic form $\omega = \mathcal{L}^*\omega_0$. It should be noted that the content of the theory is independent of the choice of description. On the other hand, the tangent bundle approach simplifies the transition to situations where the Legendre transformation is degenerate. On (P, ω) introduce the energy function

$$E = \tfrac{1}{2}\, g(v, v) + V(q). \tag{12}$$

We can write the Lagrangian equations of motion (3) in Hamiltonian form as

$$\xi \mathbin{\lrcorner} \omega = dE + \lambda^a \pi^* \phi_a. \tag{13}$$

Here ξ is a vector field on P which is tangent to the motions and $\pi : P \to Q$ is the tangent bundle projection. As in [1] we introduce the constraint manifold

$$M = \{(q, v) \in P \,|\, \langle \phi_a(q) \,|\, v\rangle = 0 \quad \forall a = 1, 2, \ldots, k\} \tag{14}$$

and the distribution

$$H = \{u \in TM \,|\, \langle \pi^*\phi_a \,|\, u\rangle = 0 \quad \forall a = 1, \ldots, k\}. \tag{15}$$

The following theorem gives the basic results of the Hamiltonian formulation of nonholonomically constrained systems.

THEOREM 1. [1]
1. *The restriction ω_H of ω to H is nondegenerate. Moreover, at points of M*

$$TP|M = H \oplus H_\omega \,. \tag{16}$$

Here H_ω denotes the ω-symplectic perpendicular of H and $TP|M$ is the space of tangent vectors to P whose base point lies in M.

2. *The motion of the nonholonomic system (M, H, ω_H, E) takes place on the constraint manifold M. The vector field ξ describing this motion lies in H.*

3. *The restriction of (13) to H gives*

$$\xi \mathbin{\lrcorner} \omega_H = d_H E \,. \tag{17}$$

Here $d_H E$ denotes the restriction of dE to vectors in H. Equation (17) uniquely determines the vector field ξ, since $\xi \in H$.

4. *The Lagrange multipliers λ^a are uniquely determined by the restriction of (13) to H_ω.*

3. Symmetries and conservation laws

In the absence of nonholonomic constraints there is a correspondence between symmetries and conservation laws given by the first Noether theorem [2]. In the presence of nonholonomic constraints the relationship between symmetries and conservation laws seems to be much more obscure. In [1] we have found conserved quantities generating canonical transformations which do not preserve the constraints as well as symmetries not giving rise to conserved quantities. In several examples of completely integrable systems with nonholonomic constraints there are constants of motion which appear as constants of integration of a system of linear differential equations (see §6 equation (36)). There is no apparent relation between these constants and any obvious symmetry, despite the fact that they are linear in momenta. An attempt to understand constants of motion in a nonholonomic system has led us to the following nonholonomic Noether theorem.

For every function f on (P, ω) we denote by ξ_f the Hamiltonian vector field of f, namely,

$$\xi_f \mathbin{\lrcorner} \omega = df \,. \tag{18}$$

Using (16) the restriction of ξ_f to M can be decomposed into components ξ_f^H in H and $\xi_f^{H_\omega}$ in H_ω, namely,

$$\xi_f = \xi_f^H + \xi_f^{H_\omega} . \tag{19}$$

THEOREM 2. *A function f on P is a constant of motion if and only if ξ_f^H preserves the energy function,*

$$L_{\xi_f^H} E = 0 \,. \tag{20}$$

Proof: Since H is symplectic and ξ lies in H, it follows that $\xi_f^{H_\omega} \lrcorner (\xi \lrcorner \omega) = 0$. Hence

$$\begin{aligned} L_\xi f &= \xi \lrcorner df = \xi \lrcorner (\xi_f \lrcorner \omega) = -\xi_f \lrcorner (\xi \lrcorner \omega) \\ &= -(\xi_f^H + \xi_f^{H_\omega}) \lrcorner (\xi \lrcorner \omega) = -\xi_f^H \lrcorner (\xi \lrcorner \omega) \\ &= -\xi_f^H \lrcorner (dE + \lambda^a \pi^* \phi_a) = -\xi_f^H \lrcorner dE = -L_{\xi_f^H} E . \end{aligned}$$

Thus $L_\xi f = 0$ if and only if $L_{\xi_f^H} E = 0$. □

4. Reduction of symmetries

The problem of reduction of symmetries of nonholonomic systems has been studied in [1]. Here we summarize the main results.

Let G be the symmetry group of the theory, that is, G is a Lie group which preserves the symplectic form ω, the energy function E, the constraint manifold M and the distribution H. Assume that the space $\overline{M} = M/G$ of G-orbits on M is a manifold with projection map $\rho : M \to \overline{M}$. Performing reduction amounts to pushing forward all the structures on M to the orbit space $\overline{M}$ via the mapping ρ. The nondegenerate 2-form ω_H does not push forward under ρ unless it annihilates all vectors tangent to the fibers of ρ.

Therefore let us introduce the following distributions:

$$\begin{aligned} V &= \ker T\rho , \\ U &= \{u \in H \,|\, \omega_H(u, v) = 0 \ \forall v \in V \cap H\} , \end{aligned} \tag{21}$$

and $\overline{H} = T\rho(U)$.

THEOREM 3. [1]

1. *The vector field ξ describing the motion lies in the distribution U and projects to a vector field $\overline{\xi}$ which lies in $\overline{H}$.*
2. *The 2-form ω_U given by restricting ω_H to U pushes forward under the map ρ to a nondegenerate 2-form $\omega_{\overline{H}}$ on $\overline{H}$.*
3. *The energy function E pushes forward under ρ to a function $\overline{E}$ on $\overline{M}$ such that*

$$\overline{\xi} \lrcorner \omega_{\overline{H}} = d_{\overline{H}} \overline{E} . \tag{22}$$

In Section 2 we have described a mechanical system with nonholonomic constraints in terms of a quadruple (M, H, ω_H, E) with distributional Hamiltonian equations of motion (17). Reduction of symmetry leads to an analogous quadruple $(\overline{M}, \overline{H}, \omega_{\overline{H}}, \overline{E})$ with a smaller number of degress of freedom. Moreover, this nonholonomic reduction preserves the form of the equations of motion, see (22).

5. Reconstruction

One of the main uses of reduction of symmetries is to find solutions of the equations of motion. Since the reduced equations have fewer dependent variables, the hope is that they may be easier to solve than the original equations. Once solutions of the reduced equations are known, it remains to lift them to solutions of the original equations. This lifting problem is called the problem of reconstruction. In order to discuss reconstruction let us introduce the distributions $K = U \cap V$ and $W_{\omega_H} = \{u \in H \mid \omega_H(u,w) = 0 \ \forall w \in W\}$. In other words, W_{ω_H} is the ω_H-perpendicular of W in H. In proposition 1 below we give the symplectic relations linking the distributions K, U and U_{ω_H}.

PROPOSITION 1.

1. *The distribution K is ω_H-isotropic and G-invariant.*
2. *$K \subseteq \ker T\rho$. For each $p \in M$ the map $T_p\rho : U_p/K_p \to \overline{H}_{\rho(p)}$ is an isomorphism.*
3. *$(U + U_{\omega_H})_{\omega_H} = U_{\omega_H} \cap U = K$. Moreover, $(U + U_{\omega_H})/K$ is ω_H-symplectic and $K \subseteq U \subseteq (U + U_{\omega_H}) \subseteq H$. If $K = U$ then U is ω_H-isotropic, that is, $U \subseteq U_{\omega_H}$. If $U = (U + U_{\omega_H})$, then U is ω_H-coisotropic, that is, $U_{\omega_H} \subseteq U$.*

Proof:

1. Since $U \subseteq H$, it follows that $K = U \cap V \cap H = (V \cap H)_{\omega_H} \cap (V \cap H)$. Hence K is ω_H-isotropic. K is also G-invariant since it is defined in terms of V, H and ω_H which are G-invariant.
2. $K \subseteq \ker T\rho = V$ by definition. Since $\overline{H} = T\rho(U)$, it follows that the mapping $T_p\rho : U_p/(U_p \cap \ker T\rho) \to \overline{H}_{\rho(p)}$ is an isomorphism for every $p \in M$. But $U_p \cap \ker T\rho = U_p \cap V_p = K_p$. Hence $T_p\rho : U_p/K_p \to \overline{H}_{\rho(p)}$ is an isomorphism.
3. $K = (V \cap H)_{\omega_H} \cap (V \cap H) = U \cap U_{\omega_H} = (U + U_{\omega_H})_{\omega_H}$. Hence $(U + U_{\omega_H})/K$ is ω_H-symplectic.
4. That $K \subseteq U \subseteq (U + U_{\omega_H}) \subseteq H$ is obvious. Also $U = K = U \cap U_{\omega_H}$ implies that $U \subseteq U_{\omega_H}$, that is, U is ω_H-isotropic. If $U = (U + U_{\omega_H})$, then $U_{\omega_H} \subseteq U$, that is, U is ω_H-coisotropic. □

Suppose that we have a solution curve $\overline{c} : [t_0, t_1) \to \overline{M}$ of the reduced equations (22), that is,

$$\frac{d\overline{c}}{dt}(t) = \overline{\xi}(\overline{c}(t)) \qquad \forall t \in [t_0, t_1) . \tag{23}$$

Given $p \in \rho^{-1}(\overline{c}(t_0))$ we want to lift $\overline{c}$ to a curve $c : [t_0, t_1) \to M$ through p, that is, $c(t_0) = p$ and

$$\frac{dc}{dt}(t) = \xi(c(t)) \qquad \forall t \in [t_0, t_1) , \tag{24}$$

where ξ satisfies (17). We shall discuss this problem under the following

Hypotheses.

1. $\rho : M \to \overline{M}$ is (locally) a left principal G bundle.
2. K is an involutive distribution on M.

Let us first consider the special case when $V \cap H = \{0\}$. Then $U = H$. The distribution U can be interperted as a partial connection for the bundle $\rho : M \to \overline{M}$ covering $\overline{H}$. In this case the curve c is the horizontal lift of the curve $\overline{c}$ through p. Thus the reconstruction problem is equivalent to that of finding a horizontal lift of a curve in $\overline{H}$.

Suppose now that $V \cap H \neq \{0\}$. In order to find the required lift c of $\bar{c}$ let us first choose any lift $\widetilde{c} : [t_0, t_1) \to M$ of $\bar{c}$ through p, that is, $\widetilde{c}(t_0) = p$ and the tangent vector to $\widetilde{c}$ is contained in U,

$$\frac{d\widetilde{c}}{dt}(t) \in H \qquad \forall t \in [t_0, t_1) . \tag{25}$$

Along $\rho^{-1}(\bar{c}([t_0, t_1))$ choose a G-invariant complement B to K in U, that is,

$$(B \oplus K)|\rho^{-1}(\bar{c}([t_0, t_1)) = U|\rho^{-1}(\bar{c}([t_0, t_1)) \tag{26}$$

and

$$\frac{d\widetilde{c}}{dt}(t) \in B \qquad \forall t \in [t_0, t_1) . \tag{27}$$

Such a choice is always possible for t_1 sufficiently close to t_0. We also choose a G-invariant complement D of K in $V \cap H$, that is,

$$(D \oplus K)|\rho^{-1}(\bar{c}([t_0, t_1)) = V \cap H|\rho^{-1}(\bar{c}([t_0, t_1)) . \tag{28}$$

Since D and B are complementary to K in $V \cap H$ and U respectively and $K = V \cap H \cap U$, it follows that $D \cap B = \{0\}$. The idea here is that B is a model for $\overline{H}$ and D is a model for the "symplectic piece" of V.

PROPOSITION 2. *D, B and $D \oplus B$ are ω_H-symplectic.*

Proof: Since $\overline{H}_{\rho(p)}$ is isomorphic to U_p/K_p for every $p \in M$, it follows that B is ω_H-symplectic. Similarly D is ω_H-symplectic since $K = (V \cap H) \cap (V \cap H)_{\omega_H}$ and D_p is isomorphic to $(V_p \cap H_p)/K_p = (V_p \cap H_p)/\Big((V_p \cap H_p) \cap (V_p \cap H_p)_{\omega_H}\Big)$.

To show that $D \oplus B$ is ω_H-symplectic, suppose that $x \in D$ and $z \in B$ are vectors such that $x + z \in (D \oplus B)_{\omega_H} \cap (D \oplus B)$. Thus for every $u \in D$ and $v \in B$, $\omega_H(u+v, x+z) = 0$. Since $D \subseteq V \cap H$, $U_{\omega_H} \subseteq B$ and $B \subseteq D_{\omega_H}$, it follows that $\omega_H(u, z) = \omega_H(v, x) = 0$. Hence $\omega_H(u, x) + \omega_H(v, z) = 0$ for every $u \in D$ and every $v \in B$. Hence $x = z = 0$ because D and B are ω_H-symplectic. □

Let C be a complementary space to the ω_H-perpendicular of $D \oplus B$ in H, that is, $C = (D \oplus B)_{\omega_H}$. Clearly C is ω_H-symplectic and complementary to $D \oplus B$ in H. Thus we get a G-invariant decomposition

$$H|\rho^{-1}(\bar{c}([t_0, t_1)) = B \oplus C \oplus D . \tag{29}$$

Moreover $K|\rho^{-1}(\bar{c}([t_0, t_1))$ is a maximal ω_H-isotropic subspace of C. The reader will notice that the above construction of B, C, and D is just a distributional, G-invariant version of the familiar Witt decomposition from symplectic linear algebra.

By hypothesis K is involutive. Let θ_i $i = 1, \dots k$ be local coordinates for a typical integral manifold of K which have be extended to a local coordinate system for M. This amounts to giving a local product structure $N \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ for an open set N in M. Here m is the dimension of the manifold M. Let ζ^a be a basis of the Lie algebra of G and let ζ^a_M be vector fields on M corresponding to the action of the one-parameter subgroup generated by ζ^a. The vector fields $(\partial_{\theta_1}, \dots, \partial_{\theta_k})$ span K and can be expressed as linear combinations of the vector fields ζ^a_M, namely, $\partial_{\theta_j} = f_{ja}\zeta^a_M$. In $N \cap \rho^{-1}(\overline{c}([t_0, t_1))$ we can extend the vector fields $(\partial_{\theta_1}, \dots, \partial_{\theta_k})$ to a ω_H-symplectic basis $(\partial_{\theta_1}, \dots, \partial_{\theta_k}, \eta_1, \dots, \eta_k)$ of C.

Using the decomposition of H given by (29) we can split ω_H into components ω_B, ω_C and ω_D along B, C and D, that is, $\omega_H = \omega_B + \omega_C + \omega_D$. Since the vector field ξ, which determines the motion of the system, lies in H it can also be split into components ξ_B, ξ_C and ξ_D along B, C, and D, namely, $\xi = \xi_B + \xi_C + \xi_D$. Since ξ lies in U, it follows that $\xi_D = 0$.

The solution curve c which we want is contained in $\rho^{-1}(\overline{c}([t_0, t_1))$. Hence we can decompose equation (17) into its B, C and D components obtaining

$$\xi_B \lrcorner\, \omega_B = d_B E \,, \tag{30}$$

$$\xi_C \lrcorner\, \omega_C = d_C E \,, \tag{31}$$

$$\xi_D \lrcorner\, \omega_D = d_D E \,. \tag{32}$$

The right hand side of the above equations denotes the restriction of dE to the distributions B, C, and D, respectively. Since the orginal curve $\overline{c}$ satisfies the reduced equations (22), equations (23), (27) and (29) imply that $\xi_B(\widetilde{c}(t)) = \frac{d\widetilde{c}}{dt}(t)$ satisfies equation (30). Both sides of equation (32) are identically zero, because $\xi_D = 0$ and E is G-invariant. Thus we need only to satisfy equation (31), which we call the *reconstruction equation.* We can express ξ_C in terms of the symplectic basis $(\partial_{\theta_1}, \dots, \partial_{\theta_k}, \eta_1, \dots, \eta_k)$ of C. Since ξ, B and D lie in TM and the vectors $\eta_1, \dots, \eta_k$ do not, it follows that ξ_C is a linear combination of the vectors $(\partial_{\theta_1}, \dots, \partial_{\theta_k})$ spanning K. In other words

$$\xi_C = \dot{\theta}_1 \partial_{\theta_1} + \dots + \dot{\theta}_k \partial_{\theta_k} \,, \tag{33}$$

where the coefficients $\dot{\theta}_i$ are the time derivatives of the coordinates θ_i. Evaluating the reconstruction equation (31) on the basis $(\partial_{\theta_1}, \dots, \partial_{\theta_k}, \eta_1, \dots, \eta_k)$ of C we obtain

$$\dot{\theta}_i = \langle dE, \eta_i \rangle \tag{34}$$

for $i = 1, \dots, k$.

THEOREM 5. *$\langle dE, \eta_i \rangle$ are independent of $\theta_1, \dots, \theta_k$ for every $i = 1, \dots, k$.*

Proof: We compute

$$\begin{aligned} \frac{d}{d\theta_j}\langle dE, \eta_i \rangle &= L_{\partial_{\theta_j}} \langle dE, \eta_i \rangle = \langle L_{\partial_{\theta_j}} dE, \eta_i \rangle + \langle dE, L_{\partial_{\theta_j}} \eta_i \rangle \\ &= \langle d(\partial_{\theta_j} \lrcorner\, dE), \eta_i \rangle + \langle dE, [\partial_{\theta_j}, \eta_i] \rangle \\ &= L_{\eta_i}(L_{f_{ja}\zeta^a_M} E) + L_{[f_{ja}\zeta^a_M, \eta_i]} E \\ &= 2\, L_{\eta_i}(L_{f_{ja}\zeta^a_M} E) - L_{f_{ja}\zeta^a_M}(L_{\eta_i} E) \\ &= 0 \,. \end{aligned}$$

The last equality above is a consequence of the following argument. Since E is G-invariant, $L_{f_{ja}\zeta^a_M} E = 0$. Because η_i is also G-invariant, $L_{\eta_i} E$ is G-invariant. Hence $L_{f_{ja}\zeta^a_M}(L_{\eta_i} E) = 0$. □

Hence the reconstruction equation (34) decouples and may be solved by quadrature. We note that in all cases which we are aware of, the distribution K is involutive. For many classical examples, K is either zero or one-dimensional, so the hypothesis is vacuous.

6. The rolling disc

We illustrate the above theory by discussing the problem of a rolling disc. Consider a disc of radius r and uniformly distributed mass m rolling without slipping on a horizontal plane under the influence of a uniform downward vertical gravitational field of strength g. The configuration space of the disc is the open subset Q of $\mathbb{R}^2 \times SO(3)$ defined by the inequality $0 < \theta < \pi$, where θ is the angle between the horizontal plane and the plane of the disc. The presentation here follows Kemppainen [3]. Let (x, y) be rectangular coordinates in the horizontal plane and let (θ, φ, ψ) be Euler angles parameterizing $SO(3)$ (see Goldstein [4]). The Lagrangian of the rolling disc is

$$\begin{aligned} L &= \tfrac{1}{2}\, m\, (v_x^2 + v_y^2 + r^2\, v_\theta^2 \cos^2\theta) + \tfrac{1}{2}\, A\, (v_\theta^2 + v_\varphi^2 \sin^2\theta) \\ &\quad + \tfrac{1}{2}\, C\, (v_\psi + v_\varphi \cos\theta)^2 - mgr \sin\theta\,, \end{aligned}$$

where A and C are the moments of inertia of the disc relative to an axis in the plane of the disc and an axis perpendicular to the disc, respectively.

The constraint functions are

$$\begin{aligned} f_1 &= v_x \cos\varphi + v_y \sin\varphi - r\, v_\theta \sin\theta\,, \\ f_2 &= -v_x \sin\varphi + v_y \cos\varphi + r\, v_\varphi \cos\theta + r\, v_\psi\,. \end{aligned}$$

The constraint manifold M is defined by $\{f_1 = 0 \;\&\; f_2 = 0\}$. Since on M we express the velocities v_x and v_y in terms of the remaining variables, it follows that M is parameterized by $(x, y, \theta, \varphi, \psi, v_\theta, v_\varphi, v_\psi)$. The constraint 1-forms φ_a are given by

$$\begin{aligned} \varphi_1 &= \cos\varphi\, dx + \sin\varphi\, dy - r \sin\theta\, d\theta\,, \\ \varphi_2 &= -\sin\varphi\, dx + \cos\varphi\, dy + r \cos\theta\, d\varphi + r\, d\psi\,. \end{aligned}$$

The distribution H on M is spanned by the vector fields

$$\begin{aligned} \zeta_\theta &= r \sin\theta \cos\varphi\, \partial_x + r \sin\theta \sin\varphi\, \partial_y + \partial_\theta\,, \\ \zeta_\varphi &= r \cos\theta \sin\varphi\, \partial_x - r \cos\theta \cos\varphi\, \partial_y + \partial_\varphi\,, \\ \zeta_\psi &= r \sin\varphi\, \partial_x - r \cos\varphi\, \partial_y + \partial_\psi\,, \end{aligned}$$

and ∂_{v_θ}, ∂_{v_φ}, ∂_{v_ψ}. A calculation shows that H is not integrable and that the smallest integrable distribution containing H is the tangent bundle TM of M.

The rolling disc is invariant under the action of the group $G = E(2) \times SO(2)$, where $E(2)$ is the group of Euclidean motions on $\mathbb{R}^2$ and $SO(2)$ acts on the plane of the disc by rotations. The distribution V tangent to the orbits of G on M is spanned by the vector fields ∂_x, ∂_y, ∂_φ and ∂_ψ. The space $\overline{M}$ of G orbits on M is parameterized by $(\theta, v_\theta, v_\varphi, v_\psi)$ and the orbit mapping is

$$\rho : M \to \overline{M} : (x, y, \theta, \varphi, \psi, v_\theta, v_\varphi, v_\psi) \to (\theta, v_\theta, v_\varphi, v_\psi)\,.$$

Because the energy function E is invariant under the action of G, we find that the reduced energy function is

$$\overline{E} = \frac{1}{2(A+mr^2)}\rho_\theta^2 + \frac{1}{2(C+mr^2)}\rho_\psi^2 + \frac{1}{2A\sin^2\theta}(\rho_\varphi - \rho_\psi\,\cos\theta)^2 + mgr\,\sin\theta\,,$$

where

$$\begin{aligned}
\rho_\theta &= (A+mr^2)v_\theta\,,\\
\rho_\varphi &= (C+mr^2)(v_\psi + v_\varphi\,\cos\theta)\cos\theta + A\,v_\varphi\sin^2\theta\,,\\
\rho_\psi &= (C+mr^2)(v_\psi + v_\varphi\cos\theta)\,.
\end{aligned}$$

The intersection of V with H is a distribution spanned by the vector fields ζ_φ and ζ_ψ. The distribution U defined in (21) projects onto $T\overline{M}$. Hence $\overline{H} = T\overline{M}$. Thus $\omega_{\overline{H}}$ on $\overline{H}$ is a 2-form on $\overline{M}$, which we denote by $\overline{\omega}$. A calculation shows that

$$\overline{\omega} = d\theta \wedge d\rho_\theta - \frac{A+mr^2}{mr^2\,\rho_\theta\,\sin\theta}\,d\rho_\varphi \wedge d\rho_\psi\,.$$

Note that $\overline{\omega}$ is almost symplectic, that is, it is nondegenerate but *not* closed. The reduced equations of motion are given by

$$\overline{\xi} \,\lrcorner\, \overline{\omega} = d\overline{E}\,.$$

Setting $\overline{\xi} = \dot{\theta}\,\partial_\theta + \dot{\rho_\theta}\partial_{\rho_\theta} + \dot{\rho_\varphi}\partial_{\rho_\varphi} + \dot{\rho_\psi}\partial_{\rho_\psi}$, we obtain

$$\begin{aligned}
\dot{\theta} &= \frac{1}{A+mr^2}\,\rho_\theta\,,\\
\dot{\rho_\theta} &= \frac{\cos\theta}{A\sin^3\theta}(\rho_\varphi - \rho_\psi\,\cos\theta)^2 - \frac{1}{A\,\sin\theta}\rho_\psi(\rho_\varphi - \rho_\psi\,\cos\theta) - mgr\,\cos\theta\,,\\
\dot{\rho_\varphi} &= \frac{mr^2\,\rho_\theta}{A(A+mr^2)}\left[\cot\theta(\rho_\varphi - \rho_\psi\,\cos\theta) - \frac{A}{C+mr^2}\,\sin\theta\,\rho_\psi\right],\\
\dot{\rho_\psi} &= \frac{mr^2\,\rho_\theta}{A(A+mr^2)}\,\csc\theta(\rho_\varphi - \rho_\psi\,\cos\theta)\,.
\end{aligned}$$

Using θ as an independent variable we obtain

$$\begin{aligned}
\frac{d\rho_\varphi}{d\theta} &= \frac{mr^2}{A}\cot\theta\,(\rho_\varphi - \rho_\psi\cos\theta) - \frac{mr^2\sin\theta}{C+mr^2}\,\rho_\psi\,,\\
\frac{d\rho_\psi}{d\theta} &= \frac{mr^2}{A}\csc\theta\,(\rho_\varphi - \rho_\psi\cos\theta)\,.
\end{aligned} \tag{35}$$

Differentiating the second equation in (35) and substituting both equations into the equations of motion gives the second order linear equation

$$\frac{d^2\rho_\psi}{d\theta^2} + \cot\theta \frac{d\rho_\psi}{d\theta} - \frac{Cmr^2}{A(C+mr^2)}\rho_\psi = 0\,,$$

which is changed into the standard form of a hypergeometric equation by the substitution $z = \cos^2\theta$, namely

$$z(1-z)\frac{d^2\rho_\psi}{dz^2} + \left(\frac{1-3z}{2}\right)\frac{d\rho_\psi}{dz} - \frac{Cmr^2}{4A(C+mr^2)}\rho_\psi = 0\,. \tag{36}$$

The solution $p_\psi(\theta)$ of (36) is a linear combination of two hypergeometric functions. The coefficients of these linear combinations are the constants of motion refered to in section 3. The function $\rho_\varphi(\theta)$ is found by differentiating $\rho_\psi(\theta)$ because

$$\rho_\varphi = \frac{A}{mr^2}\sin\theta\,\frac{d\rho_\psi}{d\theta} + \rho_\psi\cos\theta\,. \tag{37}$$

It remains to find the function $\theta(t)$. We use the fact that $\rho_\theta = (A+mr^2)\dot\theta$ and that the reduced energy $\overline{E}$ is conserved to obtain a first order differential equation for $\theta(t)$, which we then integrate.

Having solved the reduced dynamics on $\overline{M}$ (at least up to quadrature), it remains to find the motion on M. Compare the discussion in [5]. We know that the original dynamics is tangent to the distribution U, that is $\xi(m) \in U_m$ for every $m \in M$. Moreover we know that $\dim U = \dim \overline{M}$ and $U \cap V = \{0\}$. Thus the distribution U may be viewed as a connection on the principal G-bundle $\rho : M \to \overline{M}$. The problem of reconstructing the dynamics is precisely the problem of finding the horizontal lift of a parameterized curve in $\overline{M}$. First, change variables back to the velocity variables. Given that we are writing $\rho : M \to \overline{M}$ as a trivial bundle with fiber coordinates $\{x, y, \varphi, \psi\}$ and that we have expressed U as $\operatorname{span}\{\eta_\theta, \eta_\varphi, \eta_\psi, \partial_{v_\theta}\}$, where the vector fields η_θ, η_φ, and η_ψ are given by

$$\begin{aligned}
\eta_\theta &= r\sin\theta\cos\varphi\,\partial_x + r\sin\theta\sin\varphi\,\partial_y + \partial_\theta + \dots\\
\eta_\varphi &= r\cos\theta\sin\varphi\,\partial_x - r\cos\theta\cos\varphi\,\partial_y + \partial_\varphi + \dots\\
\eta_\psi &= r\sin\varphi\,\partial_x - r\cos\varphi\,\partial_y + \partial_\psi + \dots
\end{aligned} \tag{38}$$

and the omitted terms involve only the variables $\theta, v_\theta, v_\varphi, v_\psi$. Since $\xi(m) \in U_m$ we have

$$\xi = \dot\theta\,\eta_\theta + \dot\varphi\,\eta_\varphi + \dot\psi\,\eta_\psi + \dot v_\theta\,\partial_{v_\theta}\,.$$

If $\bar c(t)$ is an integral curve of $\bar\xi$ in $\overline{M}$ then the lifting problem is solved once we compute the integral given by the separable ordinary differential equations

$$\begin{aligned}
\dot\psi &= \langle d\psi, \xi\rangle\,,\\
\dot\varphi &= \langle d\varphi, \xi\rangle\,,
\end{aligned}$$

and then the integrals given by the separable equations

$$\begin{aligned}
\dot{x} &= \langle dx, \xi \rangle = \langle dx, \dot{\theta}\,\eta_\theta + \ldots \rangle \\
&= r \sin\theta \cos\varphi\, \dot{\theta} + r \cos\theta \sin\varphi\, \dot{\varphi} + r \sin\varphi\, \dot{\psi}\,, \\
\dot{y} &= \langle dy, \xi \rangle \\
&= r \sin\theta \sin\varphi\, \dot{\theta} - r \cos\theta \cos\varphi\, \dot{\varphi} - r \cos\varphi\, \dot{\psi}\,.
\end{aligned}$$

Finally, we observe that the order in which the above integrations are performed reflects the solvability of the group G.

REFERENCES

[1] L. Bates and J. Śniatycki: *Rep. Math. Phys.* **32** (1993), 99.
[2] R. Abraham and J. Marsden: *Foundations of Mechanics*, second edition, Addison-Wesley, Reading, MA 1978.
[3] D. Kemppainen: *The geometry of the rolling disk*, Masters thesis, University of Calgary, 1993.
[4] H. Goldstein: *Classical Mechanics*, Addison-Wesley, Reading, MA 1950.
[5] R. Rosenberg: *Analytical Dynamics*, Plenum Press, New York 1977.