WHAT IS A COMPLETELY INTEGRABLE NONHOLONOMIC DYNAMICAL SYSTEM?*

LARRY BATES

Department of Mathematics and Statistics, University of Calgary.
2500 University Dr. N. W.. Calgary, Alberta T2N 1N4, Canada
(e-mail: bates@math.ucalgary.ca)

RICHARD CUSHMAN

Mathematics Institute, Budapestlaan 6, University of Utrecht,
3508TA Utrecht, the Netherlands
(e-mail: cushman@math.uu.nl)

(*Received September 22, 1998*)

We compare the geometry of a toral fibration defined by the common level sets of the integrals of a Liouville integrable Hamiltonian system with a toral fibration coming from a completely integrable nonholonomic system. We illustrate their differences using the following examples: the nonholonomic oscillator. Chaplygin's skate, Routh's sphere and the rolling oblate ellipsoid of revolution.

1. Review of Liouville integrable systems

At the most elementary level, a completely integrable Hamiltonian system is one in which one can integrate the equations of motion, say, by quadrature. In the most famous case of integrability, due to Liouville, for an n-degree of freedom Hamiltonian system, we need to find n functions $f_1, \ldots, f_n$ that

(i) Poisson commute, that is, the Poisson brackets $\{f_i, f_j\} = 0$;

(ii) are typically independent, that is, $df_1 \wedge \ldots \wedge df_n \neq 0$ almost everywhere.

Liouville's theorem asserts that if (i) and (ii) hold, we may integrate the equations of motion by quadrature. When Liouville proved this theorem (1853) it was thought of as an integrability condition for the 1-form $\sum_{i=1}^{n} p_i \, dq_i$. Since then, we know that his analytic criteria give rise to some very pretty geometry. To wit, we have criteria that enable us to give precise notions of what it means to have action-angle variables, integrability at

*Invited lecture of the XXX Symposium on Mathematical Physics, Toruń, May 26–30, 1998, delivered by R. Cushman.

singular points, etc. (see for instance [7, 12, 11]). For example, the theorem on the local existence of action-angle variables has the following precise statement.

THEOREM 1. *Let (M, ω) be a 2n-dimensional symplectic manifold. Suppose that $(f_1, \ldots, f_n)$ are Poisson commuting functions, and that the integral map*

$$f : M \to \mathbb{R}^n : m \mapsto (f_1(m), \ldots, f_n(m))$$

is proper. Suppose that q is a regular value of f which lies in its image and that $f^{-1}(q)$ has a compact connected component F_q. Then there is a neighbourhood V of F_q, an open set U of $\mathbb{R}^n$ and a diffeomorphism

$$V \to U \times \mathbb{T}^n : m \mapsto (j_1, \ldots, j_n, \varphi_1, \ldots, \varphi_n),$$

where $\mathbb{T}^n$ is the n-torus $\mathbb{R}^n/\mathbb{Z}^n$. The coordinates j_k, called the actions, are smooth functions of the integrals f_j. The coordinates φ_j are called the angles. In action-angle variables the symplectic form ω may be written as

$$\omega = \sum_{k=1}^{n} d\varphi_k \wedge dj_k .$$

In other words, V has the structure of a symplectic principal bundle with structure group $\mathbb{T}^n$, Lagrangian fibers, and a Hamiltonian action of the structure group whose momentum mapping is the projection map of the bundle. Furthermore, a section may be chosen so that it is Lagrangian and the action-angle variables form a symplectic chart.

For a proof see [7, Appendix D, p. 377] or [4].

Action angle variables essentially solve our problem, because we can read off the rotation numbers of the flow by inspection (and hence the existence of periodic orbits).

Geometrically, the Hamiltonian vector fields of the action variables define a flat, torsion free affine structure on the tori. This comes about by making a frame at each point of a torus out of the Hamiltonian vector fields of the integrals, and then declaring that this framing is an absolute parallelism, that is, the covariant derivative of one of the Hamiltonian vector fields with respect to another vanishes. The fact that the actions Poisson commute and that the symplectic form is closed, allow us to conclude that the Hamiltonian vector fields of the actions commute and thus that the parallelism has no torsion. Consequently, we can identify a torus, defined as the intersection of the level sets of the integrals, with the standard affine torus $\mathbb{R}^n/\mathbb{Z}^n$, and thus integrate the flows of the Hamiltonian vector fields associated to the integrals.

Moving on to more global considerations, we observe that the action variables give rise to another torsion free, affine connection. This time the connection is defined on $\mathcal{R}$, the set of regular values in the image of the integral map $f : M \to \mathbb{R}^n$ and is defined as follows. In each open subset U of $\mathcal{R}$ on which the function $j : U \to \mathbb{R}^n : u \mapsto (j_1(u), \ldots j_n(u))$ whose components are the values of the actions, is defined, we define a covariant derivative ∇ by

$$\nabla_X Y = DY \cdot X + (Dj)^{-1} \cdot D^2 j(X, Y) ,$$

where X and Y are vector fields on U. The local covariant derivatives ∇ agree on overlaps because the change of variables for the action variables is given by a locally constant element of the group of affine linear transformations of determinant ± 1 of $\mathbb{R}^n$ into itself, (see [7, p. 386–87 and Chapter 4, Section 5]). Suppose that $\Gamma : S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n : t \mapsto \Gamma(t)$ is a smooth curve in $\mathcal{R}$ such that the fiber of the integral map f at $\Gamma(t)$ is a smooth n-torus $\mathbb{R}^n/\mathbb{Z}^n$. Let $\mathcal{X}_0(p_0) = \{X_1(p_0), \ldots, X_n(p_0)\}$ be a frame for $\mathbb{R}^n$ at $p_0 = \Gamma(0)$. Using the connection ∇, parallel transport the frame $\mathcal{X}_0$ along Γ obtaining a frame $\mathcal{X}_1(p_1)$ at $p_1 = \Gamma(1)$. The linear transformation $\mathcal{M}$ of determinant 1 which maps $\mathcal{X}_0(p_0)$ onto $\mathcal{X}_1(p_1)$ is called the monodromy of the connection ∇ along Γ. It is the classifying map of the n-torus bundle $\pi : f^{-1}(\Gamma(S^1)) \to \Gamma(S^1)$. In other words, $\mathcal{M}$ is the map which glues together the end n-tori of the trivial bundle $\pi | f^{-1}(\Gamma(S^1) - \Gamma(0))$. We say that a Liouville integrable system has nontrivial monodromy if the bundle π is nontrivial. The dynamical importance of monodromy is that it gives constraints on the topology of the singular fibers of the integral map, and consequently says something about the stable and unstable manifolds of hyperbolic equilibrium points, (see [8] and [19]).

It is also possible to theoretically construct other obstructions to the existence of global angles (see [1, 10], and [20]) but as yet there are no physical examples where we might examine their subtle dynamical meaning.

2. Nonholonomically constrained systems

While studying nonholonomic mechanical systems we found several examples where we could integrate the equations of motion by quadrature. Despite the fact that these systems are not Hamiltonian, they still possess many features analogous to the Hamiltonian case. For example, such systems need fewer integrals than one would expect to be integrated, or they may possess foliations by affine tori, which have nontrivial global topology. We wish to stress that these examples provide a compelling case for trying to understand the geometric basis of what a completely integrable nonholonomic system should be.

A (linear) nonholonomic system (M, ω, h, H) consists of a constraint manifold M embedded in a symplectic manifold (P, ω) with a Hamiltonian function h and a symplectic distribution $H \subseteq TM$. The equation of motion is given by a vector field X that lies in H and for which $X \lrcorner \omega$ agrees with dh when evaluated on vectors in H, (see [5]). There is a theory of reduction by symmetries for such systems, (see [2, 18] and also [13, 14]). Perhaps more surprisingly, the reduced system has the same structure as the original. This is just like the Hamiltonian case in that a Hamiltonian system reduced by symmetry is still Hamiltonian. Sometimes the reduced nonholonomic system has a reduced distribution which is equal to the tangent space of the reduced manifold. In this case the equations of motion look like $X \lrcorner \omega = dh$, but the 2-form ω need not be closed.

3. Integrable nonholonomic systems

We now look at some examples of integrable nonholonomic systems.

EXAMPLE 2. The nonholonomic oscillator.
Let $(x, y, z) \in S^1 \times \mathbb{R} \times S^1$ be the configuration variables for the nonholonomic oscillator and $(p_x, p_y, p_z) \in T^*_{(x,y,z)}(S^1 \times \mathbb{R} \times S^1) = \mathbb{R}^3$ the momentum variables. Suppose that the Hamiltonian h is given by

$$h = \tfrac{1}{2}\,(p_x^2 + p_y^2 + p_z^2) + \tfrac{1}{2}\,y^2$$

and the nonholonomic constraint is

$$p_z = y\,p_x\,.$$

The equations of motion are

$$\begin{aligned} \dot{x} &= p_x\,, & \dot{p}_x &= -\lambda\,y\,,\\ \dot{y} &= p_y\,, & \dot{p}_y &= -\,y\,,\\ \dot{z} &= p_z\,, & \dot{p}_z &= \lambda\,, \end{aligned}$$

where λ is an undetermined multiplier. Using the first two equations and the constraint to write $\dot{z} = y\,p_x$ and $\lambda = \frac{p_x\,p_y}{1+y^2}$ we can eliminate λ and z. We obtain a reduced phase space $P = T^*(S^1 \times \mathbb{R})$ and the reduced equations of motion

$$\begin{aligned} \dot{x} &= p_x\,, & \dot{p}_x &= -\frac{y}{1+y^2}\,p_x\,p_y\,,\\ \dot{y} &= p_y\,, & \dot{p}_y &= -y\,. \end{aligned} \tag{1}$$

It follows that the function $k = p_x\sqrt{1+y^2}$ is a conserved quantity. The reduced equations (1) can be written in the form $X_h \lrcorner\, \omega = dh$ where the nonholonomic "Hamiltonian vector field" is

$$X_h = \dot{x}\,\partial_x + \dot{y}\,\partial_y + \dot{p}_x\,\partial p_x + \dot{p}_y\,\partial p_y\,,$$

the almost symplectic form (= nondegenerate but not necessarily closed) is

$$\omega = dx \wedge dp_x + dy \wedge dp_y + y\,dx \wedge d(y\,p_x),$$

and the reduced Hamiltonian is

$$h = \tfrac{1}{2}\left((1+y^2)p_x^2 + p_y^2\right) + \tfrac{1}{2}\,y^2.$$

We have the following

PROPOSITION 3. *Consider the integral map of the reduced nonholonomic oscillator*

$$f : P \to \mathbb{R}^2 : (x, y, p_x, p_y) \mapsto (h, k)\,.$$

If q is a regular value of f in $f(P)$, then $f^{-1}(q)$ is a 2-torus. Furthermore, the nonholonomic Hamiltonian vector fields X_k and X_h are tangent to this torus and are pointwise linearly independent.

Observe that $L_{X_h} k = 0$ implies that $\omega(X_k, X_h) = 0$. Thus the torus $f^{-1}(q)$ is Lagrangian. The Lie bracket of the two vector fields X_h and X_k is

$$[X_k, X_h] = -\frac{y\, p_y}{1+y^2}\, X_k\,.$$

Here we see a difference with the Hamiltonian case. The vector fields X_h and X_k on the torus do not commute, because the 2-form ω is not closed. However, we can find an integrating factor g so that $[g\, X_k, X_h] = 0$, so we can certainly "solve" the equations of motion.

From the geometric point of view, it is natural to follow what one did in the Hamiltonian case, namely, declare the frame $\{X_h, X_k\}$ to be an absolute parallelism on the torus $f^{-1}(q)$. The associated flat connection with covariant derivative ∇ has torsion as the following calculation shows

$$T(X_h, X_k) = \nabla_{X_h} X_k - \nabla_{X_k} X_h - [X_h, X_k] \;=\; -[X_k, X_h] \;=\; \frac{y\, p_y}{1+y^2}\, X_k\,.$$

Thus $f^{-1}(q)$ is not the standard affine torus. This is a significant difference from the Hamiltonian case.

There are interesting global aspects to this. In the plane, it is always possible locally to rescale two transverse vector fields so that their flows commute. However, globally there may be no cross section for either of the vector fields, (see [3] for an example). In addition, they may have no smooth non-constant integral. (Recall that in general the only integrals a nonvanishing smooth complete vector field in the plane (so no recurrence and no equilibria) has are constant functions.) We are unaware of any global integrability results for vector fields on a torus. Perhaps a more promising line of investigation is to examine the affine structure of the torus to see if there are more affine or projective transformations than one might normally expect. This might be a clue to integrability.

What seems to be lacking here is a good notion of solvability of the module of vector fields generated by X_h and X_k. However, Nelson's idea of a Lie module [15, p. 27] seems promising. Perhaps it is deeper to ask why we want to examine Lagrangian tori. The answer will have to wait until we more fully understand what happens to Hamilton–Jacobi theory in nonholonomic systems.

EXAMPLE 4. Chaplygin's skate.
Chaplygin's skate is a rigid body with a skate edge at one end and having its center of mass at a point not directly above the point of contact of the skate with a horizontal plane. The skate is constrained to move only in the direction of the skate edge. In other words, the skate edge follows a pursuit curve relative to the motion of its center of mass, (see [17, p. 334]).

The phase space for this problem is $T^*(\mathbb{R}^2 \times S^1)$ with $(x, y, \theta, p_x, p_y, p_\theta)$ being the canonical variables. In particular (x, y) are the coordinates of the center of mass of the skate and θ is the angle the skate edge makes with the x-axis. The Hamiltonian is $h = \frac{1}{2}\,(p_x^2 + p_y^2 + p_\theta^2)$ with the nonholonomic constraint $d\theta + \sin\theta\, dx - \cos\theta\, dy = 0$. Performing singular nonholonomic reduction of the SE(2) (the 2-dimensional special

Euclidean group) symmetry (see [2, p. 243–245]) yields a reduced nonholonomic system on $\mathbb{R}^2$ with coordinates (p_ℓ, p_t), where p_ℓ is the longitudinal and p_t the transverse component of the momentum relative to the skate. The reduced Hamiltonian is $h = \frac{1}{2}\,(2\,p_t^2 + p_\ell^2)$ and the reduced equations of motion are

$$\dot{p}_\ell = 2\,p_t^2 \quad \text{and} \quad \dot{p}_t = -p_\ell\, p_t\,. \tag{2}$$

The reduced equations have a line of equilibrium points, namely $p_t = 0$. Reflection in this singular line, given by $(p_\ell, p_t) \rightarrow (p_\ell, -p_t)$, is a symmetry of the reduced system. A glance at the phase portrait shows that a generic integral curve of (2) is asymptotically stable forward and backward in time to a point on the singular line. Note that energy is conserved, that is, h is an integral of (2). If we remove the singular line from the phase space, then the reduced system is Hamiltonian with respect to the symplectic form $\frac{1}{p_t}\, dp_\ell \wedge dp_t$.

A similar phenomenon appears to be happening in the rolling disc, see [9]. This is a specific instance of a general fact about reduction and Lie theory.

EXAMPLE 5. Routh's sphere.
Routh's sphere is a sphere which is constrained to roll without slipping on a horizontal plane and which moves under the influence of a constant vertical gravitational force. The center of mass of Routh's sphere is not its geometric center and the moment of inertia tensor is symmetric about the line joining the center of mass with the geometric center, (see [6]). After reducing the SE(2) symmetry and restricting to a level set of Jellet's integral, one is left with a two degree of freedom almost Hamiltonian system. One can show that an open subset of phase space is foliated by Lagangian tori that are invariant manifolds for the flow. Globally these tori are twisted up which means that there is monodromy. Physically this corresponds to the fact that Routh's sphere undergoes gyroscopic stabilization. In particular, when the center of mass of the nonspinning sphere is vertically above its geometric center, the position of the sphere is unstable. However, when the sphere spins fast enough about the vertical axis, its position is stable.

EXAMPLE 6. An oblate ellipsoid of revolution.
Here an oblate ellipsoid of revolution is rolls without slipping on a horizontal plane under the influence of a constant vertical gravitational force. There is a physically obvious heteroclinic pair of hyperbolic equilibria, namely when the ellipsoid is standing on either of its ends. The sign of the upper right hand entry of the monodromy matrix $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$ is not determined by the dynamics as in the Hamiltonian case. One can show that the monodromy matrix of a 2-torus bundle over a curve in the set of regular values in the image of the integral map which bounds a region containing the image of both hyperbolic equilibria is the identity, see [8]. This cannot happen for a Hamiltonian system because the sign of the upper right hand entry of the monodromy matrix must be the same for both hyperbolic equilibria. Thus monodromy is not a purely Hamiltonian phenomenon. This is the first example we are aware of which proves the non-Hamiltonian character of a time reversible system using a global topological invariant.

EXAMPLE 7. Sphere on a turntable.
Even when one has solved the reduced problem, there still remains the task of reconstructing the full motion. Here one often runs into problems that one cannot solve. For example, in the problem of a sphere rolling on a turntable (see [16, p. 207–209]), the reduced problem is a harmonic oscillator, which is certainly integrable. However, to perform reconstruction one must solve a differential equation of the form $\dot{A} = \xi(t)\,A$, where $A \in \mathrm{SO}(3)$, $\xi \in \mathrm{so}(3)$ and $t \longrightarrow \xi(t)$ is an affine circle. The full solution of the reconstruction equations depends on evaluating a function which is not computable in the sense of differential Galois theory.

It would be nice to have some criteria that would distinguish an integrable nonholonomic system from an nonintegrable one. For example, the rattle back is almost surely nonintegrable, yet we have no such theorem. We suspect that this question is so difficult that it will remain open for quite a while.

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