Removing the cocycle in a momentum map

Larry Bates[1] and Richard Cushman[2]

Abstract

We show that for a momentum map with a cocycle corresponding to a Hamiltonian G-action we can find a Hamiltonian action of a larger group whose momentum map extends the original one and is coadjoint equivariant.

The cocycle

Suppose we are given a Hamiltonian action ϕ of a connected Lie group G on the symplectic manifold (P, ω) with momentum mapping j, that is, for each vector ξ in the Lie algebra $\mathfrak{g}$ of G, the infinitesimal generator Y_ξ of the action of G on P satisfies $Y_\xi \mathbin{\lrcorner} \omega = \mathrm{d}j_\xi$. The Poisson brackets of the components of j satisfy

$$\{j_\xi, j_\eta\} = j_{[\xi,\eta]} + \Sigma(\xi, \eta).$$

The momentum map j is equivariant with respect to an affine algebra action [7, p.42] of $\mathfrak{g}$ on $\mathfrak{g}^*$. Σ is a two-cocycle because it is antisymmetric, bilinear and satisfies the cocycle identity

$$\Sigma([\xi, \eta], \zeta) + \Sigma([\eta, \zeta], \xi) + \Sigma([\zeta, \xi], \eta) = 0.$$

We are using the sign conventions of [1] and [2].

The extension

We try to make a new Lie algebra by first choosing a basis $\{X_a\}_{a=1}^n$ for $\mathfrak{g}$. The structure constants c_{ab}^c for $\mathfrak{g}$ are defined by

$$[X_a, X_b] = c_{ab}^c X_c.$$

[1]Department of Mathematics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Research partially supported by NSERC.

[2]Mathematics Institute, University of Utrecht, 3508TA Utrecht, the Netherlands. Research partially supported by European Community funding for the Research and Training Network MASIE (HPRN-CT-2000-00113).

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Now add a new element Z and define the bracket relations for our new Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \times_\Sigma \mathbb{R}$ by

$$\begin{aligned}
[X_a, X_b] &= c^c_{ab} X_c + \Sigma_{ab} Z, \\
[X_a, Z] &= 0.
\end{aligned}$$

We need to show that these bracket relations indeed define a Lie algebra. Since antisymmetry and bilinearity are clear, we need to check that the Jacobi identity holds. This follows because the cyclic sum

$$\mathfrak{S}[[X_a, X_b], X_c] = \mathfrak{S}(c^e_{ab} c^d_{ec} X_d + c^e_{ab} \Sigma_{ec}) = 0$$

by virtue of the Jacobi identity in $\mathfrak{g}$ and the cocycle identity for Σ. The other possible brackets involve either one or two copies of Z, and these are easily seen to be zero since $[Z, *] = 0$. Hence we conclude that the bracket relations define a one-dimensional central extension of $\mathfrak{g}$. Note that $\mathfrak{z} = \mathbb{R}Z$ is an ideal in $\hat{\mathfrak{g}}$ and that $\hat{\mathfrak{g}}/\mathfrak{z} = \mathfrak{g}$.

Coadjoint equivariance

In order to show that the extended momentum map $\hat{\jmath} = (j_1, \ldots, j_n, j_Z)$ is coadjoint equivariant, first choose the basis (χ^a, ζ) for $\hat{\mathfrak{g}}^*$ dual to the basis (X_a, Z) for $\hat{\mathfrak{g}}$. The coadjoint action of $\hat{\mathfrak{g}}$ on $\hat{\mathfrak{g}}^*$ is given by

$$\begin{aligned}
\mathrm{ad}^t_{X_b} \zeta &= \Sigma_{bc} \chi^c, \\
\mathrm{ad}^t_{X_b} \chi^a &= c^a_{bc} \chi^c, \\
\mathrm{ad}^t_Z * &= 0.
\end{aligned}$$

Let the extended connected, simply connected group $\widehat{G}$ with Lie algebra $\hat{\mathfrak{g}}$ act on P by $\hat{\phi}(\hat{g}, p) = \phi(\rho(\hat{g}), p)$ where $\rho : \widehat{G} \to G$ is the quotient map. Infinitesimal equivariance may be seen from the following calculation:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \hat{\jmath}(\hat{\phi}(\exp t X_b, p))|_{t=0} &= \langle \mathrm{d}\hat{\jmath}, Y_b \rangle \\
&= \omega(Y_a, Y_b) \chi^a \\
&= \{j_a, j_b\} \chi^a \\
&= c^c_{ab} j_c \chi^a + \Sigma_{ab} j_Z \chi^a \\
&= -\mathrm{ad}^t_{X_b}(j_c \chi^c + j_Z \zeta).
\end{aligned}$$

We conclude that the extended momentum map $\hat{\jmath}$ is coadjoint equivariant with respect to the extended algebra $\hat{\mathfrak{g}}$. Since $\widehat{G}$ is connected, this proves the coadjoint equivariance of $\hat{\jmath}$ with respect to $\widehat{G}$ as well.

Examples

The first example is the simplest. Let $P = \mathbb{R}^2$, $\omega = \mathrm{d}q \wedge \mathrm{d}p$, and $G = \mathbb{R}^2$ acting by translation:

$$\phi((x, y), (q, p)) = (q + x, p + y).$$

Then the cocycle is $\Sigma(\binom{1}{0}, \binom{0}{1}) = 1$. The extended group $\widehat{G}$ is the Heisenberg group with multiplication

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy' - yx')$$

(note the cocycle term $\Sigma(\binom{x}{y}, \binom{x'}{y'})$). The most illuminating representation for this group is as a subgroup of $\mathrm{Sp}(4, \mathbb{R})$, given by matrices of the form

$$\begin{pmatrix} 1 & x & y & z \\ & 1 & 0 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix}.$$

This example is also given in [4, p.272]. It is much harder to see the symplectic role of the extension there because the authors employ the usual Heisenberg representation

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}.$$

The next example is the extension of the affine symplectic group of the symplectic vector space $\mathbb{R}^{2n}$ (the semidirect product $\mathrm{Sp}(2n, \mathbb{R}) \rtimes \mathbb{R}^{2n}$) acting by symplectic affine transformations on $\mathbb{R}^{2n}$. In this case, the extended group may be identified with the odd symplectic group $\mathrm{Sp}(2n + 1, \mathbb{R})$ [3], namely the group of matrices of form

$$\begin{pmatrix} 1 & & \\ v & A & \\ z & (JA^{-1}v)^t & 1 \end{pmatrix},$$

where $v \in \mathbb{R}^{2n}$, $A \in \mathrm{Sp}(2n, \mathbb{R})$, $z \in \mathbb{R}$, and J is the almost complex multiplication

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Remarks

A proof of this extension theorem (sans examples!) may also be found in Wallach [6]. However, it seems that (for reasons that escape us) the index calculation given here is more direct and more transparent than his coordinate-free one. The primary rationale for the new proof is that it helps one understand the affine symplectic group example given above more easily.

We further note that a discussion related to ours occurs in Woodhouse [7, p.58]. Moreover, this result was surely known to experts such as Tuynman [5], but they do not appear to have written it down.

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