COMPLETE INTEGRABILITY BEYOND LIOUVILLE–ARNOL'D

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Completely integrable Hamiltonian systems with fibres not of cylindrical type are shown to arise naturally from holomorphic functions of two variables. They can have Hamiltonian monodromy of finite order.

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It is a standard result in Hamiltonian mechanics that a system with n degrees of freedom and n functionally independent Poisson commuting integrals is integrable by quadrature. A more careful statement of this result is that if all of the flows of the first integrals are complete, then the connected components of the joint level sets have the topology of a generalized cylinder [5]. The flow is rectilinear on the generalized cylinder. It sometimes happens that the joint level sets of the integrals are compact and connected. Then the generalized cylinder is a torus and we even have action angle variables. In this case, all of the important *local* qualitative information about the Hamiltonian system may be seen by inspection.

Flaschka [6] gave some interesting examples of two degrees of freedom integrable systems where the joint level sets have a more interesting topology than a generalized cylinder and posed the problem of explaining just what integrability meant. Of course, because the topology is not that of a cylinder or $\mathbb{R}^2$, the flow cannot be complete, but this does not present insurmountable difficulties in describing what is going on.

1. Holomorphic functions as completely integrable systems

Consider a holomorphic function of two variables

$$F : \mathbb{C}^2 \to \mathbb{C} : (z_1, z_2) \mapsto F(z_1, z_2), \tag{1}$$

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where $\mathbb{C}^2$ has the complex symplectic form $\Omega = dz_1 \wedge dz_2$. The holomorphic Hamiltonian vector field X_F is the vector field satisfying $X_F \lrcorner\, \Omega = dF$. The complex integral curves of X_F satisfy Hamilton's equations

$$\begin{aligned}\frac{dz_1}{d\tau} &= \frac{\partial F}{\partial z_2},\\ \frac{dz_2}{d\tau} &= -\frac{\partial F}{\partial z_1},\end{aligned} \tag{2}$$

where τ is a *complex* time parameter. From mechanics, we would expect that this is a completely integrable system, as we could pretend in our quadrature that all variables were real. More formally, let us treat (2) as a real system by setting $z_a = x_a + iy_a$, $F = u + iv$, $\tau = t - is$. Then the complex Hamilton's equations (2) become

$$\frac{dx_1}{d\tau} + i\frac{dy_1}{d\tau} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial y_2}\right) + \frac{1}{2}i\left(\frac{\partial v}{\partial x_2} - \frac{\partial u}{\partial y_2}\right) \tag{3}$$

and

$$\frac{dx_2}{d\tau} + i\frac{dy_2}{d\tau} = -\frac{1}{2}\left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y_1}\right) - \frac{1}{2}i\left(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial y_1}\right). \tag{4}$$

Since F is holomorphic, it satisfies the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \quad \text{and} \quad \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}. \tag{5}$$

Setting $\tau = t$, equating real and imaginary parts, and using (5), Eqs. (3) and (4) become

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{\partial u}{\partial x_2}, & \frac{dx_2}{dt} &= -\frac{\partial u}{\partial x_1},\\ \frac{dy_1}{dt} &= -\frac{\partial u}{\partial y_2}, & \frac{dy_2}{dt} &= \frac{\partial u}{\partial y_1}.\end{aligned} \tag{6}$$

These differential equations are the Hamiltonian differential equations of the Hamiltonian vector field X_u on $\mathbb{R}^4$ if the symplectic form is given by

$$\omega = dx_1 \wedge dx_2 - dy_1 \wedge dy_2 = \mathrm{Re}\,(dz_1 \wedge dz_2)$$

and the Hamiltonian is $u = \mathrm{Re}(F)$. Furthermore, if $\tau = -is$, we obtain

$$\begin{aligned}\frac{dx_1}{ds} &= \frac{\partial v}{\partial x_2}, & \frac{dx_2}{ds} &= -\frac{\partial v}{\partial x_1},\\ \frac{dy_1}{ds} &= -\frac{\partial v}{\partial y_2}, & \frac{dy_2}{ds} &= \frac{\partial v}{\partial y_1},\end{aligned} \tag{7}$$

which are Hamilton's equations for the Hamiltonian vector field X_v on $(\mathbb{R}^4, \omega)$ with Hamiltonian $v = \mathrm{Im}\,F$. We need only one more fact to complete the real description.

LEMMA 1.1. *The Lie bracket* $[X_u, X_v] = 0$ *if* $F = u + iv$ *is holomorphic.*

Proof: This is most easily seen by examining the complex Poisson bracket

$$\{u, v\} = \frac{1}{4i}\{F + \bar{F}, F - \bar{F}\} = \frac{1}{2i}\{\bar{F}, F\} = \frac{1}{2i}\left(\frac{\partial \bar{F}}{\partial z_1}\frac{\partial F}{\partial z_2} - \frac{\partial \bar{F}}{\partial z_2}\frac{\partial F}{\partial z_1}\right).$$

But

$$\frac{\partial \bar{F}}{\partial z} = \overline{\frac{\partial F}{\partial \bar{z}}}$$

and $\bar{\partial} F = 0$ since F is holomorphic. Therefore $\{u, v\} = 0$. □

We summarize the above discussion by saying that a holomorphic function F on $\mathbb{C}^2$ gives rise to a Liouville completely integrable system on $(\operatorname{Re} F, \operatorname{Im} F, \mathbb{R}^4, \omega)$ with $\omega = \operatorname{Re}(dz_1 \wedge dz_2)$ and the real Poisson bracket $\{\operatorname{Re} F, \operatorname{Im} F\} = 0$.

2. Flows on surfaces and Flaschka's questions

As Flaschka observed [6], the interest in such systems arises because for regular values of F, we obtain a surface with a pair of nowhere zero commuting vector fields. The topology of this surface need not be of the cylindrical (or toral) type that one has as a conclusion of the Liouville–Arnol'd theorem for completely integrable systems (see [5, Appendix D]). Flaschka illustrated this with a twice punctured genus three surface which is a level set of $F(z_1, z_2) = z_1^2 + p_8(z_2)$, where p_8 is a polynomial of eighth degree. The main thrust of Flaschka's article was to point out that one really wants equivalent statements (or understanding) about the qualitative nature of the (local) flows on these surfaces that parallel known statements about straight line quasi-periodic flows on tori or cylinders in the Liouville–Arnol'd case.

We attempt to answer this by starting with the possibly more abstract question. Given an orientable surface with a vector field, can it be realized as a Hamiltonian vector field on a regular level set of a holomorphic function in $\mathbb{C}^2$?

The approach here is to find a sequence of obstructions, the passing of which will permit such a realization. These obstructions should be viewed more as being in principle rather than concrete enough to actually solve. Nevertheless it is still a useful exercise because it demonstrates how exceptional these integrable flows are.

If our surface S is to be a regular level set of a holomorphic function, the first requirement is that it be orientable, because the complex structure induced on the surface provides an orientation. The surface must also be open, as the level set of a regular value of a holomorphic function cannot be compact. This follows by looking at the coordinate functions z_1 and z_2 and using the maximum principle.

Next, the vector field should have no zeroes. This requirement is forced by the Hamiltonian structure of the problem: a Hamiltonian vector field can have a zero only at a critical point of the Hamiltonian, and we want a regular value. A slightly different point of view is to see that the given vector field should have no zeroes as a necessary condition for the existence of a transverse vector field. This vector field does not have to have integral curves which are the inverse image of a regular value of some Hamiltonian that has critical points elsewhere.

The next requirement is that the given vector field X has a commuting transverse vector field Y. The transversality is easy: once again, the orientation provides us with such a vector field. The first nontrivial obstacle we run into is whether a transverse field Y can be chosen to commute with our given one. For reasons that will become apparent soon, we pass to the universal cover M of our surface S and pull back X by the covering projection. We will denote the pulled back vector field by X. It is a fact that the universal cover of any open surface is diffeomorphic to the plane $\mathbb{R}^2$. If we could find the vector field Y, then we could find two one-forms ϕ and ψ with the property that $\phi(X) = 1$, $\psi(X) = 0$, $\phi(Y) = 0$ and $\psi(Y) = 1$. The 1-form ϕ is closed as

$$d\phi(X, Y) = X(\phi(Y)) - Y(\phi(X)) - \phi([X, Y]) = 0,$$

since we are assuming $[X, Y] = 0$. Similarly $d\psi = 0$ as well. Hence, by the converse to the Poincaré lemma, there exists a function f such that $\phi = df$, and a function g so that $\psi = dg$. Now $dg(X) = \psi(X) = 0$, so g is an integral for X. Also, $dg \neq 0$ anywhere as $\psi(Y) = 1$.[1] Furthermore, if such a g exists, it is a Hamiltonian for X as $X \,\lrcorner\, (\phi \wedge \psi) = dg$.

One way of viewing the obstruction to the existence of a transverse commuting vector field Y is to examine the quotient space $M = \mathbb{R}^2/\sim$, where the equivalence relation $\sim$ is defined by saying that two points are equivalent if they are on the same integral curve of X. Endow M with the quotient topology. In general M is a non-Hausdorff manifold with local charts given by parametrized integral curves of Y. The interesting feature here is that the differential structure of M may force the existence of critical points of a smooth function. These analytic invariants are obstructions to finding a function g with no critical points on M, and typically show up when examining the integrals around Reeb-like components of the flow. More details may be found in [4] and [11]. To be more concrete, we look at the

EXAMPLE. Consider the topological manifold M defined by the charts $\{x \mid x \in \mathbb{R}\}$ and $\{y \mid y \in \mathbb{R}\}$ with transition map on $\{x \mid x > 0\} \cap \{y \mid y > 0\}$ given by $y = x^2$. Thus the model for our non-Hausdorff manifold M is two copies of the real line glued together on the positive numbers. Let f be a smooth function on M. This means that f looks like a classical smooth function in any local chart, which is to say that the local representatives $f_x(x)$ and $f_y(y)$ are smooth functions of x and y respectively. We now compute

$$\begin{aligned} f_y'(y) &= \frac{d}{dy} f_x(x(y)), && \text{by definition of local representative} \\ &= \frac{d}{dx} f_x(x(y)) \cdot \frac{dx}{dy}, && \text{by the chain rule} \end{aligned}$$

[1] It is known that there are C^∞-smooth nowhere zero vector fields on the plane whose only C^1 integrals are constant [15].

$$= \frac{d}{dx} f_x(x(y)) \cdot \frac{1}{2\sqrt{y}} = f_x'(x(y)) \cdot \frac{1}{2\sqrt{y}}.$$

Since $\lim_{y\downarrow 0} \frac{1}{2\sqrt{y}} = +\infty$ and $f_y(y)$ is a smooth function, it follows $f_x'(0) = 0$. From this point of view Wazewski's example [15] may be thought of as having a dense set of critical points and so any smooth function is necessarily constant. □

Suppose that we can find a vector field Y which is transverse and commutes with X. Hence the coframe $\{\phi, \psi\}$ exists. Then we can define a Riemannian metric γ on M by

$$\gamma = \phi \otimes \phi + \psi \otimes \psi.$$

The metric γ is flat, which is easily seen using the Koszul formula

$$\begin{aligned} 2\gamma(\nabla_Z Y, X) &= Z\gamma(Y, X) + Y\gamma(X, Z) + X\gamma(Z, Y) \\ &\quad -\gamma(Z, [Y, X]) + \gamma(Y, [X, Z]) + \gamma(X, [Z, Y]), \end{aligned}$$

$\gamma(X, Y) = 0$, and $[X, Y] = 0$. This enables us to develop the manifold (M, γ) onto the Euclidean plane $\mathbb{R}^2$ with Euclidean metric $\delta = dx \otimes dx + dy \otimes dy$. In other words, there is a *developing map*[2] $\Phi : (M, \gamma) \to (\mathbb{R}^2, \delta)$ with the following properties:

(1) Φ is a local diffeomorphism into $\mathbb{R}^2$,
(2) Φ is an isometry: $\Phi^*(\delta) = \gamma$.

The existence of this developing map allows us to answer one of Flaschka's questions. Since the commuting vector fields X and Y may be chosen to be Φ-related to the vector fields ∂_x and ∂_y, we find that the vector field X has a *globally straight line flow*. So, modulo the fact that the map Φ need not be one-to-one,[3] we obtain the same result as in the case of a complete flow.

Note that there is nothing inherently two-dimensional about this argument. Given n commuting fields we get a straight line flow on the universal cover of an n-dimensional manifold. Furthermore, this argument provides a topological restriction in the case of a complete flow, for in that case we have an identification with the quotient of $\mathbb{R}^n$ by a discrete (lattice) subgroup.

The next obstruction we could meet is related to realizing the surface M as a regular level set of a holomorphic function on $\mathbb{C}^2$. Given the surface S and the two vector fields X and Y, define an almost-complex structure J by setting $JX = Y$, $JY = -X$, and extending linearly. This makes S into an open Riemann surface. We try to find a proper holomorphic embedding of (S, J) into $\mathbb{C}^2$. By a theorem of Narasimhan [9], there exists such an embedding into $\mathbb{C}^3$. It is a difficult

[2] The developing map as envisaged by Levi-Civita took a flat surface in $\mathbb{R}^3$ and rolled it out isometrically onto the plane. Standard references for developing maps are [3, 8, 13].

[3] If Φ is not one-to-one then the flow of X is necessarily incomplete. From the theory of Lie groups, the flow of two commuting, complete, nonzero vector fields on a surface S realizes S as a quotient $\mathbb{R}^2/\Lambda$, where Λ is a lattice. Since we are working on the universal cover, the only simply-connected case is the entire plane $\mathbb{R}^2$.

and unsolved problem in complex analysis as to whether or not we can push this embedding down into $\mathbb{C}^2$. It is conjectured that this can always be done.[4]

Assuming that we have got this far, the last obstruction is to realize the properly embedded surface S as a regular level set for a holomorphic function. To see that we can always do this, we give the following argument. Suppose S has a proper holomorphic embedding in $\mathbb{C}^2$. Find an open cover of $\mathbb{C}^2$ by sets U_a with the property that in each U_a we can find a holomorphic function f_a so that $S \cap U_a = f_a^{-1}(0)$, and 0 is a regular value of f_a. This we can do by the holomorphic implicit function theorem. If $S \cap U_a = \emptyset$, set $f_a \equiv 1$. On $U_a \cap U_b$, set $c_{ab} = f_a/f_b$.

CLAIM 2.1. *c_{ab} is holomorphic and nonzero.*

Proof: In U_a, we can find local coordinates w_{1a}, w_{2a} so that $f_a(w_{1a}, w_{2a}) = w_{2a}$ and similarly in U_b. Expanding w_{2a} in terms of (w_{1b}, w_{2b}), we get

$$w_{2a} = k_{ab} w_{2b} + \text{h.o.t.},$$

where $k_{ab} = k_{ab}(w_{1b})$ is a holomorphic function with nonzero constant term in the expansion

$$k_{ab} = a_{0ab} + a_{1ab} w_{1b} + \cdots .$$

Thus

$$\frac{f_a}{f_b} = \frac{w_{2a}}{w_{2b}} = \frac{(a_{0ab} + a_{1ab} w_{1b} + \cdots) w_{2b} + \text{h.o.t.}}{w_{2b}} = a_{0ab} + \text{h.o.t.}$$

The claim now follows. □

Now $\{c_{ab}\}$ forms a cocycle with coefficients in the sheaf of germs of nonzero holomorphic functions. The short exact exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

leads to the long exact sequence in cohomology

$$\cdots \to H^1(\mathbb{C}^2, \mathbb{Z}) \to H^1(\mathbb{C}^2, \mathcal{O}) \to H^1(\mathbb{C}^2, \mathcal{O}^*) \to H^2(\mathbb{C}^2, \mathbb{Z}) \to \cdots .$$

Now $H^1(\mathbb{C}^2, \mathbb{Z}) = H^2(\mathbb{C}^2, \mathbb{Z}) = 0$ by de Rham theory, and $H^1(\mathbb{C}^2, \mathcal{O}) = 0$ by the $\bar{\partial}$-Poincaré lemma. Hence, the cocycle is cohomologically trivial and we can find a global f so that $S = f^{-1}(0)$.

3. Singularities of type A_k

The singularity[5] of type A_k is modelled by the function

$$F(z_1, z_2) = z_1^{k+1} + z_2^2.$$

[4]Stehle [14] has shown that there is a proper embedding of the open unit disc in $\mathbb{C}$ into $\mathbb{C}^2$. Explicit embeddings of the disk have been given by Alexander [1]. Globevnik and Stensones [7] show that there is an embedding of any finitely connected planar domain with nondegenerate boundary components into $\mathbb{C}^2$.

[5]The reason for the similarity of this notation and that of Lie groups is that the morsification of the singularity yields a finite group isomorphic to the Weyl group of the Lie group with the same name. This is explained in [2].

The complex Hamilton's equations for the Hamiltonian F are

$$\begin{aligned} \dot{z}_1 &= 2z_2, \\ \dot{z}_2 &= -(k+1)z_1^k, \end{aligned} \tag{8}$$

which implies that all nonzero values of F are regular values. Since far from the origin $z_2 \sim \sqrt{-z_1^{k+1}}$, it follows that the Riemann surface $S = F^{-1}(c)$, $c \neq 0$, has two punctures if k is odd and a single puncture if k is even.

Triangulate the compact surface $\overline{S}$, which is S with the punctures filled in, by drawing lines between the neighbouring roots of unity, between the roots of unity to the origin, and rays from the roots of unity to the point(s) at infinity. Counting the number of vertices V of this triangulation as the number of double points plus the number of points at infinity plus the centres of the polygons yields $(k+1) + (1 + k \bmod 2) + 2$. The number of edges E is $2 \cdot 3(k+1)$, since there are two sheets. The number of faces is seen to be $2 \cdot 2(k+1)$. From Euler's formula $\chi = V - E + F$ it follows that the Euler characteristic of the surface is $2-k$ if k is even and $3-k$ if k is odd. Thus the genus of the surface is $k/2$ if k is even and $(k-1)/2$ if k is odd. This proves the following result.

LEMMA 3.1. *For a nonzero value c, the level set $F^{-1}(c)$ is not only connected but has the topology of a surface of genus $\lfloor k/2 \rfloor$, which is singly punctured if k is even and doubly punctured if k is odd.*

3.1. The case A_2

Consider the A_2 singularity

$$F(z_1, z_2) = z_1^3 + z_2^2.$$

On the level set $F^{-1}(c)$,

$$(\dot{z}_1)^2 = 4z_2^2 = 4(c - z_1^3).$$

Letting $w = -z_1$, we get

$$(\dot{w})^2 = 4w^3 + 4c = 4w^3 - g_3, \tag{9}$$

if we set $g_2 = 0, g_3 = -4c$. The solution of (9) is

$$w(s) = \wp(s + a_0; 0, -4c),$$

where $\wp$ is the Weierstrass $\wp$ function. Here we are using standard notation. Results about the $\wp$-function may be found in [12]. The constant term a_0 is chosen so that the initial condition w_0 in (9) is $w(0)$. In terms of our original variables z_1, z_2 we have

$$\begin{aligned} z_1(s) &= -\wp(s + a_0; 0, -4c), \\ z_2(s) &= -\tfrac{1}{2}\wp'(s + a_0; 0, -4c). \end{aligned} \tag{10}$$

Of course, having an explicit formula (10) for the solutions of (9) does not give us the qualitative information we would like about the flow of X_F, namely, the incomplete orbits, the itinerary of trajectories, and so on. Also, it does not yield topological information on the way that all the punctured tori $F^{-1}(c)$, $c \neq 0$, fit together. In other words, we would like to know what the structure of the bundle $F : F^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$ is.

In order to do this, it helps to have a good model for a single fibre. The nicest model is the one we get from the developing map. Recall that the developing map is the one that develops the flat metric defined on the fibre to the standard Euclidean metric of the plane. Furthermore, it does this by isometrically mapping the integral curves of the first vector field X_u to the parallel family of lines $y = \text{const}$ in the xy-plane and the integral curves of the other vector field X_v to the orthogonal family of parallel lines $x = \text{const}$ in the xy-plane. Because the construction is only local, it is technically easier to work with the universal covering of the fibre, and push the awkwardness associated with the topology of the fibre into the discrete group of covering transformations.

In the case at hand, the developing map from the universal cover of the fibre

$$\Phi : \widetilde{F^{-1}(c)} \to \mathbb{C}$$

naturally extends to a map $\overline{\Phi}$ from the universal cover of the torus T^2 to the plane together with the point at infinity. This yields qualitative information about the flow of X_u on $F^{-1}(c)$. By inspection, one sees that there is a single trajectory that comes from infinity and returns there in finite time if and only if all other trajectories of X_u are periodic. This happens when one of the basis vectors for the period lattice $L_{g_2,g_3} = \text{span}_{\mathbb{Z}}\{\omega_1, \omega_2\}$ corresponding to $(g_2, g_3) = (0, -4c)$ satisfies the condition that $\text{Im}(\omega_1)/\text{Re}(\omega_1)$ is a rational number. Otherwise, all trajectories but two wind around forever.

To extract information about how all the tori fit together, choose initial conditions in (9) to depend on θ as follows

$$z_1(\theta) = e^{2\theta i} \qquad \text{and} \qquad z_2(\theta) = e^{3\theta i},$$

then $F(z_1(\theta), z_2(\theta)) = 2e^{6\theta i}$. Thus all the quantities $\omega_1, \omega_2, g_2, g_3$ now depend on the parameter θ. Now as θ runs from 0 to $2\pi/6$, we see that $F(z_1(\theta), z_2(\theta))$ runs once around the circle of radius two. Since $\{\omega_1, \omega_2\}$ is a basis for the period lattice of the $\wp$ function at $\theta = 0$, it is known that[6]

$$g_2 = \sum{}' \frac{1}{(m\omega_1 + n\omega_2)^4}, \qquad g_3 = \sum{}' \frac{1}{(m\omega_1 + n\omega_2)^6},$$

where the summation is over all pairs $(m, n) \neq (0, 0)$. Letting θ vary, we see that $g_2(\theta) = 0$ and $g_3(\theta) = -8e^{6\theta i}$. Hence

$$\omega_1(\theta) = e^{-i\theta}\omega_1(0) \quad \text{and} \quad \omega_2(\theta) = e^{-i\theta}\omega_2(0).$$

Since a loop in the base of the bundle F is traced out when θ increases from 0 to $2\pi/6$, the lattice L_{g_2,g_3} returns to itself, even though the basis has rotated clockwise

[6]These are standard results about elliptic functions, and may be found in [12].

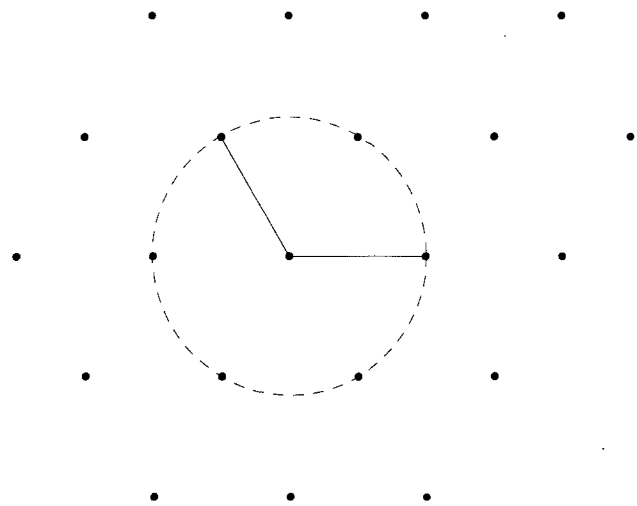

Fig. 1: A lattice with multiplication $e^{2\pi i/6}$

by $\pi/3$. In classical texts $\{\omega_1, \omega_2\}$ is often chosen to have the standard form (up to multiplication by an element of $\mathbb{C}^*$) $\{1, \omega_2\}$, with ω_2 lying in the modular region. This implies that the vectors $e^{2\pi ik/6}$ are also in the lattice L_{g_2,g_3}. In other words, the lattice L_{g_2,g_3} has a multiplication of order six (see Fig. 1). Note that the lattice points represent the puncture in the torus fiber of F.

Taking a small loop around the origin in the c-plane starting at $c_0 \neq 0$, it follows that the Hamiltonian monodromy of the lattice has order six. In the homology basis $\{\omega_1, \omega_2\}$ of the fiber $F^{-1}(c_0)$, the monodromy has matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Observe that the induced map on the end of the punctured torus $F^{-1}(c_0)$, which may be thought of as a $(2, 3)$ torus knot in S^3, is a rotation by $2\pi/6$, which also has order six.

In this example we are following the convention of presenting the Hamiltonian monodromy as an automorphism of the first homology group of the fibre. Nothing forbids us from defining the monodromy as an automorphism of the fundamental group of the fibre. It is then slightly more involved to write it down in the noncylindrical case. The question then arises as to whether it is possible to have an automorphism of the fibre that induces a nontrivial map in π_1, but is the identity on homology.

For example consider the case when the fiber S is a Riemann surface which is a singly punctured torus. Its fundamental group is isomorphic to the free group on two generators a and b. The map $(a, b) \to (a, aba^{-1})$ is a non-identity isomorphism of the fundamental group. But since it is conjugation, it induces the identity on first homology.

For finite genus greater than one there is no problem, as the following theorem demonstrates.[7]

THEOREM 3.2. *Let S be a compact Riemann surface of finite genus greater than one with a finite number of punctures. We may consider S as having a finite number of marked points. Suppose that $f : S \to S$ is a holomorphic map that induces the identity on the first homology group $\mathrm{H}_1(S)$. Then f is the identity map.*

Proof: The Lefschetz number of f is $2-2g$, which is negative for genus g greater than one. But if f is not the identity map, the fixed points of f are isolated (since f is holomorphic) and each such point makes a positive contribution to the Lefschetz number. So f is the identity and therefore induces the identity map on the fundamental group. □

This theorem does not completely answer our original question. But it likely suffices for any example we could realistically expect to compute. In particular, since the action variables are typically given as the integral of the fundamental one-form ϑ_0, we consider $\vartheta_0 = \mathrm{Re}(z_2 dz_1)$ and its periods (integrals) on a homology basis to get a set of action variables $j_1, \ldots, j_n$. Since the dimension of the first homology group of a Riemann surface of genus g with p punctures is $2g+p-1$, which is greater than 2 when $g \geq 2$, the actions $j_1, \ldots, j_n$ may not be used as coordinates, but still may be analytically continued in order to examine the monodromy.

LEMMA 3.3. *The actions $j_1, \ldots, j_n$ Poisson commute.*

Proof: The proof of the lemma follows from the observation that the j_ks are functions of the Poisson commuting functions $\mathrm{Re}(F)$ and $\mathrm{Im}(F)$, since they are constant on the level sets of F, which are Lagrangian submanifolds. □

3.2. The singularity A_5

The singularity A_5 is represented by the Hamiltonian

$$F(z_1, z_2) = z_1^6 + z_2^2.$$

The discussion at the beginning of Section 3 shows that the level set of F corresponding to a regular value c is a doubly punctured surface of genus two. Similarly to the A_2 singularity, choose $c = 1$ as a regular value of F, then eliminating z_2 on the level set $F^{-1}(1)$ and separating variables we obtain the integral

$$\int \frac{dz}{\sqrt{1-z^6}} = \int \frac{dz}{\sqrt{(1-z^2)(1-\omega^2 z^2)(1-\omega^4 z^2)}} = \mathrm{sn}^{-1}(z; \omega, \omega^2).$$

Here $z_1 = z$ and ω is a primitive sixth root of unity. The function $\mathrm{sn}(z; \omega, \omega^2)$ is a Jacobi hyperelliptic function, which is discussed in Appendix.

[7] We thank Geoff Mess for this argument.

To extract qualitative information, we examine the developing map Φ. The image of the unit disc under the developing map

$$z \to \int_0^z \frac{1}{\sqrt{1-z^6}}dz \tag{11}$$

is a curvilinear hexagon, whose vertices are the image of the six roots of unity. Continuity of the map (11) implies that the boundary of the curvilinear hexagon may be slightly deformed (while keeping the six roots of unity fixed) so that the image is a regular hexagon. Similarly, the image of the exterior of the unit disc under the developing map

$$z \to \int_\infty^z \frac{1}{\sqrt{1-z^6}}dz$$

is a double cover of a curvilinear triangle, with the double point at the centre of the triangle corresponding to the point at infinity. If we deform the boundary of the disc in exactly the same way as above, the image will be a regular triangle with side length the same as the hexagon. Continuing the integral will produce a picture that gives multiple copies of the hexagons and triangles. Of course, since the centres of the triangles correspond to the point at infinity, they are removed from our 'tiling'. Thus the developing map produces a model for the compactified Riemann surface that is a double cover of the entire plane with ramification points at the centres of the triangles in the tiling, see Fig. 2.

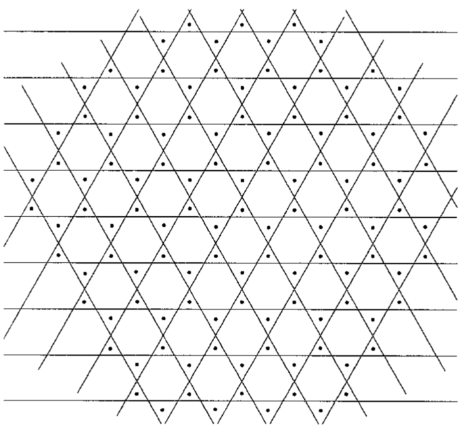

Fig. 2: The hexagonal-triangular tiling for A_5

Once more we may compute the monodromy and find that it has order six as well. We give some details. First we determine the number of punctures. Far from the origin $y \sim ix^3$, so a loop on the extended plane stays on the same sheet. Therefore there are two punctures. From Lemma 3.1 we find that the genus of the corresponding compact Riemann surface is 2. Now set $F(z_1, z_2) = e^{i\theta}$, that is, $z_2^2 = e^{i\theta}(1-(e^{-\frac{i\theta}{6}}z_1)^6)$. Then $e^{-\frac{i\theta}{6}}z_1$ are sixth roots of unity. Therefore, as θ increases from 0 to 2π, a cycle containing adjacent sixth roots of unity rotates through an angle of $\frac{2\pi}{6}$. Therefore the monodromy has order six.

Our treatment of the developing map is *different* from the usual approach in function theory where uniformization yields a tiling of the hyperbolic disk, and not a ramified covering of the plane. The reason for this is that one loses dynamical information in the hyperbolic case, as the flow is no longer rectilinear. At any rate, it is interesting to realize the punctured Riemann surface that is the fibre as a quotient of a ramified covering of the plane by a discrete group of the Euclidean group as opposed to a quotient of the hyperbolic disc by a discrete subgroup of $\mathrm{SU}(1,1)$. In fact, for the A_5 singularity, the discrete group is even a lattice subgroup of the translation subgroup, and is the same group as in the case of A_2 (the lattice with multiplication of order six).

Virtually identical considerations as above allow the tabulation of the monodromy of the singularities of type A, D, E as[8]

Singularity	Parity	Genus	Punctures	Normal form	Monodromy group
A_k	k even	$k/2$	1	$z_1^{k+1}+z_2^2$	$\mathbb{Z}_{2k}$
A_k	k odd	$(k-1)/2$	2	$z_1^{k+1}+z_2^2$	$\mathbb{Z}_{k+1}$
D_k	k even	$(k-2)/2$	3	$z_1^{k-1}+z_1z_2^2$	$\mathbb{Z}_{k-1}$
D_k	k odd	$(k-1)/2$	3	$z_1^{k-1}+z_1z_2^2$	$\mathbb{Z}_{2k-2}$
E_6		3	1	$z_1^4+z_2^3$	$\mathbb{Z}_{12}$
E_7		3	2	$z_1^3+z_1z_2^3$	$\mathbb{Z}_9$
E_8		4	1	$z_1^5+z_2^3$	$\mathbb{Z}_{15}$

4. Remarks

(1) Flaschka was also quite interested in the relation of these integrable systems to Painlevé analysis. It appears to us that the occurrence of singular solutions of the form

$$z(t) = t^{\alpha}(c_0 + c_1 t + \cdots)$$

with rational, but not integral, exponent $\alpha < 0$ follows directly from the occurrence of ramification points (corresponding to the point at infinity) in the developing map. The degree of the ramification point then immediately gives the negative reciprocal of α.

[8] We remind the reader that the notation and basic results of singularity theory may be found in [2].

(2) A different way to view these systems is to think of a real-analytic integrable system on $\mathbb{R}^2$, and then complexify to get a holomorphic system, whose real and imaginary parts Poisson commute. Thus the original real system is embedded as a two-dimensional slice of the complex one. One may note, however, that the same conclusion applies for a real-analytic system on $\mathbb{R}^{2n}$. The complexification then yields a completely integrable system on $\mathbb{R}^{4n}$, with the real and imaginary parts of the n Poisson commuting holomorphic functions being the integrals. It is interesting to note that the restriction of the real parts to the imaginary slice can still form a completely integrable system on $\mathbb{R}^{2n}$.

(3) In earlier work Richens and Berry [10] looked at billiard flows on rational angled polygons, which give flows on surfaces of higher genus. The authors gave such systems the unfortunate name *pseudointegrable*, because they viewed the failure of the Liouville–Arnol'd theorem as due to singularity in the description of the vector field, not as incompleteness.

5. Appendix

Consider the system of differential equations

$$\begin{aligned}\dot{x}_1 &= x_2x_3,\\ \dot{x}_2 &= -x_3x_1,\\ \dot{x}_3 &= -k^2x_1x_2.\end{aligned}$$

The solution of this system with initial value $(0, 1, 1)$ is

$$\begin{aligned}x_1(t) &= \mathrm{sn}(t;k),\\ x_2(t) &= \mathrm{cn}(t;k),\\ x_3(t) &= \mathrm{dn}(t;k),\end{aligned}$$

where sn, cn and dn are Jacobi elliptic functions. It is easy to check from the differential equations that $x_1^2+x_2^2$ and $k^2x_1^2+x_3^2$ are first integrals, which yields a pair of identities for the Jacobi functions.

This may be generalized as follows. Consider the system of differential equations

$$\begin{aligned}\dot{x}_1 &= \hat{x}_1x_2\dots x_n,\\ \dot{x}_2 &= -x_1\hat{x}_2\dots x_n,\\ \dot{x}_3 &= -k_1^2x_1x_2\hat{x}_3\dots x_n,\\ &\vdots\\ \dot{x}_n &= -k_{n-2}^2x_1x_2\dots x_{n-1}\hat{x}_n.\end{aligned}$$

Here the notation $\hat{x}$ means that the term is omitted. The solution of this system with the initial value $(0, 1, 1, \dots, 1)$ is denoted by

$$x_1(t) = \mathrm{sn}(t;k_1,\dots,k_{n-2}),$$

$$\begin{aligned} x_1(t) &= \mathrm{cn}(t; k_1, \ldots, k_{n-2}), \\ x_3(t) &= \mathrm{dn}_1(t; k_1, \ldots, k_{n-2}), \\ &\vdots \\ x_n(t) &= \mathrm{dn}_{n-2}(t; k_1, \ldots, k_{n-2}). \end{aligned}$$

We call these functions the *Jacobi hyperelliptic functions.*

PROPOSITION A.1. *The system of differential equations have the following first integrals*

$$x_1^2 + x_2^2, \quad k_1^2 x_1^2 + x_3^2, \quad \ldots, \quad k_{n-2}^2 x_1^2 + x_n^2.$$

Once again the verification is easy; just differentiate and substitute into the differential equations.

That the Jacobi hyperelliptic functions have some bearing on hyperelliptic integrals may be seen as follows. Squaring the differential equation $\dot{x}_1 = \hat{x}_1 x_2 \ldots x_n$ and substituting the first integrals gives

$$\dot{x}_1^2 = (1 - x_1^2)(1 - k_1^2 x_1^2) \ldots (1 - k_{n-2}^2 x_1^2).$$

Separating the variables and integrating yields

$$\int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k_1^2 x^2) \ldots (1 - k_m^2 x^2)}} = \mathrm{sn}^{-1}(x; k_1, \ldots, k_m),$$

setting $x_1 = x, n - 2 = m$). Observe that nothing so far forbids $x_1, \ldots, x_n, k_1, \ldots, k_{n-2}, t$ from being complex variables, one just needs to be careful with the choice of square root and the path of integration. In the integral make the substitutions $z = x^2$, $k_1^2 = a, \ldots, k_m^2 = m$. This yields

$$\frac{1}{2} \int \frac{dz}{\sqrt{z(1 - z)(1 - az) \ldots (1 - mz)}},$$

so it follows that the Jacobi hyperelliptic function sn inverts the general hyperelliptic integral of the form $\int \frac{dz}{\sqrt{p(z)}}$ with $p(z)$ a polynomial with distinct roots because we still have the freedom of moving all the roots by an affine transformation $z \to az + b$ (in other words we perform an affine transformation to map two of the roots of $p(z)$ to 0 and 1 and then invert the integral).

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