

CHARACTER SHEAVES AND DEPTH-ZERO REPRESENTATIONS

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ABSTRACT. In this paper we provide a geometric framework for the study of characters of depth-zero representations of unramified groups over local fields with finite residue fields which is built directly on Lusztig's theory of character sheaves for groups over finite fields and uses ideas due to Schneider-Stuhler. Specifically, we introduce a class of coefficient systems on Bruhat-Tits buildings of perverse sheaves on affine algebraic groups over an algebraic closure of a finite field, and to each supercuspidal depth-zero representation of an unramified p -adic group we associate a formal sum of these coefficient systems, called a model for the representation. Then, using a character formula due to Schneider-Stuhler and a fixed-point formula in étale cohomology we show that each model defines a distribution which coincides with the Harish-Chandra character of the corresponding representation on the set of regular elliptic elements. The paper includes a detailed treatment of $SL(2)$, $Sp(4)$ and $GL(n)$ as examples of the theory.

22E50 (Representations of Lie and linear algebraic groups over local fields)

INTRODUCTION

In their 1997 article in the *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, Peter Schneider and Ulrich Stuhler defined a functor from the category of certain smooth representations over \mathbb{C} of a connected reductive p -adic group $G(\mathbb{Q}_p)$ to the category of $G(\mathbb{Q}_p)$ -equivariant coefficient systems of vector spaces over \mathbb{C} and subsequently obtained a new formula for the characters of such representations on the set of regular elliptic elements of $G(\mathbb{Q}_p)$. In this article we restrict our attention to supercuspidal depth-zero representations and consider a related construction. We pass from \mathbb{Q}_p to an unramified closure \mathbb{Q}_p^{nr} and replace the category of vector spaces over \mathbb{C} by a triangulated

category of ℓ -adic sheaf complexes. This gives rise to a category of coefficient systems on the Bruhat-Tits building for $G(\mathbb{Q}_p^{nr})$ of ℓ -adic sheaves, with $\ell \neq p$. Our category is equipped with parabolic restriction functors, an action of $G(\mathbb{Q}_p^{nr})$, a notion of parabolic induction and an action of Frobenius; we use all these to define *Frobenius-stable admissible coefficient systems*.

We then show that Frobenius-stable admissible coefficient systems are directly related to ℓ -adic representations of p -adic groups. For example, we show there is a formal linear combination of Frobenius-stable admissible coefficient systems (an element of a Grothendieck group tensored with \mathbb{Q}_ℓ) canonically associated to each supercuspidal depth-zero representation of $G(\mathbb{Q}_p)$; we call this a *model for the representation*. We also show that each Frobenius-stable admissible coefficient system defines a distribution on the set of elliptic elements of $G(\mathbb{Q}_p)$. Finally, we show that the distribution associated to the model of a representation coincides with the character of the representation on the set of regular elliptic elements.

Although our focus here is on models for depth-zero supercuspidal representations, the distributions associated to admissible coefficient systems themselves are very interesting. In general, these distributions are neither orbital integrals nor characters of representations; however, they appear to generalize the distributions in [Wal01].

We have recently found that admissible coefficient systems admit a geometric description: they may be interpreted as objects in a triangulated category of ℓ -adic sheaf complexes on the étale site of a rigid analytic space associated to the group $G_{\mathbb{Q}_p}$. However, since the theory of derived categories of ℓ -adic étale sheaves on rigid analytic spaces over \mathbb{Q}_p is, as far as the authors are aware, in some sense still under development, we have opted to restrict ourselves to ℓ -adic étale sheaves on schemes over $\overline{\mathbb{F}}_p$ for the moment. While the benefit of this decision is that we can provide a rigorous argument using ideas readily available in the literature, the cost of this decision is that several arguments in this paper are rather unpleasant due to the fact that we are essentially working with objects and morphisms defined by local data. The rigid analytic perspective is also the point of departure for expanding the scope of this paper to a larger class of admissible representations; in particular, we view the present paper as the depth-zero part of a nascent theory involving ℓ -adic sheaves on the étale site of a rigid analytic space which is compatible with the theory of character sheaves, via vanishing cycles functors, on the reductive quotients (over $\overline{\mathbb{F}}_p$) of special fibres of affinoid spaces formed from canonical integral models for parahoric subgroups. In fact, that is where this story began, but the authors were surprised to find that much of the depth-zero story could be told without rigid analytic geometry. Hence this paper.

* * *

We now describe the sections and principal results of this paper in more detail. In Sections 1 through 2, \mathbb{K} denotes a field equipped with a non-trivial discrete valuation such that \mathbb{K} is strictly henselian and such that the residue field \mathbb{k} of \mathbb{K} is algebraically closed with non-zero characteristic. We let G be a connected

reductive linear algebraic group over \mathbb{K} satisfying a hypothesis described in Section 1.3. In Sections 1.2 through 1.6 we review certain basic constructions associated to parahoric subgroups of the group $G(\mathbb{K})$. We denote facets of the extended Bruhat-Tits building $I(G, \mathbb{K})$ for $G(\mathbb{K})$ by i, j, k or l . For each such facet we consider a canonical integral model G_i such that $G_i(\mathfrak{o}_{\mathbb{K}}) = G(\mathbb{K})_i$, where $\mathfrak{o}_{\mathbb{K}}$ is the ring of integers in \mathbb{K} . We are particularly interested in the maximal reductive quotient \bar{G}_i of the special fibre of G_i , which is a connected linear algebraic group over \mathbb{k} because of the conditions placed on G in Section 1.3. After recalling some important facts concerning equivariant perverse sheaves in Section 1.7 we introduce cohomological parabolic induction functors on these reductive quotients in Section 1.8. We then recall some important facts concerning character sheaves in Section 1.9. In Section 1.10 we introduce a new category, denoted $\bar{\mathcal{D}}G$, formed roughly by attaching the categories $D_c^b(\bar{G}_i; \mathbb{Q}_\ell)$ using the Bruhat order on facets of $I(G, \mathbb{K})$, where $D_c^b(\bar{G}_i; \mathbb{Q}_\ell)$ denotes the bounded derived category of constructible ℓ -adic étale sheaves on \bar{G}_i . See Definition 1.12 for the details. We end Section 1 by defining an additive subcategory $\bar{\mathcal{C}}G$ (See Definition 1.12) of $\bar{\mathcal{D}}G$. Admissible coefficient systems are objects in this category with special properties.

In order to define admissible coefficient systems we first define cuspidal coefficient systems in Section 2. We begin by defining a cohomological parabolic restriction functor res_P^G in Section 2.1. Then, we describe an action of $G(\mathbb{K})$ on the category $\bar{\mathcal{C}}G$ in Section 2.2 and say that an object of $\bar{\mathcal{C}}G$ is *weakly-equivariant* if its isomorphism class is fixed by the action of $G(\mathbb{K})$. A *cuspidal coefficient system* is, roughly, a weakly-equivariant simple object C of $\bar{\mathcal{C}}G$ such that $\text{res}_P^G C = 0$ for every proper parabolic subgroup P of G . Our first main result is Theorem 2.11, which, together with Corollary 2.12, provides a complete description of cuspidal coefficient systems in $\bar{\mathcal{C}}G$. We find that every cuspidal coefficient system may be produced, in a manner similar to compact induction, from some cuspidal character sheaf on the reductive quotient of the special fibre of the canonical integral scheme for a maximal parahoric subgroup of $G(\mathbb{K})$.

In Sections 3 we assume \mathbb{K} is a maximal unramified extension of a p -adic field. Note that such a field is strictly henselian and the residue field of that extension is an algebraic closure of a finite field. In Section 3.3 we define a weakly-equivariant object $\text{ind}_P^G A$ associated to any weakly-equivariant object A in $\bar{\mathcal{C}}L$, where L is the Levi component for P . Using this we define *admissible coefficient systems* as those simple coefficient systems appearing in $\text{ind}_P^G C$ for some parabolic subgroup P and some cuspidal coefficient system C (see Definition 3.7).

In Sections 4 through 6 we fix a p -adic field \mathbb{K}_1 and let \mathbb{K}_1^{nr} denote a maximal unramified extension of \mathbb{K}_1 . Thus, \mathbb{K}_1^{nr} plays the role of \mathbb{K} above. Let $G_{\mathbb{K}_1}$ be a connected, quasi-split unramified linear algebraic group. Then $G_{\mathbb{K}_1} \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr})$ is a split connected reductive linear algebraic group over \mathbb{K}_1^{nr} and so we may let $G_{\mathbb{K}_1} \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr})$ play the role of G above. The main idea of Section 4 is to use the action of the Galois group $\text{Gal}(\mathbb{K}_1^{nr}/\mathbb{K}_1)$ on the extended Bruhat-Tits building $I(G, \mathbb{K}_1^{nr})$ to define a notion of (geomet-

ric) *frobenius-stable* objects of $\bar{\mathcal{C}}G$; roughly, $C \in \text{obj}\bar{\mathcal{C}}G$ is frobenius-stable if its isomorphism class is fixed by the action of Frobenius. See Proposition 4.2 for details. In the rest of Section 4 we briefly revisit the main ideas from Sections 2 and 3 with this Galois action in mind.

Section 5 relates frobenius-stable admissible coefficient systems to depth-zero representations through the notion of a *model for a representation*; a model is an element of the $\bar{\mathbb{Q}}_\ell$ -vector space obtained by tensoring $\bar{\mathbb{Q}}_\ell$ with the subgroup of the Grothendieck group for $\bar{\mathcal{D}}G$ generated by admissible coefficient systems (*cf.* Definition 5.3). Our second main result is Theorem 5.4, which shows that every supercuspidal depth-zero representation admits a model. In Section 5.3 we use the character formula of [SS97] to associate a distribution to each admissible coefficient system. Our third main result is Theorem 5.6, which shows that the distribution associated to a model of a depth-zero representation coincides with the character of the representation, in the sense of Harish-Chandra, on the set of regular elliptic elements of $G(\mathbb{K}_1)$. In this way we suggest that the classification and character theory of depth-zero supercuspidal representations may be studied through the theory of admissible coefficient systems. Section 6 applies the machinery of the paper to the groups of p -adic points on $\text{SL}(2)$ and $\text{Sp}(4)$ in order to illustrate this suggestion.

In summary, the main features of this paper are:

- Theorem 2.11 and Corollary 2.12, where cuspidal coefficient systems are classified;
- Theorem 5.4, where models for supercuspidal depth-zero representations are constructed;
- Theorem 5.6, where we show that the distribution associated to a model of a representation equals the character of that representation on the set of regular elliptic elements;
- Section 6, where we give models for all supercuspidal depth-zero representations of $\text{SL}(2)$, $\text{Sp}(4)$ and $\text{GL}(n)$.

* * *

A.-M.A. wishes to thank the University of Calgary for providing the opportunity for the authors to work together on this project. She also wishes to thank the Pacific Institute of Mathematical Science and the Institute for Advanced Study (Princeton) for their support. This work was presented by A.-M.A. at Ann Arbor at the invitation of Julee Kim and Gopal Prasad and in Berlin (von Humboldt University) at the invitation of Volker Heiermann and Wilhelm Zink, and also at the meeting on K -types for p -adic groups at Münster organized by Guy Henniart, Philip Kutzko and Peter Schneider. She thanks Roger Carter, Ivan Fesenko and Michel Gros for their interest in this work.

C.C. wishes to thank the École Normale Supérieure, the Centre National de la Recherche Scientifique, the Institut des Hautes Études Scientifiques and the

National Science and Engineering Research Council (Canada) for support during various parts of the writing of this paper. He also wishes to thank Jonathan Korman for help and Sam Evans for his encouragement and hospitality. This paper was presented in abbreviated form by C.C. in the Summer of 2004 at Harvard at the invitation of Laurent Berger and then at a Fields Institute Workshop at the University of Ottawa at the invitation of Monica Nevins. Finally, both authors thank Peter Schneider for helpful conversations.

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1 FUNDAMENTAL NOTIONS

1.1 FIELDS AND ALGEBRAIC GROUPS

Let \mathbb{K} be a field equipped with a non-trivial discrete valuation, let $\mathfrak{o}_{\mathbb{K}}$ be the ring of integers of \mathbb{K} and let \mathbb{k} be the residue field of $\mathfrak{o}_{\mathbb{K}}$. We assume that \mathbb{K} is strictly henselian and that \mathbb{k} is algebraically closed with non-zero characteristic. Examples of such fields \mathbb{K} include \mathbb{Q}_p^{nr} (a maximal unramified extension of the field \mathbb{Q}_p of p -adic numbers) and $\mathbb{F}_p((t))$ (formal Laurent series in t with coefficients from an algebraic closure of a field with p elements). Note that \mathbb{K} is neither complete nor locally compact.

Let G be a connected reductive linear algebraic group over \mathbb{K} . We assume G splits over \mathbb{K} . Let $G(\mathbb{K})$ be the group of \mathbb{K} -rational points on G .¹

1.2 INTEGRAL MODELS

The enlarged Bruhat-Tits building for $G(\mathbb{K})$ will be denoted $I(G, \mathbb{K})$. Recall that $I(G, \mathbb{K})$ is the product of the semi-simple Bruhat-Tits building for $G(\mathbb{K})$ (the building for the derived group) by a real affine space. We denote polyfacets of $I(G, \mathbb{K})$ by i, j or k and refer to these as facets.

For each facet i of $I(G, \mathbb{K})$, the parahoric subgroup of Bruhat-Tits will be denoted $G(\mathbb{K})_i$ (*cf.* [BT84, 4.6.28]). Let G_i denote the integral model of G associated to i by [BT84, 5.1.30, 5.2.1]; see also [Yu02, 7.3.1]. Thus, G_i is a smooth group scheme over $\mathfrak{o}_{\mathbb{K}}$ equipped with an isomorphism between the generic fibre of G_i and G such that $G_i(\mathfrak{o}_{\mathbb{K}})$ corresponds to $G(\mathbb{K})_i$ under that isomorphism (*cf.* [Yu02, 7.2]). The special fibre of G_i will be denoted \tilde{G}_i ; thus, $\tilde{G}_i = G_i \times_{\text{Spec}(\mathfrak{o}_{\mathbb{K}})} \text{Spec}(\mathbb{k})$. Then \tilde{G}_i is a smooth connected affine group scheme over \mathbb{k} (*cf.* [Yu02, 7.2]). Although \tilde{G}_i is reduced as a scheme, it need not be reductive as a group scheme; let $\nu_i : \tilde{G}_i \rightarrow \bar{G}_i$ be the maximal reductive quotient of \tilde{G}_i . Then \bar{G}_i is a linear algebraic group over \mathbb{k} which is both connected and reductive. In fact, \bar{G}_i is a closed subscheme (over \mathbb{k}) of \tilde{G}_i , which is a closed subscheme of G_i . In summary, we have the following commutative diagramme.

$$\begin{array}{ccccc}
 G & \longrightarrow & G_i & \longleftarrow & \tilde{G}_i & \xrightarrow{\nu_i} & \bar{G}_i & (1) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \text{Spec}(\mathbb{K}) & \longrightarrow & \text{Spec}(\mathfrak{o}_{\mathbb{K}}) & \longleftarrow & \text{Spec}(\mathbb{k}) & & &
 \end{array}$$

¹In Section 3 we apply the results of Sections 1 and 2 to the case when \mathbb{K} is an unramified closure of a local field with finite residue field. In Section 4 we fix the local field, denoted \mathbb{K}_1 , and consider a form $G_{\mathbb{K}_1}$ for G ; thus, in Sections 4, 5 and 6, $G_{\mathbb{K}_1}$ is a connected reductive algebraic group over the local field \mathbb{K}_1 and $G_{\mathbb{K}_1}$ splits over an unramified extension of \mathbb{K}_1 .

Let $\rho_i : G_i(\mathfrak{o}_{\mathbb{K}}) \rightarrow \bar{G}_i(\mathbb{k})$ denote the composition of the group homomorphism $G_i(\mathfrak{o}_{\mathbb{K}}) \rightarrow G_i(\mathbb{k})$ defined by composition with the canonical map $\text{Spec}(\mathbb{k}) \rightarrow \text{Spec}(\mathfrak{o}_{\mathbb{K}})$, the identification $G_i(\mathbb{k}) = \bar{G}_i(\mathbb{k})$, and the map of \mathbb{k} -rational points $\bar{G}_i(\mathbb{k}) \rightarrow \bar{G}_i(\mathbb{k})$ induced from ν_i . Observe that ρ_i is a map of points; it is not a map of ringed spaces.

EXAMPLE 1.1. Let $G = SL(2)_{\mathbb{K}}$; thus, the global sections of this affine scheme are

$$\mathcal{O}_G(G) = \mathbb{k}[X_{11}, X_{12}, X_{21}, X_{22}]/(\det X - 1),$$

where $\det X = X_{11}X_{22} - X_{12}X_{21} - 1$. Let $i = (01)$ be the maximal facet (grande cellule) of the chamber corresponding to the Iwahori subgroup

$$G(\mathbb{K})_{(01)} = \left\{ \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \mid \begin{array}{l} h_{11}, h_{12}, h_{22} \in \mathfrak{o}_{\mathbb{K}}; h_{21} \in \mathfrak{p}_{\mathbb{K}} \\ h_{11}h_{22} - h_{12}h_{21} = 1 \end{array} \right\}.$$

In this case, $G_{(01)}$ is the affine $\mathfrak{o}_{\mathbb{K}}$ -scheme with global sections

$$\mathcal{O}_{G_{(01)}}(G_{(01)}) = \mathfrak{o}_{\mathbb{K}}[X_{11}, X_{12}, X_{21}, X_{22}, X'_{21}]/(\det X - 1, X_{21} - \varpi X'_{21}),$$

where ϖ is a generator for $\mathfrak{p}_{\mathbb{K}}$. (Of course, the scheme $G_{(01)}$ is independent of this choice.) The isomorphism of the generic fibre of $G_{(01)}$ with G is given by $X_{nm} \mapsto X_{nm}$, for $1 \leq n, m \leq 2$. Since $G_{(01)}$ is a group scheme, $\mathcal{O}_{G_{(01)}}(G_{(01)})$ is a Hopf algebra; the co-multiplication is given by $X_{nm} \mapsto \sum_k X_{nk} \otimes X_{km}$ and $X'_{21} \mapsto X'_{21} \otimes X_{12} + X_{22} \otimes X'_{21}$. The \mathbb{k} -algebra of sections on the special fibre $\tilde{G}_{(01)}$ of $G_{(01)}$ is

$$\mathcal{O}_{G_{(01)}}(G_{(01)}) \otimes \mathbb{k} = \mathbb{k}[X_{11}, X_{12}, X_{22}, X'_{21}]/(X_{11}X_{22} - 1),$$

with co-multiplication given by

$$\begin{aligned} X_{11} &\mapsto X_{11} \otimes X_{11} \\ X_{12} &\mapsto X_{11} \otimes X_{12} + X_{12} \otimes X_{22} \\ X'_{21} &\mapsto X'_{21} \otimes X_{11} + X_{22} \otimes X'_{21} \\ X_{22} &\mapsto X_{22} \otimes X_{22}. \end{aligned}$$

Evidently, $\tilde{G}_{(01)}$ is not reductive. The reductive quotient of $\tilde{G}_{(01)}$ is $GL(1)_{\mathbb{k}}$, and the map $\nu_{(01)} : \tilde{G}_{(01)} \rightarrow \bar{G}_{(01)}$ is induced from the inclusion

$$\mathbb{k}[X_{11}, X_{22}]/(X_{11}X_{22} - 1) \hookrightarrow \mathbb{k}[X_{11}, X_{12}, X_{22}, X'_{21}]/(X_{11}X_{22} - 1).$$

In this example, $\rho_{(01)} : G_{(01)}(\mathfrak{o}_{\mathbb{K}}) \rightarrow \bar{G}_{(01)}(\mathbb{k})$ is given by

$$\rho_{(01)} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \bar{h}_{11},$$

where \bar{h}_{11} is the image of h_{11} under the canonical map $\mathfrak{o}_{\mathbb{K}} \rightarrow \mathbb{k}$.

EXAMPLE 1.2. Continuing with $G = SL(2)_{\mathbb{K}}$, let (0) and (1) be the vertices in the closure of the facet considered above. Let $G_{(0)}$ be the model for G with global section given by

$$\mathcal{O}_{G_{(0)}}(G_{(0)}) = \mathfrak{o}_{\mathbb{K}}[X_{11}, X_{12}, X_{21}, X_{22}]/(\det X - 1).$$

As above, the isomorphism of the generic fibre of $G_{(0)}$ with G is the obvious one. On the other hand, $G_{(1)}$ is the integral affine scheme with global sections

$$\mathcal{O}_{G_{(1)}}(G_{(1)}) = \frac{\mathfrak{o}_{\mathbb{K}}[X_{11}, X_{12}, X_{21}, X_{22}, X'_{12}, X'_{21}]}{(\det X - 1, \varpi X_{12} - X'_{12}, X_{21} - \pi X'_{21})},$$

where ϖ is a uniformizer for \mathbb{K} . (As above, the scheme $G_{(1)}$ is independent of this choice.) The isomorphism from the generic fibre of $G_{(1)}$ to G is determined by $X_{nm} \mapsto X_{nm}$ for $1 \leq n, m \leq 2$. In both cases, the special fibre is $SL(2)_{\mathbb{K}}$; since this group scheme is reductive, the maps $\nu_{(0)}$ and $\nu_{(1)}$ are identities.

1.3 STABILIZERS

When the time comes to relate admissible coefficient systems to characters of depth-zero representations of p -adic groups we will see that it is natural to study stabilizers of facets rather than parahoric subgroups. The stabiliser of any facet of $I(G, \mathbb{K})$ under the action of $G(\mathbb{K})$ is a compact group (recall that $I(G, \mathbb{K})$ refers to the *enlarged* Bruhat-Tits building) which admits a canonical smooth integral model (*cf.* [Yu02, 9.3.2]). However, in general, the maximal reductive quotient of the special fibre of this integral model need not be connected, so we cannot use the theory of character sheaves as developed in [Lus85]. We therefore now impose a condition on the groups G that we study: we assume that the stabilizer of each facet in $I(G, \mathbb{K})$ is a parahoric subgroup, and further that the reductive quotient of the special fibre of the canonical integral model of that parahoric subgroup is connected, as an algebraic group over \mathbb{k} ; we also demand that the same property hold for all cuspidal Levi subgroups of G (*cf.* Definition 2.13). When G is simply connected, this condition is satisfied (*cf.* [Tit79, 3.5.2]); it also holds for general linear groups and symplectic groups.

Remarkably, Lusztig has recently extended the definition of character sheaves to the disconnected group case so there is good reason to expect the main results of this paper can be extended, *mut. mut.*, to a larger class of groups.

1.4 RESTRICTION BETWEEN REDUCTIVE QUOTIENTS

Let i and j be facets of $I(G, \mathbb{K})$ such that $i \leq j$ in the Bruhat order. Let $f_{i \leq j} : G_j \rightarrow G_i$ be the morphism of group schemes over $\mathfrak{o}_{\mathbb{K}}$ obtained by extending the identity morphism id_G in the category of group schemes over $\mathfrak{o}_{\mathbb{K}}$ (*cf.* [Lan96, 6.2]). By restriction to special fibres, this defines a morphism $\tilde{f}_{i \leq j} : \tilde{G}_j \rightarrow \tilde{G}_i$

of group schemes over \mathbb{k} , making the following diagramme commute.

$$\begin{array}{ccc} G_j & \xrightarrow{f_{i \leq j}} & G_i \\ \uparrow & & \uparrow \\ \tilde{G}_j & \xrightarrow{\tilde{f}_{i \leq j}} & \tilde{G}_i \end{array} \quad (2)$$

In fact, this diagramme is cartesian. Let $\tilde{G}_{i \leq j}$ denote the schematic image of $\tilde{f}_{i \leq j}$ in \tilde{G}_i . Let $\bar{G}_{i \leq j}$ be the schematic image of $\tilde{G}_{i \leq j}$ under ν_i and let $\nu_{i \leq j}$ denote the restriction of ν_i to $\bar{G}_{i \leq j}$. Next, let

$$\tilde{f}_{i \leq j} = h_{i \leq j} \circ g_{i \leq j} \quad (3)$$

be the factorization given by the Isomorphism Theorem. The kernel of $\tilde{f}_{i \leq j}$, which equals the kernel of $g_{i \leq j}$, is contained in the kernel of ν_j ; thus, $g_{i \leq j}$ factors through ν_j to give a map $\bar{G}_{i \leq j} \rightarrow \bar{G}_j$; since the kernel of $\nu_{i \leq j}$ is contained in the kernel of this new map, it too factors, this time through $\nu_{i \leq j}$, thus defining $r_{i \leq j} : \bar{G}_{i \leq j} \rightarrow \bar{G}_j$. Notice that

$$\nu_j = r_{i \leq j} \circ \nu_{i \leq j} \circ g_{i \leq j}. \quad (4)$$

Let $s_{i \leq j} : \bar{G}_{i \leq j} \rightarrow \bar{G}_i$ be the obvious inclusion; this is an affine closed immersion. By [Lan96, 9.22], $s_{i \leq j} : \bar{G}_{i \leq j} \rightarrow \bar{G}_i$ is a parabolic subgroup with Levi component \bar{G}_j given by the reductive quotient map $r_{i \leq j} : \bar{G}_{i \leq j} \rightarrow \bar{G}_j$, which is a smooth projective map; we also have $\bar{G}_i = \bar{G}_i \times_{\bar{G}_{i \leq j}} \bar{G}_{i \leq j}$. In summary we have the following commutative diagramme, in which the square on the bottom right is cartesian.

$$\begin{array}{ccccc} \tilde{G}_j & & \xrightarrow{\tilde{f}_{i \leq j}} & & \tilde{G}_i \\ \nu_j \downarrow & \searrow^{g_{i \leq j}} & & \nearrow^{h_{i \leq j}} & \downarrow \nu_i \\ \tilde{G}_j & & \tilde{G}_{i \leq j} & & \tilde{G}_i \\ & \swarrow^{r_{i \leq j}} & \downarrow \nu_{i \leq j} & \searrow^{s_{i \leq j}} & \\ & & \bar{G}_{i \leq j} & & \end{array} \quad (5)$$

Recall the derived category $D_c^b(\bar{G}_i, \bar{\mathbb{Q}}_\ell)$ of cohomologically bounded constructible ℓ -adic sheaves with $\ell \neq p$, introduced in [Del80, 1.1.1-1.1.5] and [BBD, 2.2.9, 2.2.14, 2.2.18] (*cf.* [SGA5, exposés VI, V, XV]). We will follow the notational conventions of [BBD] regarding derived functors.

DEFINITION 1.3. Let i and j be facets of $I(G, \mathbb{K})$ such that $i \leq j$. Define $\text{res}_{i \leq j} : D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell) \rightarrow D_c^b(\bar{G}_j; \bar{\mathbb{Q}}_\ell)$ by

$$\text{res}_{i \leq j} = r_{i \leq j}^* s_{i \leq j}^* (d_{i \leq j}), \quad (6)$$

where $(d_{i \leq j})$ denotes Tate twist by $d_{i \leq j} = \dim \ker r_{i \leq j}$.

Remark 1.4. Thus, $\text{res}_{i \leq j} = \text{res}_{\bar{G}_{i \leq j}}^{\bar{G}_i}$, where the right-hand side refers to the parabolic restriction functor defined in [Lus85, Sect.3.8]. We will sometimes write $\text{res}_{\bar{G}_j}^{\bar{G}_i}$ for $\text{res}_{i \leq j}$ to emphasize the fact that it is a functor from $D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell)$ to $D_c^b(\bar{G}_j; \bar{\mathbb{Q}}_\ell)$. It must be understood that the definition of the functor makes reference to a *specific* parabolic subgroup of \bar{G}_i with Levi component \bar{G}_j .

Remark 1.5. Note also that $\text{res}_{i \leq i}$ is an identity functor.

PROPOSITION 1.6. *If i, j, k, l are facets of $I(G, \mathbb{K})$ such that $i \leq j \leq k \leq l$ then there are canonical isomorphisms of functors*

$$\text{res}_{i \leq j \leq k} : \text{res}_{j \leq k} \text{res}_{i \leq j} \rightarrow \text{res}_{i \leq k}$$

such that the diagramme

$$\begin{array}{ccc} \text{res}_{k \leq l} \text{res}_{j \leq k} \text{res}_{i \leq j} & \xrightarrow{\text{res}_{j \leq k \leq l} \text{res}_{i \leq j}} & \text{res}_{j \leq l} \text{res}_{i \leq j} \\ \downarrow \text{res}_{k \leq l} \text{res}_{i \leq j \leq k} & & \downarrow \text{res}_{i \leq j \leq l} \\ \text{res}_{k \leq l} \text{res}_{i \leq k} & \xrightarrow{\text{res}_{i \leq k \leq l}} & \text{res}_{i \leq l} \end{array}$$

commutes.

Proof. Observe that $i \leq j \leq k \leq l$ implies there is an apartment containing all of i, j, k and l . We begin by defining $\text{res}_{i \leq j \leq k}$. Let $r_{i \leq j \leq k} : \bar{G}_{i \leq k} \rightarrow \bar{G}_{j \leq k}$ and $s_{i \leq j \leq k} : \bar{G}_{i \leq k} \hookrightarrow \bar{G}_{i \leq j}$ be the pull-back of $r_{i \leq j} : \bar{G}_{i \leq j} \rightarrow \bar{G}_j$ and $s_{j \leq k} : \bar{G}_{j \leq k} \rightarrow \bar{G}_k$ with domain $\bar{G}_{i \leq k}$; in particular,

$$r_{i \leq k} = r_{j \leq k} \circ r_{i \leq j \leq k} \quad (7)$$

and

$$s_{i \leq k} = s_{i \leq j} \circ s_{i \leq j \leq k}. \quad (8)$$

See [Lan96, Prop.9.22] for the existence of such a pull-back. The situation is summarized by the following diagramme, in which the square is cartesian and all triangles commute.

$$\begin{array}{ccccc} & & \bar{G}_{i \leq k} & & \\ & \swarrow s_{i \leq k} & \text{---} & \searrow r_{i \leq k} & \\ & & \bar{G}_{i \leq j} & \xrightarrow{s_{i \leq j \leq k} \quad r_{i \leq j \leq k}} & \bar{G}_{j \leq k} \\ & \swarrow s_{i \leq j} & \text{---} & \searrow s_{j \leq k} & \\ \bar{G}_i & & \bar{G}_j & & \bar{G}_k \end{array} \quad (9)$$

Observe that all maps r . are smooth projective and all maps s . are affine closed immersions.

To define $\text{res}_{i \leq j \leq k}$ we begin by observing that $d_{i \leq k} = d_{i \leq j} + d_{j \leq k}$ and that Tate twists commute with everything below. Thus,

$$\begin{aligned} \text{res}_{j \leq k} \text{res}_{i \leq j} &= r_{j \leq k!} s_{j \leq k}^* (d_{j \leq k}) r_{i \leq j!} s_{i \leq j}^* (d_{i \leq j}) \\ &= r_{j \leq k!} s_{j \leq k}^* r_{i \leq j!} s_{i \leq j}^* (d_{i \leq k}). \end{aligned}$$

Applying the smooth base-change theorem for direct images with compact supports for ℓ -adic sheaves (see [SGA4, Exposé XVII, §5.2]) to the cartesian square in Diagramme 9, it follows that the base-change natural transformation

$$\varphi_{i \leq j \leq k} : s_{j \leq k}^* r_{i \leq j!} \rightarrow r_{i \leq j \leq k!} s_{i \leq j \leq k}^* \quad (10)$$

is an isomorphism of functors. Thus,

$$\begin{array}{c} r_{j \leq k!} s_{j \leq k}^* r_{i \leq j!} s_{i \leq j}^* (d_{i \leq k}) \\ \downarrow r_{j \leq k!} \varphi_{i \leq j \leq k} s_{i \leq j}^* (d_{i \leq k}) \\ r_{j \leq k!} r_{i \leq j \leq k!} s_{i \leq j \leq k}^* s_{i \leq j}^* (d_{i \leq k}) \end{array} \quad (11)$$

is a natural isomorphism. Let $\rho_{i \leq j \leq k} : r_{j \leq k!} r_{i \leq j \leq k!} \rightarrow r_{i \leq k!}$ be the natural isomorphism determined by Equation 7; these isomorphisms satisfy a cocycle condition (see [SGA4, Exposé XVII, Thm 5.1.8(a)(i)]). Likewise, let $\sigma_{i \leq j \leq k} : s_{i \leq j \leq k}^* s_{j \leq k}^* \rightarrow s_{i \leq k}^*$ be the natural isomorphism determined by Equation 8; these isomorphisms satisfy the analogous cocycle condition. Now, the following diagramme commutes.

$$\begin{array}{ccc} r_{j \leq k!} r_{i \leq j \leq k!} s_{i \leq j \leq k}^* s_{i \leq j}^* (d_{i \leq k}) & \xleftarrow{\rho_{i \leq j \leq k} s_{i \leq j \leq k}^* s_{i \leq j}^* (d_{i \leq k})} & r_{i \leq k!} s_{i \leq j \leq k}^* s_{i \leq j}^* (d_{i \leq k}) \\ \uparrow r_{j \leq k!} r_{i \leq j \leq k!} \sigma_{i \leq j \leq k} (d_{i \leq k}) & & \uparrow r_{i \leq k!} \sigma_{i \leq j \leq k} (d_{i \leq k}) \\ r_{j \leq k!} r_{i \leq j \leq k!} s_{i \leq k}^* (d_{i \leq k}) & \xleftarrow{\rho_{i \leq j \leq k} s_{j \leq k}^* (d_{i \leq k})} & r_{i \leq k!} s_{i \leq k}^* (d_{i \leq k}) \end{array} \quad (12)$$

Define $\text{res}_{i \leq j \leq k}$ by composing Diagramme 11 with Diagramme 12 in the obvious manner. It is clearly a natural isomorphism as it is defined by composing natural isomorphisms.

Having defined $\text{res}_{i \leq j \leq k}$ we now turn to the remaining part of Proposition 1.6. Using the same procedure as above, let $r_{j \leq k \leq l} : \bar{G}_{j \leq l} \rightarrow \bar{G}_{j \leq k}$ and $s_{j \leq k \leq l} : \bar{G}_{j \leq l} \hookrightarrow \bar{G}_{j \leq k}$ be the pull-back of $r_{j \leq k} : \bar{G}_{j \leq k} \rightarrow \bar{G}_k$ and $s_{k \leq l} : \bar{G}_{k \leq l} \rightarrow \bar{G}_l$ with domain $\bar{G}_{j \leq l}$; in particular, $r_{j \leq l} = r_{k \leq l} \circ r_{j \leq k \leq l}$ and $s_{j \leq l} = s_{j \leq k \leq l} \circ s_{j \leq k \leq l}$. Likewise, let $r_{i \leq j \leq k \leq l} : \bar{G}_{i \leq l} \rightarrow \bar{G}_{i \leq j}$ and $s_{i \leq j \leq k \leq l} : \bar{G}_{i \leq l} \hookrightarrow \bar{G}_{i \leq k}$ be the pull-back of $r_{i \leq j \leq k} : \bar{G}_{i \leq k} \rightarrow \bar{G}_{j \leq k}$ and $s_{j \leq k \leq l} : \bar{G}_{j \leq l} \rightarrow \bar{G}_{j \leq k}$ with domain $\bar{G}_{i \leq l}$; in particular, $r_{i \leq l} = r_{j \leq k \leq l} \circ r_{i \leq j \leq k \leq l}$ and $s_{i \leq k} = s_{i \leq j \leq k \leq l} \circ s_{i \leq j \leq k}$. The

situation is summarized by Diagramme 13, in which all squares are cartesian. (Please note that the group scheme \bar{G}_l and all related morphisms are omitted in this diagramme for reasons of space.) As above, observe that all maps r . are smooth projective and all maps s . are affine closed immersions. The bottom part of Diagramme 13 is obtained by pushing-out, which is possible exactly because the maps r . are smooth projective and all maps s . are affine closed immersions!

$$\begin{array}{ccccc}
 & & \bar{G}_{i \leq k} & & \\
 & \swarrow^{s_{i \leq j \leq k}} & & \searrow^{r_{i \leq j \leq k}} & \\
 & \bar{G}_{i \leq j} & & \bar{G}_{j \leq k} & \\
 \swarrow^{s_{i \leq j}} & & & & \searrow^{r_{j \leq k}} \\
 \bar{G}_i & & \bar{G}_j & & \bar{G}_k \\
 \swarrow^{r_{i \leq j}} & & \swarrow^{s_{j \leq k}} & & \\
 \bar{G}_i / \bar{G}_{i \leq j} & & \bar{G}_j / \bar{G}_{j \leq k} & &
 \end{array} \tag{13}$$

The cocycle relation for the restriction functors is obtained by repeated application of [SGA4, Exposé XVII, Thm 4.4] to Diagramme 13 (with the group scheme \bar{G}_l and all related morphisms replaced). \square

1.5 PARABOLIC RESTRICTION ON THE LEVEL OF REDUCTIVE QUOTIENTS

In Section 2.1 we will need the following consequence of Proposition 1.6. Let $P \subseteq G$ be a parabolic subgroup with reductive quotient L . We fix an imbedding of buildings $I(L, \mathbb{K}) \rightarrow I(G, \mathbb{K})$ (cf. [Lan00]). We will write i_G for the image of a facet i of $I(L, \mathbb{K})$ under this embedding. Let i be any facet of $I(L, \mathbb{K})$. Then \bar{L}_i is a Levi subgroup of \bar{G}_{i_G} . By [Lan96, 9.22] there is a unique facet i_P in $I(G, \mathbb{K})$ such that $i_G \leq i_P$ and $\bar{L}_i = \bar{G}_{i_P}$ and $\bar{G}_{i_G \leq i_P}$ is the schematic intersection of \bar{G}_{i_G} with P in G_{i_G} . (See Section 1.4 for the definition of $\bar{G}_{i_G \leq i_P}$.)

LEMMA 1.7. *Let P be a parabolic subgroup of G with levi component L . With notation as above, there is an isomorphism of functors*

$$\text{res}_{i \leq j}^P : \text{res}_{j_P \leq i_P} \text{res}_{i_G \leq i_P} \rightarrow \text{res}_{j_G \leq j_P} \text{res}_{i_G \leq j_G}$$

such that

$$\begin{array}{ccc}
 \text{res}_{j_P \leq k_P} \text{res}_{i_P \leq j_P} \text{res}_{i_G \leq i_P} & \xrightarrow{\text{res}_{i_P \leq j_P \leq k_P} \text{res}_{i_G \leq i_P}} & \text{res}_{i_P \leq k_P} \text{res}_{i_G \leq i_P} \\
 \downarrow \text{res}_{j_P \leq k_P} \text{res}_{i \leq j}^P & & \downarrow \text{res}_{i \leq k}^P \\
 \text{res}_{j_P \leq k_P} \text{res}_{j_G \leq j_P} \text{res}_{i_G \leq j_G} & & \text{res}_{i_G \leq k_G} \\
 \downarrow \text{res}_{j \leq k}^P \text{res}_{j_P \leq k_P} & & \downarrow \\
 \text{res}_{k_G \leq k_P} \text{res}_{j_G \leq k_G} \text{res}_{i_G \leq j_G} & \xrightarrow{\text{res}_{k_G \leq k_P} \text{res}_{i_G \leq j_G \leq k_G}} & \text{res}_{k_G \leq k_P} \text{res}_{i_G \leq k_G}
 \end{array}$$

commutes for all facets i, j and k of $I(L, \mathbb{K})$ such that $i \leq j \leq k$.

Proof. As the notation perhaps suggests, the natural transformation $\text{res}_{i \leq j}^P$ is defined using Definition 1.3; specifically,

$$\text{res}_{i \leq j}^P := \text{res}_{i_G \leq j_G \leq j_P}^{-1} \circ \text{res}_{i_G \leq i_P \leq j_P}. \quad (14)$$

This is clearly an isomorphism of functors. The property appearing in Lemma 1.7 follows from Proposition 1.6. \square

1.6 CONJUGATION

Let $m : G \times G \rightarrow G$ be conjugation over \mathbb{K} . Recall that the Bruhat-Tits building $I(G, \mathbb{K})$ is equipped with an action of $G(\mathbb{K})$ which we indicate by

$$\begin{aligned}
 G(\mathbb{K}) \times I(G, \mathbb{K}) &\rightarrow I(G, \mathbb{K}) \\
 (g, i) &\mapsto gi.
 \end{aligned}$$

We will also write ig for $g^{-1}i$.

Fix an element g of $G(\mathbb{K})$ and let $m(g) : G \rightarrow G$ be the morphism given by $m(g)(h) = m(g, h)$ for $h \in G(\mathbb{K})$ (recall that \mathbb{k} is algebraically closed, see Section 1.1). Fix a facet i and recall that G_i and G_{gi} are smooth integral models of G . Since $m(g)(G_i(\mathfrak{o}_{\mathbb{K}})) = G_{gi}(\mathfrak{o}_{\mathbb{K}})$, it follows from the Extension Principle (cf. [BT84, 1.7]) that the isomorphism $m(g) : G \rightarrow G$ of group schemes over \mathbb{K} extends to an isomorphism $m(g)_i : G_i \rightarrow G_{gi}$ of group schemes over $\mathfrak{o}_{\mathbb{K}}$. Restricting to special fibres, $m(g)_i$ defines an isomorphism $\bar{m}(g)_i : \bar{G}_i \rightarrow \bar{G}_{gi}$ of reductive quotients. Restricting to case when g in an element of $G_i(\mathfrak{o}_{\mathbb{K}})$ we obtain a family of isomorphisms

$$\bar{m}(g)_i : \bar{G}_i \rightarrow \bar{G}_i$$

which together define conjugation $\bar{m}_i : \bar{G}_i \times \bar{G}_i \rightarrow \bar{G}_i$ on the level of reductive quotients. If h is an element of $G_i(\mathfrak{o}_{\mathbb{K}})$ then $\bar{m}_i(\rho_i(h)) = \bar{m}(h)_i$, with ρ_i as defined in Section 1.2.

LEMMA 1.8. *Let g be an element of $G(\mathbb{K})$ and let i, j be facets of $I(G, \mathbb{K})$ with $i \leq j$. Then there is an isomorphism of functors in $D_c^b(\bar{G}_{gi}; \bar{\mathbb{Q}}_\ell)$*

$$\text{res}_{i \leq j}^g : \text{res}_{i \leq j} \bar{m}(g)_i^* \cong \bar{m}(g)_j^* \text{res}_{gi \leq gj}$$

such that

$$\begin{array}{ccc} \text{res}_{j \leq k} \text{res}_{i \leq j} \bar{m}(g)_i^* & \xrightarrow{\text{res}_{i \leq j \leq k} \bar{m}(g)_i^*} & \text{res}_{i \leq k} \bar{m}(g)_i^* \\ \downarrow \text{res}_{j \leq k} \text{res}_{i \leq j}^g & & \downarrow \text{res}_{i \leq k}^g \\ \text{res}_{j \leq k} \bar{m}(g)_j^* \text{res}_{gi \leq gj} & & \\ \downarrow \text{res}_{j \leq k}^g \text{res}_{gi \leq gj} & & \\ \bar{m}(g)_k^* \text{res}_{gj \leq gk} \text{res}_{gi \leq gj} & \xrightarrow{\bar{m}(g)_k^* \text{res}_{gi \leq gj \leq gk}} & \bar{m}(g)_k^* \text{res}_{gi \leq gk} \end{array}$$

commutes for all facets i, j and k of $I(G, \mathbb{K})$ such that $i \leq j \leq k$.

Proof. Observe that

$$\text{res}_{i \leq j} \bar{m}(g)_i^* = r_{i \leq j!} s_{i \leq j}^* \bar{m}(g)_i^* (d_{i \leq j}), \quad (15)$$

by Definition 1.3. Now, consider the following commutative diagramme, where $\bar{m}(g)_{i \leq j}$ is the isomorphism of special fibres obtained by restricting $\bar{m}(g)_j$ to $\bar{G}_{i \leq j}$.

$$\begin{array}{ccccc} \bar{G}_i & \xleftarrow{s_{i \leq j}} & \bar{G}_{i \leq j} & \xrightarrow{r_{i \leq j}} & \bar{G}_j \\ \bar{m}(g)_i \downarrow & & \downarrow \bar{m}(g)_{i \leq j} & & \downarrow \bar{m}(g)_j \\ \bar{G}_{gi} & \xleftarrow{s_{gi \leq gj}} & \bar{G}_{gi \leq gj} & \xrightarrow{r_{gi \leq gj}} & \bar{G}_{gj} \end{array} \quad (16)$$

Since the left-hand square in Diagramme 16 commutes (by construction) we have the following natural isomorphisms in $D(\bar{G}_{gi}; \bar{\mathbb{Q}}_\ell)$.

$$s_{i \leq j}^* \bar{m}(g)_i^* \cong (\bar{m}(g)_i \circ s_{i \leq j})^* = (s_{gi \leq gj} \circ \bar{m}(g)_{i \leq j})^* \cong \bar{m}(g)_{i \leq j}^* s_{gi \leq gj}^* \quad (17)$$

Applying the smooth base-change theorem for direct images with compact support for ℓ -adic sheaves (see [SGA4, Exposé XVII, §5.2], see also [Eke90, Thm 6.3 (c)]) to the right-hand square in Diagramme 16 (which is indeed cartesian) it follows that the base-change natural transformation

$$\varphi_{i \leq j}^g : \bar{m}(g)_j^* r_{gi \leq gj!} \rightarrow r_{i \leq j!} \bar{m}(g)_{i \leq j}^* \quad (18)$$

is an isomorphism of functors. Since

$$r_{gi \leq gj!} s_{gi \leq gj}^* (d_{gi \leq gj}) = \text{res}_{gi \leq gj}, \quad (19)$$

by Definition 1.3 again, we define $\text{res}_{i \leq j}^g$ by composing the isomorphisms appearing in Equations 15, 17, 18 and 19 in the obvious manner. The property appearing in Lemma 1.8 is now a direct result Proposition 1.6 (which in turn follows from [SGA4, Exposé XVII, §5]). \square

1.7 EQUIVARIANT PERVERSE SHEAVES

In this Subsection we discuss a fundamental result concerning the category of equivariant perverse sheaves which plays a key role in the definition of parabolic induction *as a functor*.

Let X be an algebraic variety over \mathbb{k} and let $\mathcal{M}X$ denote the category of perverse sheaves on X . Let $m : P \times X \rightarrow X$ be an action of a connected algebraic group on X over \mathbb{k} . Recall from [Lus84, §0] that a perverse sheaf F on X is equivariant if there is an isomorphism

$$\mu_F : m^* F \rightarrow \text{pr}^* F \quad (20)$$

in $D_c^b(P \times X; \bar{\mathbb{Q}}_\ell)$ such that $e^* \mu_F = \text{id}_F$, where $e : X \rightarrow P \times X$ is defined by $x \mapsto (1, x)$ and $\text{pr} : P \times X \rightarrow X$ is projection onto the second component. As observed in [Lus84, §0], if F is an equivariant perverse sheaf, then there is exactly one such isomorphism μ_F .

Fix $h \in P(\mathbb{k})$ and let $e_x : X \rightarrow P \times X$ be the morphism determined by $e_x(y) = (x, y)$ (recall that \mathbb{k} is algebraically closed, see Section 1.1). Define

$$\mu_F(x) = e_{x^{-1}}^* \mu_F. \quad (21)$$

Set $m(x^{-1}) := m \circ e_{x^{-1}}$. Then $\text{pr} \circ e_{x^{-1}} = \text{id}$ and Equation 21 defines a family of isomorphisms

$$\forall x \in P(\mathbb{k}), \quad \mu_F(x) : m(x^{-1})^* F \rightarrow F, \quad (22)$$

such that $\mu_F(1) = \text{id}_F$ and

$$\begin{array}{ccc} m(x^{-1})^* m(y^{-1})^* F & \xrightarrow{m(x^{-1})^* \mu_F(y)} & m(x^{-1})^* F \\ \downarrow & & \downarrow \mu_F(x) \\ m((xy)^{-1})^* F & \xrightarrow{\mu_F(xy)} & F \end{array} \quad (23)$$

commutes, for all $h, h' \in P(\mathbb{k})$, where the left-hand arrow refers to the inverse of the isomorphism in $D_c^b(P, \bar{\mathbb{Q}}_\ell)$ determined by the isomorphism of functors $m((xy)^{-1})^* \rightarrow m(x^{-1})^* m(y^{-1})^*$. (We will use Equations 21, 22 and 23 in Section 2.2.)

We will also say that a morphism $\phi : F_1 \rightarrow F_2$ in $\mathcal{M}X$ is *equivariant* if F_1 and F_2 are equivariant perverse sheaves and the following diagramme commutes.

$$\begin{array}{ccc} m^* F_1 & \xrightarrow{m^* \phi} & m^* F_2 \\ \mu_{F_1} \downarrow & & \downarrow \mu_{F_2} \\ \text{pr}^* F_1 & \xrightarrow{\text{pr}^* \phi} & \text{pr}^* F \end{array}$$

Note that this definition makes implicit use of the uniqueness of the isomorphisms $m^*F_1 \rightarrow \text{pr}^*F_1$ and $m^*F_2 \rightarrow \text{pr}^*F_2$ as above. Since id_F is equivariant if F is equivariant and since the composition of equivariant morphisms is equivariant, it follows that equivariant perverse sheaves define a category, with morphisms as above, henceforth denoted $\mathcal{M}_P X$.

PROPOSITION 1.9. *Let $f : X \rightarrow Y$ be a locally trivial principal fibre space with group P and suppose P is connected. Let F_X be a perverse sheaf on X . Then F_X is equivariant if and only if $F_X \cong f^*[\dim P]F_Y$ for some perverse sheaf F_Y on Y .*

Proof. (This result is presented in [Lus85, 1.9.3].) By the definition of a locally trivial principal fibre space there is an open covering $Y = \cup_n Y_n$ and isomorphisms $t_n : f^{-1}Y_n \rightarrow P \times Y_n$ such that f is given locally by $f_n = \text{pr} \circ t_n$ (so $f_n : f^{-1}Y_n \rightarrow Y_n$) and the action m_X of P on X is given locally by $t_n \circ m_n = (m_P \times \text{id}) \circ (\text{id} \times t_n)$ (so $m_n : P \times f^{-1}Y_n \rightarrow f^{-1}Y_n$). To simplify notation slightly, let X_n denote $f^{-1}Y_n$; also, let $j_n : Y_n \rightarrow Y$ and $i_n : X_n \rightarrow X$ denote inclusions.

First, suppose F_Y is a perverse sheaf on Y . Let $F_X = f^*[\dim P]F_Y$. Since f is smooth with fibres isomorphic to P (so the relative dimension of f is $\dim P$) and since P is geometrically connected (recall that \mathbb{k} is algebraically closed), it follows from [BBD, Prop 4.2.5] that F_X is a perverse sheaf. To show that F_X is equivariant we must find an isomorphism $\mu : m^*F_X \rightarrow \text{pr}^*F_X$ in $D_c^b(P \times X; \mathbb{Q}_\ell)$ such that $e^*\mu = \text{id}_{F_X}$, where $e : X \rightarrow P \times X$ is the section defined by $x \mapsto (1, x)$. To see this, consider the restriction of m^*F_X to $P \times X_n$:

$$\begin{aligned} (m^*F_X)|_{P \times X_n} &= (\text{id} \times i_n)^* m^* f^* F_Y[\dim P] \\ &\cong (f \circ m \circ \text{id} \times i_n)^* F_Y[\dim P] \\ &= (f \circ i_n \circ m_n)^* F_Y[\dim P]. \end{aligned}$$

On the other hand, the restriction of pr^*F_X to $P \times X_n$ is

$$\begin{aligned} (\text{pr}^*F_X)|_{P \times X_n} &= (\text{id} \times i_n)^* \text{pr}^* f^* F_Y[\dim P] \\ &\cong (f \circ \text{pr} \circ \text{id} \times i_n)^* F_Y[\dim P] \\ &= (f \circ i_n \circ \text{pr})^* F_Y[\dim P]. \end{aligned}$$

Since $f \circ i_n \circ m_n = f \circ i_n \circ \text{pr}$, we have $(m^*F_X)|_{P \times X_n} \cong (\text{pr}^*F_X)|_{P \times X_n}$. Since $\cup_n P \times X_n$ is an open cover for $P \times X$, this gives the isomorphism we seek.

Next, suppose $F_X \in \text{obj} \mathcal{M} X$ is equivariant; thus, $F_X \in \text{obj} \mathcal{M}_P X$. Let F_n denote the restriction of F_X to X_n . Since $\text{id} \times i_n$ satisfies the hypotheses of [BBD, Prop 4.2.5], it follows that F_n is a perverse sheaf on X_n . Recall that $m : P \times X \rightarrow X$ is given locally by $m_n : P \times X_n \rightarrow X_n$, as above. Restricting the isomorphism $\mu_{F_X} : m^*F_X \rightarrow \text{pr}^*F_X$ to $P \times X_n$ yields the isomorphism $\mu_{F_n} : m_n^*F_n \rightarrow \text{pr}^*F_n$. Thus, F_n is an equivariant perverse sheaf. Now, let $v_n : Y_n \rightarrow X_n$ be the section of $f_n : X_n \rightarrow Y_n$ corresponding to $1 \in P(\mathbb{k})$ (so v_n is the unique morphism of varieties such that $(t_n \circ v_n)(y) = (1, y)$). Define

$$F'_n := v_n^*F_n[-\dim P].$$

Then $F'_n \in \text{obj} D_c^b(Y_n; \bar{\mathbb{Q}}_\ell)$. By standard glueing arguments, the collection of $F'_n \in \text{obj} D_c^b(Y_n; \bar{\mathbb{Q}}_\ell)$, as Y_n ranges over the open cover of Y fixed above, uniquely determines an object F_Y of $D_c^b(Y; \bar{\mathbb{Q}}_\ell)$.

It remains to be shown that F_Y is a perverse sheaf. Again, we work locally. For each such n ,

$$\begin{aligned} f_n^*[\dim P]F'_n &= f_n^*[\dim P]v_n^*F_n[-\dim P] \\ &\cong (v_n \circ f_n)^*F_n. \end{aligned}$$

Let $u_n : X_n \rightarrow P \times X_n$ be the section of $m_n : P \times X_n \rightarrow X_n$ corresponding to 1 (so u_n is the unique morphism of varieties such that $(\text{id} \times t_n) \circ u_n \circ t_n^{-1}(h, y) = (h, 1, y)$). The domain of $u_n^*\mu_{F_n}$ is $u_n^*m_n^*F_n \cong (m_n \circ u_n)^*F_n = F_n$, since u_n is a section of m_n ; the codomain of $u_n^*\mu_{F_n}$ is $u_n^*\text{pr}^*F_n \cong (\text{pr} \circ u_n)^*F_n$. Since $\text{pr} \circ u_n = v_n \circ f_n$, it follows that

$$u_n^*\mu_{F_n} : F_n \rightarrow f_n^*[\dim P]F'_n$$

is an isomorphism in $D_c^b(X_n; \bar{\mathbb{Q}}_\ell)$. By [BBD, Prop 4.2.5] and the fact that $\mathcal{M}X_n$ is stable in $D_c^b(X_n; \bar{\mathbb{Q}}_\ell)$ under isomorphisms, it follows that $F'_n \in \text{obj} \mathcal{M}Y_n$. By standard glueing arguments, the collection of isomorphisms $u_n^*\mu_{F_n} \in \text{mor} \mathcal{M}X_n$, as X_n ranges over the open cover of X fixed above, uniquely determines an isomorphism in $\text{Hom}_{D_c^b(X; \bar{\mathbb{Q}}_\ell)}(F_X, f_n^*[\dim P]F_Y)$. From [BBD, Prop 4.2.5] it follows that $F_Y \in \text{obj} \mathcal{M}Y$. This completes the proof of Proposition 1.9. \square

PROPOSITION 1.10. *Let $f : X \rightarrow Y$ be a locally trivial principal fibre space with group P and suppose P is connected. Then $f^*[\dim P] : \mathcal{M}Y \rightarrow \mathcal{M}_P X$ is an equivalence of categories and $\mathcal{M}_P X$ is a thick subcategory of $\mathcal{M}X$.*

Proof. By [BBD, Prop 4.2.5] we know that $f^*[\dim P] : \mathcal{M}Y \rightarrow \mathcal{M}X$ is full and faithful. Let F_Y be a perverse sheaf on Y . From the proof of Proposition 1.9 we have seen that $F_X := f^*[\dim P]F_Y$ is an equivariant perverse sheaf on X and that $f^*[\dim P]\phi$ is an equivariant morphism in $\mathcal{M}X$ for each morphism ϕ in $\mathcal{M}Y$. Thus, $f^*[\dim P]$ is a functor from $\mathcal{M}Y$ to $\mathcal{M}_P X$. Thus, $f^*[\dim P] : \mathcal{M}Y \rightarrow \mathcal{M}_P X$ is full and faithful. Proposition 1.9 tells us that this functor is essentially surjective. Thus, $f^*[\dim P]$ is an equivalence. The last clause of Proposition 1.10 follows from [BBD, 4.2.6]. \square

1.8 PARABOLIC INDUCTION ON REDUCTIVE QUOTIENTS

Let H be a reductive algebraic group over \mathbb{k} and let P be a parabolic subgroup of H with levi component L and unipotent radical U . Let $r : P \rightarrow L$ denote the reductive quotient map and let $s : P \rightarrow H$ be inclusion. Equip $X := H \times P$ with the P -action defined by $p \cdot (g, h) = (pg^{-1}, php^{-1})$ and let Y denote the quotient by this action. (This is a variety!) Consider the diagramme

$$L \xleftarrow{a} X \xrightarrow{b} Y \xrightarrow{c} H \tag{24}$$

where $a(g, h) = r(h)$, $b(g, h) = [g, h]$ and $c[g, h] = ghg^{-1}$. Observe that a is smooth with connected fibres of equal dimension $\dim H + \dim U$, which is therefore the relative dimension of a . Observe that b is a locally trivial principal fibre bundle with group P , which is connected. Observe also that c is proper. Let F be an equivariant perverse sheaf on L with respect to conjugation m_L . It follows from [BBD, Prop 4.2.5] that $a^*[\dim a]F$ is a perverse sheaf on X . Moreover, since a is P -equivariant (with respect to the action on X defined above and the action $p \cdot l = r(p)lr(p)^{-1}$ on L) and since F is P -equivariant with respect to the action just defined on L , it follows that $a^*[\dim a]F$ is a P -equivariant perverse sheaf on X . Let

$$F_X = a^*[\dim a]F. \quad (25)$$

Since $b : X \rightarrow Y$ is a locally trivial principle fibre space with group P , and since P is connected, it follows from Proposition 1.9 that there is a some perverse sheaf F_Y on Y such that

$$F_X = b^*[\dim P]F_Y. \quad (26)$$

Note that Proposition 1.9 tells us exactly how to construct the perverse sheaf F_Y . Define

$$\mathrm{ind}_P^H F := c_* F_Y \quad (27)$$

(Since c is proper, this is equal to $c_1 F_Y$.) Notice that, *a priori*, $\mathrm{ind}_P^H F$ is an object of $D_c^b(H; \overline{\mathbb{Q}}_\ell)$; we do not claim that this is a perverse sheaf.

Next, let ϕ be a morphism of P -equivariant perverse sheaves on L . Then $a^*[\dim H]\phi$ is a morphism of P -equivariant perverse sheaves on X (by [BBD, Prop 4.2.5] and arguments as above). Using the equivalence of categories in Proposition 1.9 again, there is a unique (given the choices made above) morphism ϕ_Y of perverse sheaves on Y such that

$$\phi_X = b^*[\dim P]\phi_Y.$$

Let ϕ_X be that morphism. Define

$$\mathrm{ind}_P^H \phi := c_* \phi_Y.$$

Notice that $\mathrm{ind}_P^H \phi$ is an morphism $D_c^b(H; \overline{\mathbb{Q}}_\ell)$. Thus, we have defined a *functor*

$$\mathrm{ind}_P^H : \mathcal{M}_L L \rightarrow D_c^b(H; \overline{\mathbb{Q}}_\ell).$$

In this paper we will use the above construction with $H = \overline{G}_i$, $P = \overline{G}_{i \leq j}$ (see Section 1.4) and $L = \overline{G}_j$, where i and j are facets of $I(G, \mathbb{K})$ and $i \leq j$ in the Bruhat order. In that case we will denote the functor $\mathrm{ind}_{\overline{G}_{i \leq j}}^{\overline{G}_i}$ by $\mathrm{ind}_{i \leq j}$.

LEMMA 1.11. *Let g be an element of $G(\mathbb{K})$ and let i, j be facets of $I(G, \mathbb{K})$ such that $i \leq j$. There is an isomorphism of functors*

$$\mathrm{ind}_{i \leq j} \bar{m}(g)_j^* \cong \bar{m}(g)_i^* \mathrm{ind}_{gi \leq gj}$$

in the category of equivariant perverse sheaves on \overline{G}_j .

Proof. Work locally and use the construction appearing in the proof of Proposition 1.9. \square

1.9 CHARACTER SHEAVES

Let H be a connected reductive algebraic group over \mathbb{k} . We recall that a character sheaf is an irreducible perverse sheaf satisfying any one (and hence all) of the conditions appearing in [Lus85, Prop.2.9]. We also remind the reader that an irreducible perverse sheaf on H is *admissible* if it is an irreducible component of $\text{ind}_P^H F$ for some parabolic subgroup P and some cuspidal perverse sheaf F on the Levi component L of P (see [Lus85, 7.1.10]) (Cuspidal perverse sheaves are defined in [Lus85, 7.1.1].) In [Lus85, Th.23.1] it is shown that, under some extremely mild conditions on H (which are satisfied if $p \geq 7$, for example), these two classes of perverse sheaves coincide.

Regarding parabolic induction as defined in Section 1.8, if F is a character sheaf on L then F is equivariant for the conjugation action (see [Lus85, Prop.2.18]), in which case $\text{ind}_P^H F$ is defined. Moreover, in [Lus85, Prop.4.8] it is shown that if F is a character sheaf on L then $\text{ind}_P^H F$ is a finite direct sum of character sheaves on H , and therefore equivariant for the conjugation action on H . Thus, if F is a finite direct sum of character sheaves on L then $\text{ind}_P^H F$ is a finite direct sum of character sheaves on H . From our treatment of parabolic induction as a functor, we see further that if ϕ is an isomorphism in the category of equivariant perverse sheaves on L and the domain and codomain of ϕ are finite direct sums of character sheaves on L then $\text{ind}_P^H \phi$ is an isomorphism in the category of equivariant perverse sheaves on H . We will use this fact in Section 3.3.

In this paper we will use these facts with $H = \bar{G}_i$ and $P = \bar{G}_{i \leq j}$ where i and j are facets of $I(G, \mathbb{K})$ with $i \leq j$.

1.10 CATEGORIES

We may now introduce the main categories appearing in this paper.

DEFINITION 1.12. Let $\bar{\mathcal{D}}G$ denote the following category.

OBJ: An object A of $\bar{\mathcal{D}}G$ is a family of objects

$$\{A_i \in D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell) \mid i \text{ facet of } I(G, \mathbb{K})\},$$

equipped with a family of isomorphisms

$$\{A_{i \leq j} \in \text{Hom}_{D_c^b(\bar{G}_j; \bar{\mathbb{Q}}_\ell)}(\text{res}_{i \leq j} A_i, A_j) \mid i \leq j \text{ in } I(G, \mathbb{K})\},$$

such that $A_{i \leq i} = \text{id}_{A_i}$ for each facet i of $I(G, \mathbb{K})$, and such that the diagramme

$$\begin{array}{ccc} \text{res}_{j \leq k} \text{res}_{i \leq j} A_i & \xrightarrow{\text{res}_{j \leq k} A_{i \leq j}} & \text{res}_{j \leq k} A_j \\ \text{res}_{i \leq j \leq k} A_i \downarrow & & \downarrow A_{j \leq k} \\ \text{res}_{i \leq k} A_i & \xrightarrow{A_{i \leq k}} & A_k \end{array}$$

is commutative for each triplet i, j, k of facets of $I(G, \mathbb{K})$ such that $i \leq j \leq k$. The isomorphism of sheaves appearing on the left-hand side of this diagramme is determined by the isomorphism of functors appearing in Proposition 1.6.

MOR: A morphism $\phi \in \text{Hom}_{\bar{\mathcal{D}}G}(A, B)$ in the category $\bar{\mathcal{D}}G$ is a family

$$\left\{ \phi_i \in \text{Hom}_{D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell)}(A_i, B_i) \mid i \text{ facet of } I(G, \mathbb{K}) \right\},$$

such that the diagramme

$$\begin{array}{ccc} \text{res}_{i \leq j} A_i & \xrightarrow{\text{res}_{i \leq j} \phi_i} & \text{res}_{i \leq j} B_i \\ A_{i \leq j} \downarrow & & \downarrow B_{i \leq j} \\ A_j & \xrightarrow{\phi_j} & B_j \end{array}$$

is commutative for each pair i, j of facets of $I(G, \mathbb{K})$ such that $i \leq j$.

COM: If u and v are morphisms in $\bar{\mathcal{D}}G$ then the composition $u \circ v$ is defined in $\bar{\mathcal{D}}G$ by $(u \circ v)_i = u_i \circ v_i$ for each facet i of $I(G, \mathbb{K})$.

ID: For any object A , the identity $\text{id}_A : A \rightarrow A$ is defined by $(\text{id}_A)_i = \text{id}_{A_i}$ for each facet i of $I(G, \mathbb{K})$.

Let $\bar{\mathcal{C}}G$ denote the full subcategory of $\bar{\mathcal{D}}G$ consisting of objects $A \in \text{obj} \bar{\mathcal{D}}G$ such that A_i is a perverse sheaf for \bar{G}_i for each facet i of $I(G, \mathbb{K})$. A *coefficient system (for G)* is an object of $\bar{\mathcal{C}}G$.

2 CUSPIDAL COEFFICIENT SYSTEMS

Let \mathbb{K} , $\mathfrak{o}_{\mathbb{K}}$ and \mathbb{k} be as in Section 1. Likewise, let G be a connected reductive linear algebraic group over \mathbb{K} satisfying the conditions of Section 1 (and in particular, Section 1.3).

2.1 PARABOLIC RESTRICTION

Let $P \subseteq G$ be a parabolic subgroup with reductive quotient L . Recall the notation of Section 1.5.

PROPOSITION 2.1. *There is a canonical functor $\text{res}_P^G : \bar{\mathcal{D}}G \rightarrow \bar{\mathcal{D}}L$ such that for each facet i of $I(L, \mathbb{K})$, $(\text{res}_P^G A)_i = \text{res}_{i_G \leq i_P} A_{i_G}$, considered as an object in $D_c^b(\bar{L}_i, \bar{\mathbb{Q}}_\ell)$.*

Proof. The functor is defined as follows. Let i and j be facets of $I(L, \mathbb{K})$ with $i \leq j$. For any object A in $\bar{\mathcal{D}}G$, define

$$(\text{res}_P^G A)_i := \text{res}_{i_G \leq i_P} A_{i_G},$$

considered as an object in $D_c^b(\bar{L}_i, \bar{\mathbb{Q}}_\ell)$ using the identification $\bar{G}_{i_P} = \bar{L}_i$; also define

$$(\operatorname{res}_P^G A)_{i \leq j} := \operatorname{res}_{j_G \leq j_P} A_{i_G \leq j_G} \circ \operatorname{res}_{i \leq j}^P A_{i_G},$$

likewise considered as a morphism in $D_c^b(\bar{L}_j, \bar{\mathbb{Q}}_\ell)$. (The isomorphism of functors $\operatorname{res}_{i \leq j}^P$ is defined in Lemma 1.7; observe that $\operatorname{res}_{i \leq j}^P A_{i_G}$ is a isomorphism in $D_c^b(\bar{G}_{j_P}, \bar{\mathbb{Q}}_\ell)$ from $\operatorname{res}_{i_P \leq j_P} \operatorname{res}_{i_G \leq i_P} A_{i_G}$ to $\operatorname{res}_{j_G \leq j_P} \operatorname{res}_{i_G \leq j_G} A_{i_G}$.) For any morphism ϕ in $\bar{D}G$, define

$$(\operatorname{res}_P^G \phi)_i := \operatorname{res}_{i_G \leq i_P} \phi_{i_G},$$

considered as a morphism in $D_c^b(\bar{L}_i, \bar{\mathbb{Q}}_\ell)$.

We must verify that $\operatorname{res}_P^G A$ is an object of $\bar{D}L$ (cf. Definition 1.12(obj)). Using the definition above we see that

$$(\operatorname{res}_P^G A)_{i \leq i} = \operatorname{res}_{i_G \leq i_P} A_{i_G \leq i_G} \circ \operatorname{res}_{i \leq i}^P A_{i_G},$$

for all facets i of $I(L, \mathbb{K})$. By Definition 1.12(obj), $A_{i_G \leq i_G} = \operatorname{id}_{A_{i_G}}$. From the definition of $\operatorname{res}_{i \leq j}^P$ (see Lemma 1.7) and Remark 1.5 it follows that $\operatorname{res}_{i \leq i}^P A_{i_G} = \operatorname{id}_{\operatorname{res}_{i_G \leq i_P} A_{i_G}}$. Thus,

$$(\operatorname{res}_P^G A)_{i \leq i} = \operatorname{id}_{(\operatorname{res}_P^G A)_i}.$$

Having shown that $\operatorname{res}_P^G A$ satisfies the first condition set out in Definition 1.12(obj), we now turn to the second part of Definition 1.12(obj). Suppose i, j and k are facets of $I(L, \mathbb{K})$ with $i \leq j \leq k$. To show that $\operatorname{res}_P^G A$ satisfies the second condition appearing in Definition 1.12(obj) we must show that the following diagramme commutes.

$$\begin{array}{ccc} \operatorname{res}_{j \leq k} \operatorname{res}_{i \leq j} (\operatorname{res}_P^G A)_i & \xrightarrow{\operatorname{res}_{j \leq k} (\operatorname{res}_P^G A)_{i \leq j}} & \operatorname{res}_{j \leq k} (\operatorname{res}_P^G A)_j \\ \downarrow \operatorname{res}_{i \leq j \leq k} (\operatorname{res}_P^G A)_{i \leq j} & & \downarrow (\operatorname{res}_P^G A)_{j \leq k} \\ \operatorname{res}_{i \leq k} (\operatorname{res}_P^G A)_i & \xrightarrow{(\operatorname{res}_P^G A)_{i \leq k}} & (\operatorname{res}_P^G A)_k \end{array} \quad (28)$$

To do this, we begin by recalling (from Section 1.5) that $\bar{L}_i = \bar{G}_{i_P}$ (likewise, $\bar{L}_j = \bar{G}_{j_P}$ and $\bar{L}_k = \bar{G}_{k_P}$). Together with the definition of $(\operatorname{res}_P^G A)_i$ given in above, the top left-hand corner of Diagramme 28 may be re-written as follows:

$$\operatorname{res}_{j \leq k} \operatorname{res}_{i \leq j} (\operatorname{res}_P^G A)_i = \operatorname{res}_{j_P \leq k_P} \operatorname{res}_{i_P \leq j_P} \operatorname{res}_{i_G \leq i_P} A_{i_G},$$

and likewise for all the corners of Diagramme 28. Now, to show that Diagramme 28 commutes, consider the diagramme below, in which the outer square is exactly Diagramme 28. (To save space we temporarily write $R_{j_G}^{i_G}$ for

$\text{res}_{i_G \leq j_G}$, etc...)

$$\begin{array}{ccc}
R_{k_P}^{j_P} R_{j_P}^{i_P} R_{i_P}^{i_G} A_{i_G} & \xrightarrow{\hspace{10em}} & R_{k_P}^{j_P} R_{j_P}^{j_G} A_{j_G} \\
\downarrow & \searrow 4. & \swarrow 2. \\
& R_{k_P}^{k_G} R_{k_G}^{j_G} R_{j_G}^{i_G} A_{i_G} \longrightarrow R_{k_P}^{k_G} R_{k_G}^{j_G} A_{j_G} & \\
& \downarrow & \downarrow \\
& R_{k_P}^{k_G} R_{k_G}^{i_G} A_{i_G} \longrightarrow R_{k_P}^{k_G} A_{k_G} & \\
& \swarrow 3. & \nwarrow 1. \\
R_{k_P}^{i_P} R_{i_P}^{i_G} A_{i_G} & \xrightarrow{\hspace{10em}} & R_{k_P}^{k_G} A_{k_G}
\end{array}$$

The inner square is the result of applying the functor $\text{res}_{k_G \leq k_P}$ to the relevant form of the commuting square appearing in Definition 1.12(obj), and therefore commutes, since A is an object of $\bar{D}G$. The arrow marked 1. is the identity. The arrow marked 2. is $\text{res}_{j \leq k}^P A_{j_G}$, so the right-hand square commutes by virtue of the definition of $(\text{res}_{j \leq k}^P A)_{j \leq k}$; likewise, the arrow marked 3. is $\text{res}_{i \leq k}^P A_{i_G}$, so the bottom square commutes by virtue of the definition of $(\text{res}_{i \leq k}^P A)_{i \leq k}$. The arrow marked 4. is $\text{res}_{j \leq k}^P \text{res}_{i_G \leq j_G} A_{i_G} \circ \text{res}_{j \leq k}^P \text{res}_{i \leq j} A_{i_G}$ and the top and left-hand squares commute by Lemma 1.7. This, the outer square in the diagramme above commutes. Therefore, Diagramme 28 commutes. This concludes the demonstration that $\text{res}_{i \leq j}^G A$ is an object in $\bar{D}L$.

Suppose $\phi : A \rightarrow B$ is a morphism in $\bar{D}G$ and let i and j be facets of $I(L, \mathbb{K})$ such that $i \leq j$. In order to show that $\text{res}_{i \leq j}^G \phi$ is a morphism in $\bar{D}L$ we must show that the following diagramme commutes.

$$\begin{array}{ccc}
\text{res}_{i \leq j}(\text{res}_{i \leq j}^G A)_i & \xrightarrow{\text{res}_{i \leq j}(\text{res}_{i \leq j}^G \phi)_i} & \text{res}_{i \leq j}(\text{res}_{i \leq j}^G B)_i & (29) \\
\downarrow (\text{res}_{i \leq j}^G A)_{i \leq j} & & \downarrow (\text{res}_{i \leq j}^G B)_{i \leq j} & \\
(\text{res}_{i \leq j}^G A)_j & \xrightarrow{(\text{res}_{i \leq j}^G \phi)_j} & (\text{res}_{i \leq j}^G B)_j &
\end{array}$$

As above, we begin by translating this from a statement about morphisms of sheaves on the reductive quotients of parahoric subgroups of L to a statement about morphisms of sheaves on the reductive quotients of parahoric subgroups of G , where we will see that the diagramme commutes. Thus, for example, the top arrow becomes

$$\text{res}_{i \leq j}(\text{res}_{i \leq j}^G \phi)_i = \text{res}_{i_P \leq j_P} \text{res}_{i_G \leq i_P} \phi_{i_G}.$$

Now, consider the diagramme below, in which the outer square is exactly Diagramme 29. (To save space we temporarily write $R_{j_G}^{i_G}$ for $\text{res}_{i_G \leq j_G}$, etc..., as

above.)

$$\begin{array}{ccccc}
 R_{j_P}^{i_P} R_{i_P}^{i_G} A_{i_G} & \xrightarrow{\hspace{10em}} & R_{j_P}^{i_P} R_{i_P}^{i_G} B_{i_G} & & \\
 \downarrow & \searrow 3. & \swarrow 4. & & \downarrow \\
 & R_{j_P}^{j_G} R_{j_G}^{i_G} A_{i_G} \xrightarrow{0.} R_{j_P}^{j_G} R_{j_G}^{i_G} B_{i_G} & & & \\
 \swarrow 2. & & \searrow 1. & & \\
 R_{j_P}^{j_G} A_{j_G} & \xrightarrow{\hspace{10em}} & R_{j_P}^{j_G} B_{j_G} & &
 \end{array}$$

The arrow marked 0. is $\text{res}_{j_G \leq j_P} \text{res}_{i_G \leq j_G} \phi_{i_G}$, the arrow marked 1. is $\text{res}_{j_G \leq j_P} B_{i_G \leq j_G}$ and the arrow marked 2. is $\text{res}_{j_G \leq j_P} A_{i_G \leq j_G}$; thus, the bottom square is the result of applying the functor $\text{res}_{j_G \leq j_P}$ to the relevant form of the commuting square appearing in Definition 1.12(mor), and therefore commutes since ϕ is a morphism in $\bar{\mathcal{D}}G$. The arrow marked 3. is $\text{res}_{i_{\leq j}^P} A_{i_G}$ and the arrow marked 4. is $\text{res}_{i_{\leq j}^P} B_{i_G}$, so the upper square commutes because $\text{res}_{i_{\leq j}^P}$ is a natural transformation. The left-hand triangle commutes by virtue of the definition of $(\text{res}_{i_{\leq j}^G} A)_{i_{\leq j}}$ and likewise the right-hand triangle commutes by virtue of the definition of $(\text{res}_{i_{\leq j}^G} B)_{i_{\leq j}}$. Therefore, the outer square commutes. This concludes the demonstration that $\text{res}_{i_{\leq j}^G} : \bar{\mathcal{D}}G \rightarrow \bar{\mathcal{D}}L$ is a functor. \square

Let $P \rightarrow G$ be a parabolic subgroup containing Borel B and with Levi component (*i.e.*, maximal reductive quotient) $P \rightarrow L$. Let $Q \rightarrow L$ be a parabolic subgroup containing $B \cap L$ with Levi component $Q \rightarrow M$. Let $P \leftarrow R \rightarrow Q$ be a pull-back of $P \rightarrow L \leftarrow Q$ in the category of group \mathbb{K} -schemes. Then $R \rightarrow G$ is a parabolic subgroup with Levi component $R \rightarrow L$ and $R = QU$, where U is the kernel of $P \rightarrow L$.

PROPOSITION 2.2. *With notation as above, $\text{res}_Q^L \text{res}_P^G \cong \text{res}_R^G$.*

Proof. Proposition 2.2 is a consequence of Proposition 2.1 and Proposition 1.6. \square

Remark 2.3. It was very important for us to keep track of the isomorphisms appearing in Proposition 1.6 in order to have a good definition of $\bar{\mathcal{D}}G$ and in order for us to define parabolic restriction above. However, it is not important for us to keep track of the isomorphism in Proposition 2.2, as we will see below.

2.2 WEAKLY-EQUIVARIANT OBJECTS

In this Section we make extensive use of ideas and notation introduced in Section 1.6.

PROPOSITION 2.4. *For each $g \in G(\mathbb{K})$ there is a canonical functor from $\bar{\mathcal{D}}G$ to $\bar{\mathcal{D}}G$ such that the image of $A \in \text{obj} \bar{\mathcal{D}}$ under this functor is an object ${}^g A$ given by ${}^g A_i := \bar{m}(g^{-1})_i^* A_{g^{-1}i}$ for each facet i of $I(G, \mathbb{K})$. We shall denote this functor by $\bar{m}(g)^* : \bar{\mathcal{D}}G \rightarrow \bar{\mathcal{D}}G$*

Proof. Fix $g \in G(\mathbb{K})$. For any $A \in \text{obj} \bar{D}G$, define ${}^gA \in \text{obj} \bar{D}G$ as follows: for each pair of facets i, j of $I(G, \mathbb{K})$ with $i \leq j$,

$${}^gA_i := \bar{m}(g^{-1})_i^* A_{g^{-1}i}$$

and

$${}^gA_{i \leq j} := \bar{m}(g^{-1})_j^* A_{g^{-1}i \leq g^{-1}j} \circ \text{res}_{g^{-1}i \leq g^{-1}j}^{g^{-1}} A_{g^{-1}i}.$$

(Here, $\text{res}_{g^{-1}i \leq g^{-1}j}^{g^{-1}}$ refers to the natural transformation introduced in Lemma 1.8. Observe that $\text{res}_{g^{-1}i \leq g^{-1}j}^{g^{-1}} A_{g^{-1}i}$ is an isomorphism in $D_c^b(\bar{G}_{g^{-1}j}, \bar{Q}_\ell)$ from $\text{res}_{i \leq j} \bar{m}(g^{-1})_i^* A_{g^{-1}i}$ to $\bar{m}(g^{-1})_j^* \text{res}_{g^{-1}i \leq g^{-1}j} A_{g^{-1}i}$.) Likewise, for any $\phi \in \text{Hom}_{\bar{D}G}(A, B)$ we define ${}^g\phi \in \text{Hom}_{\bar{D}G}({}^gA, {}^gB)$ by

$${}^g\phi_i := \bar{m}(g^{-1})_i^* \phi_{g^{-1}i},$$

for each facet i of $I(G, \mathbb{K})$.

From the proof of Lemma 1.8, we see that $\text{res}_{g^{-1}i \leq g^{-1}i} A_{g^{-1}i} = \text{id}_{\bar{m}(g^{-1})_i^* A_{g^{-1}i}}$. From Definition 1.12(obj) we see that $A_{g^{-1}i \leq g^{-1}i} = \text{id}_{A_{g^{-1}i}}$. Thus, using Proposition 2.4 (and Remark 1.5) it follows that

$$\begin{aligned} {}^gA_{i \leq i} &= \bar{m}(g^{-1})_i^* A_{g^{-1}i \leq g^{-1}i} \circ \text{res}_{g^{-1}i \leq g^{-1}i}^{g^{-1}} A_{g^{-1}i} \\ &= \bar{m}(g^{-1})_i^* \text{id}_{A_{g^{-1}i}} \circ \text{id}_{\bar{m}(g^{-1})_i^* A_{g^{-1}i}} \\ &= \text{id}_{\bar{m}(g^{-1})_i^* A_{g^{-1}i}} \circ \text{id}_{\bar{m}(g^{-1})_i^* A_{g^{-1}i}} \\ &= \text{id}_{{}^gA_i} \circ \text{id}_{{}^gA_i} \\ &= \text{id}_{{}^gA_i}. \end{aligned}$$

Having shown that gA satisfies the first condition set out in Definition 1.12(obj), we now turn to the second part of Definition 1.12(obj). Suppose i, j and k are facets of $I(G, \mathbb{K})$ with $i \leq j \leq k$; suppose also that $g \in G(\mathbb{K})$ as above. To show that gA satisfies the second condition appearing in Definition 1.12(obj) we must show that the following diagram commutes.

$$\begin{array}{ccc} \text{res}_{j \leq k} \text{res}_{i \leq j} {}^gA_i & \xrightarrow{\text{res}_{j \leq k} {}^gA_{i \leq j}} & \text{res}_{j \leq k} {}^gA_j \\ \text{res}_{i \leq j \leq k} {}^gA_{i \leq j} \downarrow & & \downarrow {}^gA_{j \leq k} \\ \text{res}_{i \leq k} {}^gA_i & \xrightarrow{{}^gA_{i \leq k}} & {}^gA_k \end{array} \quad (30)$$

Consider the following diagramme, in which the outer square is exactly Dia-

gramme 30. (To save space we temporarily write R_{gj}^{gi} for $\text{res}_{gi \leq gj}$, etc...)

$$\begin{array}{ccc}
 R_k^j R_j^i {}^g A_i & \xrightarrow{\hspace{10em}} & R_k^j {}^g A_j \\
 \downarrow & \searrow 4. & \swarrow 2. \\
 \bar{m}(g)_k^* R_{gk}^{gj} R_{gj}^{gi} A_{gi} & \longrightarrow & \bar{m}(g)_k^* R_{gk}^{gj} A_{gj} \\
 \downarrow & & \downarrow \\
 \bar{m}(g)_k^* R_{gk}^{gi} A_{gi} & \longrightarrow & \bar{m}(g)_k^* A_{gk} \\
 \swarrow 3. & & \searrow 1. \\
 R_k^i {}^g A_i & \xrightarrow{\hspace{10em}} & {}^g A_k
 \end{array}$$

The inner square is the result of applying the functor $\bar{m}(g)_k^*$ to the relevant form of the commuting square appearing in Definition 1.12(obj), and therefore commutes. The arrow marked 1. is the identity. The arrow marked 2. is $\text{res}_{j \leq k}^g A_{gj}$, so the right-hand square commutes by virtue of the definition of ${}^g A_{j \leq k}$; likewise, the arrow marked 3. is $\text{res}_{i \leq k}^g A_{gi}$, so the bottom square commutes by virtue of the definition of ${}^g A_{i \leq k}$. The arrow marked 4. is $\text{res}_{j \leq k}^g \text{res}_{gi \leq gj} A_{gi} \circ \text{res}_{j \leq k} \text{res}_{i \leq j}^g A_{gi}$ and the top and left-hand squares commute by Lemma 2.5. This concludes the demonstration that ${}^g A$ is an object in $\bar{D}G$.

Suppose $\phi : A \rightarrow B$ is a morphism in $\bar{D}G$ and let i and j be facets of $I(G, \mathbb{K})$ such that $i \leq j$; also, fix $g \in G(\mathbb{K})$. In order to show that ${}^g \phi$ is a morphism in $\bar{D}G$ we must show that the following diagramme commutes.

$$\begin{array}{ccc}
 \text{res}_{i \leq j} {}^g A_i & \xrightarrow{\text{res}_{i \leq j} {}^g \phi_i} & \text{res}_{i \leq j} {}^g B_i \\
 \downarrow {}^g A_{i \leq j} & & \downarrow {}^g B_{i \leq j} \\
 {}^g A_j & \xrightarrow{{}^g \phi_j} & {}^g B_j
 \end{array} \tag{31}$$

To do this, consider the diagramme below, in which the outer square is exactly Diagramme 31. (To save space we temporarily write R_{gj}^{gi} for $\text{res}_{gi \leq gj}$, etc... , as above.)

$$\begin{array}{ccc}
 R_j^i \bar{m}(g)_i^* A_{gi} & \xrightarrow{\hspace{10em}} & R_j^i \bar{m}(g)_i^* B_{gi} \\
 \downarrow & \searrow 3. & \swarrow 4. \\
 \bar{m}(g)_j^* R_{gj}^{gi} A_{gi} & \xrightarrow{0.} & \bar{m}(g)_j^* R_{gj}^{gi} B_{gi} \\
 \swarrow 2. & & \searrow 1. \\
 \bar{m}(g)_j^* A_{gj} & \xrightarrow{\hspace{10em}} & \bar{m}(g)_j^* B_{gj}
 \end{array}$$

The arrow marked 0. is $\bar{m}(g)_j^* \text{res}_{gi \leq gj} \phi_{gi}$, the arrow marked 1. is $\bar{m}(g)_j^* B_{gi \leq gj}$ and the arrow marked 2. is $\bar{m}(g)_j^* A_{gi \leq gj}$; thus, the bottom square is the result of applying the functor $\bar{m}(g)_j^*$ to the relevant form of the commuting square appearing in Definition 1.12(mor), and therefore commutes since ϕ is a morphism in $\bar{D}G$. The arrow marked 3. is $\text{res}_{i \leq j}^g A_{gi}$ and the arrow marked 3. is $\text{res}_{i \leq j}^g B_{gi}$, so the upper square commutes because $\text{res}_{i \leq j}^g$ is a natural transformation. The left-hand triangle commutes by virtue of the definition of ${}^g A_{i \leq j}$ and likewise the right-hand triangle commutes by virtue of the definition of ${}^g B_{i \leq j}$. Therefore, the outer square commutes. This concludes the demonstration that ${}^g \phi$ is a morphism in $\bar{D}G$. \square

LEMMA 2.5. *Let A be an object in $\bar{D}G$ and let g, h be elements of $G(\mathbb{K})$. Then ${}^g({}^h A) \cong {}^{gh}A$ in $\bar{D}G$.*

Proof. By Proposition 2.4, for each facet i of $I(G, \mathbb{K})$,

$${}^g({}^h A)_i = \bar{m}(g^{-1})_i^* \bar{m}(h^{-1})_{g^{-1}i}^* A_{h^{-1}g^{-1}i}$$

and

$${}^{gh}A_i = \bar{m}((gh)^{-1})_i^* A_{(gh)^{-1}i} = \bar{m}(h^{-1}g^{-1})_i^* A_{h^{-1}g^{-1}i}.$$

Let

$$\phi_i : \bar{m}(h^{-1}g^{-1})_i^* A_{h^{-1}g^{-1}i} \rightarrow \bar{m}(g^{-1})_i^* \bar{m}(h^{-1})_{g^{-1}i}^* A_{h^{-1}g^{-1}i}$$

be the canonical isomorphism. To prove the Lemma 2.5 we show that $\phi := (\phi_i)_i$ is a morphism in $\bar{D}G$; thus, we show that ϕ satisfies the condition of Definition 1.12(mor). This follows from Lemma 1.8. \square

Recall the hypothesis of Section 1.3. The fact that \bar{G}_i is a *connected* linear algebraic group over an algebraically closed field allows us to apply the theory of character sheaves from [Lus85] to \bar{G}_i . Suppose A is an object of $\bar{D}G$ such that A_i is a finite direct sum of character sheaves for \bar{G}_i , for each facet i of $I(G, \mathbb{K})$. By [Lus85, 2.18], A_i is an equivariant perverse sheaf. For each $x \in \bar{G}_i(\mathbb{k})$, let $\mu_{A_i}(x) \in \text{mor}D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell)$ be the isomorphism defined by Equation 21 in Section 1.7.

DEFINITION 2.6. An object $A \in \text{obj}\bar{C}G$ is *weakly-equivariant* if the following conditions are met.

- (a) For each facet i of $I(G, \mathbb{K})$, the perverse sheaf A_i is equivariant.
- (b) There is a family of isomorphisms

$$\mu_A = \{\mu_A(g) \in \text{Hom}_{\bar{D}G}({}^g A, A) \mid g \in G(\mathbb{K})\}$$

such that $\mu_A(1) = \text{id}_A$ and the diagramme

$$\begin{array}{ccc} g({}^h A) & \xrightarrow{{}^g \mu_A(h)} & gA \\ \downarrow & & \downarrow \mu_A(g) \\ g^h A & \xrightarrow{\mu_A(gh)} & A \end{array}$$

commutes, for all g and h in $G(\mathbb{K})$. The arrow appearing on the left-hand side of this diagramme is the isomorphism of Lemma 2.5.

(c) For each facet i of $I(G, \mathbb{K})$ and for each $g \in G_i(\mathfrak{o}_{\mathbb{K}})$,

$$\mu_{A_i}(\rho_i(g)) = \mu_A(g)_i.$$

A morphism $\phi : A \rightarrow B$ of weakly-equivariant objects of $\bar{\mathcal{C}}G$ is itself *weakly-equivariant* if the diagramme

$$\begin{array}{ccc} {}^gA & \xrightarrow{{}^g\phi} & {}^gB \\ \mu_A(g) \downarrow & & \downarrow \mu_B(g) \\ A & \xrightarrow{\phi} & B \end{array} \quad (32)$$

commutes for all $g \in G(\mathbb{K})$.

LEMMA 2.7. *Suppose A is weakly-equivariant. For each $g \in G(\mathbb{K})$, the morphism $\mu_A(g) : {}^gA \rightarrow A$ is an isomorphism in $\bar{\mathcal{D}}G$, and*

$$\mu_A(g)^{-1} = {}^g\mu_A(g^{-1}).$$

Proof. Using Definition 2.6(b), we have $\text{id}_A = \mu_A(g) \circ {}^g\mu_A(g^{-1})$. Using the functoriality of conjugation yields

$$\begin{aligned} g^{-1}({}^g\mu_A(g^{-1}) \circ \mu_A(g)) &= {}^1\mu_A(g^{-1}) \circ g^{-1}\mu_A(g) \\ &= \mu_A(g^{-1}g) \\ &= \text{id}_A. \end{aligned}$$

Thus,

$$\begin{aligned} {}^g\mu_A(g^{-1}) \circ \mu_A(g) &= {}^g\text{id}_A \\ &= \text{id}_{{}^gA}. \end{aligned}$$

It follows that $\mu_A(g)^{-1} = {}^g\mu_A(g^{-1})$, as promised. \square

2.3 CUSPIDAL COEFFICIENT SYSTEMS

Recall the definition of the additive category $\bar{\mathcal{C}}G$ from Section 1.10. In particular, recall that objects of $\bar{\mathcal{C}}G$ are called coefficient systems for G .

DEFINITION 2.8. A non-zero coefficient system C for G is *cuspidal* if it satisfies the following conditions:

(a) C_i is a finite direct sum of character sheaves for \bar{G}_i , or 0, for each facet i of $I(G, \mathbb{K})$.

- (b) C is weakly-equivariant (see Definition 2.6).
- (c) If $C = A \oplus B$ in $\bar{C}G$ and A and B are weakly-equivariant, then $A = 0$ or $B = 0$.
- (d) $\text{res}_P^G C = 0$ for each proper parabolic Levi subgroup $L \subset G$.

Let $\bar{\mathcal{A}}^{(0)}G$ denote the set of cuspidal coefficient systems for G .

In this Section we give a complete description of the isomorphism classes in $\bar{\mathcal{A}}^{(0)}G$.

PROPOSITION 2.9. *Let i_0 be a vertex of $I(G, \mathbb{K})$ and let F be a cuspidal character sheaf for \bar{G}_{i_0} . There is a cuspidal coefficient system C for G such that $C_{i_0} = F$ and $C_i = 0$ unless i is in the $G(\mathbb{K})$ -orbit of i_0 ; moreover, up to a weakly-equivariant isomorphism, C is the unique cuspidal coefficient system for G with these properties.*

Proof. We begin by showing existence of $C \in \text{obj } \bar{C}G$ with the properties claimed above. Denote the $G(\mathbb{K})$ -orbit of the vertex i_0 in $I(G, \mathbb{K})$ by $O(i_0)$. Consider the function $G(\mathbb{K}) \rightarrow O(i_0)$ given by $g \mapsto gi_0$ and let $i \mapsto w_i$ denote a normalised section of that function; thus, $w_{i_0} = 1_{G(\mathbb{K})}$ and for each $i \in O(i_0)$ the element w_i of $G(\mathbb{K})$ satisfies that $w_i i_0 = i$. For each facet i of $I(G, \mathbb{K})$, define

$$C_i := \begin{cases} \bar{m}(w_i^{-1})_i^* F, & i \in O(i_0), \\ 0, & \text{otherwise.} \end{cases} \quad (33)$$

If i and j are facets of $I(G, \mathbb{K})$ and $i \leq j$ then we define

$$C_{i \leq j} = \begin{cases} \text{id}_{C_i}, & i = j, \\ 0, & \text{otherwise.} \end{cases} \quad (34)$$

We will show that Equations 33 and 34 define an object of $\bar{C}G$. If i and j are not facets in $O(i_0)$, or if $i = j$, then the diagramme in Definition 1.12(obj) is commutative for trivial reasons. Thus, we suppose now that i or j is contained in $O(i_0)$ and $i < j$. Since i_0 is a vertex, it follows that $i \in O(i_0)$ and $j \notin O(i_0)$; therefore $C_j = 0$. Since C_i is a cuspidal character sheaf (or 0) for each such facet i , and since \bar{G}_j is a proper Levi subgroup of \bar{G}_i , it follows that $\text{res}_{i \leq j} C_i = 0$. Thus, in all cases, $C_{i \leq i} = \text{id}_{C_i}$ and $\text{res}_{i \leq j \leq k} C_i \circ C_{i \leq k} = C_{j \leq k} \circ \text{res}_{j \leq k} C_{i \leq j}$ for each triplet i, j, k of facets of $I(G, \mathbb{K})$ such that $i \leq j \leq k$. It is clear from these definitions that C is an object of $\bar{D}G$ (see Definition 1.12) and that $C_{i_0} = F$. Since C_i is a perverse sheaf for every facet i , it follows immediately that C is an object of $\bar{C}G$ (see Definition 1.12). We must now show that C is cuspidal (see Definition 2.8).

It is clear that C satisfies the condition appearing in Definition 2.8(a). In order to demonstrate Definition 2.8(b) we define a family μ_C of isomorphisms in $\bar{D}G$ satisfying the conditions of Definition 2.6. First, recall that a cuspidal character

sheaf is strongly cuspidal (*cf.* [Lus85, 7.1.6]). Using [Lus85, 7.1.1] and [Lus85, 7.1.5], observe that the strongly cuspidal perverse sheaf F is \bar{G}_i -equivariant. For each $x \in \bar{G}_{i_0}(\mathbb{K})$, let

$$\mu_F(x) : \bar{m}_{i_0}(x^{-1})^* F \rightarrow F$$

be the isomorphism as in Equation 22. For each $g \in G(\mathbb{K})$ and for each facet i of $I(G, \mathbb{K})$ in the $G(\mathbb{K})$ -orbit of i_0 , define

$$k_{i,g} := w_i^{-1} g w_{ig}. \quad (35)$$

Then $k_{i,g}$ is an element of $G_{i_0}(\mathfrak{o}_{\mathbb{K}})$, as we now show. By definition, $i = w_i i_0$; thus, $g^{-1}i = g^{-1}(w_i i_0) = (g^{-1}w_i) i_0$. On the other hand, $g^{-1}i$ is a facet of $O(i_0)$ implies $g^{-1}i = w_{g^{-1}i} i_0$. Comparing these last two equations it follows that $w_i^{-1} g w_{ig} \in G(\mathbb{K})_{i_0}$. (Note that here we use the assumption on G described in Section 1.3.) Now, set $\bar{k}_{i,g} = \rho_{i_0}(k_{i,g})$ (*cf.* Section 1.2).

In order to define $\mu_C(g)_i : {}^g C_i \rightarrow C_i$ we first suppose $i \subset O(i_0)$. Then

$$\begin{aligned} {}^g C_i &= \bar{m}(g^{-1})^* C_{ig} \\ &= \bar{m}(g^{-1})^* w_{ig} F \\ &= \bar{m}(g^{-1})^* \bar{m}(w_{ig}^{-1})^* F. \end{aligned}$$

Now, let

$$\bar{m}(g^{-1})^* \bar{m}(w_{ig}^{-1})^* F \cong \bar{m}(w_{ig}^{-1} g^{-1})_{i_0}^* F \quad (36)$$

be the canonical isomorphism and note also that

$$\begin{aligned} \bar{m}(w_{ig}^{-1} g^{-1})_{i_0}^* F &= \bar{m}(w_{ig}^{-1} g^{-1} w_i w_i^{-1})_{i_0}^* F \\ &= \bar{m}(k_{i,g}^{-1} w_i^{-1})_{i_0}^* F. \end{aligned}$$

Let

$$\bar{m}(k_{i,g}^{-1} w_i^{-1})_{i_0}^* F \cong \bar{m}(w_i^{-1})_{i_0}^* \bar{m}(k_{i,g}^{-1})_{i_0}^* F \quad (37)$$

be the canonical isomorphism and consider the isomorphism

$$\bar{m}(w_i^{-1})_{i_0}^* \mu_F(\bar{k}_{i,g}) : \bar{m}(w_i^{-1})_{i_0}^* \bar{m}(k_{i,g}^{-1})_{i_0}^* F \rightarrow \bar{m}(w_i^{-1})_{i_0}^* F. \quad (38)$$

Since $w_i F = C_i$, we let $\mu_C(g)_i : {}^g C_i \rightarrow C_i$ be the composition of the isomorphisms above, when $i \subset O(i_0)$. Otherwise, $\mu_C(g)_i = 0$.

It is clear that the family of morphisms $\mu_C(g)_i \in \text{mor} D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell)$ defined above, as i ranges over all facets of $I(G, \mathbb{K})$, defines a morphism $\mu_C(g)$ in $\bar{D}G$ (*cf.* Definition 1.12(mor)), since

$$\mu_C(g)_j \circ {}^g C_{i \leq j} = C_{i \leq j} \circ \text{res}_{i \leq j} \mu_C(g)_i, \quad (39)$$

for $i \leq j$ in $I(G, \mathbb{K})$. We now show that the family of isomorphisms $\mu_C(g) \in \text{mor} \bar{D}G$, as g ranges over $G(\mathbb{K})$, denoted μ_C , satisfies the conditions of Definition 2.6. If i is not contained in the $G(\mathbb{K})$ -orbit of i_0 then these

conditions are trivial. We now suppose, therefore, that i is a facet of $O(i_0)$, whence $\mu_C(g)_i = {}^{w_i}\mu_F(\bar{k}_{i,g}^{-1})$. If $g = 1$, we have $k_{i,g} = k_{i,1} = w_i^{-1}w_i = 1$ and $\mu_F(1) = \text{id}_F$, so

$$\begin{aligned} \mu_C(1)_i &= {}^{w_i}\mu_F(1)_i \\ &= {}^{w_i}\text{id}_F \\ &= \text{id}_{{}^{w_i}F} \\ &= \text{id}_{C_i}. \end{aligned}$$

Applying the functor $\bar{m}(w_i^{-1})_{i_0}^*$ to Diagramme 23 with $x = \bar{k}_{i,g}$ and $y = \bar{k}_{i,g,h}$ (see Equation 35) yields the following commutative diagramme.

$$\begin{array}{ccc} \bar{m}(w_i^{-1})_{i_0}^* m(\bar{k}_{i,g}^{-1})^* \bar{m}(\bar{k}_{i,g,h}^{-1})^* F & \xrightarrow{\bar{m}(w_i^{-1})_{i_0}^* m(\bar{k}_{i,g}^{-1})^* \mu_F(\bar{k}_{i,g,h})} & \bar{m}(w_i^{-1})_{i_0}^* m(\bar{k}_{i,g}^{-1})^* F \\ \downarrow & & \downarrow \\ \bar{m}(w_i^{-1})_{i_0}^* m((\bar{k}_{i,g}\bar{k}_{i,g,h})^{-1})^* F & \xrightarrow{\bar{m}(w_i^{-1})_{i_0}^* \mu_F(\bar{k}_{i,g}\bar{k}_{i,g,h})} & \bar{m}(w_i^{-1})_{i_0}^* F \end{array} \quad (40)$$

The top arrow is ${}^{w_i}\bar{k}_{i,g}\mu_F(\bar{k}_{i,g,h})$ while the right-hand side arrow is ${}^{w_i}\mu_F(\bar{k}_{i,g})$; thus, the clockwise path is

$$\begin{aligned} {}^{w_i}\mu_F(\bar{k}_{i,g}) \circ {}^{w_i}\bar{k}_{i,g}\mu_F(\bar{k}_{i,g,h}) &= {}^{w_i}\mu_F(\bar{k}_{i,g}) \circ {}^{g w_i g}\mu_F(\bar{k}_{i,g,h}) \\ &= {}^{w_i}\mu_F(\bar{k}_{i,g}) \circ \bar{m}(g^{-1})^* {}^{w_i g}\mu_F(\bar{k}_{i,g,h}) \\ &= \mu_C(g)_i \circ \bar{m}(g^{-1})^* \mu_C(h)_{i_g} \\ &= \mu_C(g)_i \circ {}^g\mu_C(h)_i. \end{aligned}$$

On the other hand, the left-hand arrow is the canonical isomorphism, while the bottom arrow is ${}^{w_i}\mu_F(\bar{k}_{i,g}\bar{k}_{i,g,h})$, and

$$\begin{aligned} {}^{w_i}\mu_F(\bar{k}_{i,g}\bar{k}_{i,g,h}) &= {}^{w_i}\mu_F(\rho_{i_0}(k_{i,g}k_{i,g,h})) \\ &= {}^{w_i}\mu_F(\rho_{i_0}(w_i^{-1}g w_{i_g} w_{i_g}^{-1} h w_{i_g} h)) \\ &= {}^{w_i}\mu_F(\rho_{i_0}(w_i^{-1}g h w_{i_g} h)) \\ &= {}^{w_i}\mu_F(\bar{k}_{i,g} h) \\ &= \mu_C(gh)_i. \end{aligned}$$

Thus, Diagramme 40 gives us the condition appearing in Definition 2.8(b).

To show that C satisfies the condition appearing in Definition 2.8(c), suppose $C = A \oplus B$ in $\bar{C}G$ and that A and B are weakly-equivariant. Then, for each facet i , $C_i = A_i \oplus B_i$ in the category of perverse sheaves for \bar{G}_i . Since $C_i = 0$ unless i is in the $G(\mathbb{K})$ -orbit of i_0 , we have $A_i = 0$ and $B_i = 0$ unless i is in the $G(\mathbb{K})$ -orbit of i_0 . Since F is a character sheaf, it is irreducible, so $C_{i_0} = A_{i_0} \oplus B_{i_0}$ implies $A_{i_0} = 0$ or $B_{i_0} = 0$ (recall that $C_{i_0} = F$). Without loss of generality, suppose $B_{i_0} = 0$. Since B is weakly-equivariant, this implies $B_i = 0$

for each facet i in the $G(\mathbb{K})$ -orbit of i_0 . Since $B_i = 0$ when i is not in the $G(\mathbb{K})$ -orbit of i_0 also, it follows that $B = 0$. Thus, C satisfies Definition 2.8(c). We now consider the condition of Definition 2.8(d). Let P be a proper parabolic subgroup P of G and let L be the reductive quotient of P . Let i be a facet of $I(L, \mathbb{K})$. From Proposition 2.1, we see that $(\text{res}_P^G C)_i$ may be identified with $\text{res}_{i_G \leq i_P} A_{i_G}$. Since P is proper, i_P is strictly greater than i_G ; thus, \bar{G}_{i_P} is a proper Levi subgroup of \bar{G}_{i_G} . Since F is a cuspidal character sheaf (and therefore strongly cuspidal) and A is weakly-equivariant, it follows that A_{i_G} is either strongly cuspidal or 0; in either case, $i_G \lesssim i_P$ implies $\text{res}_{i_G \leq i_P} A_{i_G} = 0$. Thus, $(\text{res}_P^G C)_i = 0$. Since i was an arbitrary facet of $I(L, \mathbb{K})$, it follows that $\text{res}_P^G C = 0$. Thus, C satisfies Definition 2.8(d).

We now show uniqueness. Let A and B be cuspidal objects of $\bar{\mathcal{C}}G$ such that $A_{i_0} = F = B_{i_0}$ and $A_i = 0 = B_i$ unless i is in the $G(\mathbb{K})$ -orbit of i_0 . Note that $\mu_A(k)_{i_0} = \mu_F(\rho_{i_0}(k)) = \mu_B(k)_{i_0}$ for $k \in G_{i_0}(\mathfrak{o}_{\mathbb{K}})$ by Definition 2.8(b). Now, $A_{i_0} = B_{i_0}$ implies $A_{w_i^{-1}i} = B_{w_i^{-1}i}$, hence $\bar{m}(w_i^{-1})_i^* A_{w_i^{-1}i} = \bar{m}(w_i^{-1})_i^* B_{w_i^{-1}i}$, so ${}^w A_i = {}^w B_i$ for all facets i in the $G(\mathbb{K})$ -orbit of i_0 . Define $\phi : A \rightarrow B$ by

$$\phi_i = \begin{cases} \mu_B(w_i)_i \circ {}^w \mu_A(w_i^{-1})_i & i \subset O(i_0) \\ \text{id}_0 & \text{otherwise.} \end{cases} \quad (41)$$

We will now show that ϕ is an isomorphism in $\bar{\mathcal{C}}G$. Suppose i is a facet in the $G(\mathbb{K})$ -orbit of i_0 . First, using Equation 35 and Definition 2.6(b) (twice), we have

$$\begin{aligned} \mu_A(g) &= \mu_A(w_i k_{i,g} w_{ig}^{-1}) \\ &= \mu_A(w_i) \circ {}^w \mu_A(k_{i,g} w_{ig}^{-1}) \\ &= \mu_A(w_i) \circ {}^w (\mu_A(k_{i,g}) \circ {}^{k_{i,g}} \mu_A(w_{ig}^{-1})) \\ &= \mu_A(w_i) \circ {}^w \mu_A(k_{i,g}) \circ {}^{w_i k_{i,g}} \mu_A(w_{ig}^{-1}) \\ &= \mu_A(w_i) \circ {}^w \mu_A(k_{i,g}) \circ {}^{g w_{ig}} \mu_A(w_{ig}^{-1}). \end{aligned}$$

Thus,

$$\begin{aligned} \mu_A(g)_i &= \mu_A(w_i)_i \circ {}^w \mu_A(k_{i,g})_i \circ {}^{g w_{ig}} \mu_A(w_{ig}^{-1})_i \\ &= \mu_A(w_i)_i \circ \bar{m}(w_i^{-1})^* \mu_A(k_{i,g})_{w_i^{-1}i} \circ \bar{m}(g^{-1})^* {}^{w_{ig}} \mu_A(w_{ig}^{-1})_{ig} \\ &= \mu_A(w_i)_i \circ \bar{m}(w_i^{-1})^* \mu_A(k_{i,g})_{i_0} \circ \bar{m}(g^{-1})^* {}^{w_{ig}} \mu_A(w_{ig}^{-1})_{ig} \\ &= \mu_A(w_i)_i \circ \bar{m}(w_i^{-1})^* \mu_F(k_{i,g}) \circ \bar{m}(g^{-1})^* {}^{w_{ig}} \mu_A(w_{ig}^{-1})_{ig}. \end{aligned}$$

Likewise,

$$\mu_B(g)_i = \mu_B(w_i)_i \circ \bar{m}(w_i^{-1})^* \mu_F(k_{i,g}) \circ \bar{m}(g^{-1})^* {}^{w_{ig}} \mu_B(w_{ig}^{-1})_{ig}.$$

Using Lemma 2.7 and the definition of ϕ^i we have

$$\begin{aligned}
& \phi_i \circ \mu_A(g)_i \\
&= \mu_B(w_i)_i \circ {}^{w_i}\mu_A(w_i^{-1})_i \circ \mu_A(w_i)_i \\
&\quad \circ \bar{m}(w_i)^* \mu_F(\bar{k}_{i,g}) \circ \bar{m}(g^{-1})^* {}^{w_{ig}^{-1}}\mu_A(w_{ig}^{-1})_{ig} \\
&= \mu_B(w_i)_i \circ \bar{m}(w_i)^* \mu_F(\bar{k}_{i,g}) \circ \bar{m}(g^{-1})^* {}^{w_{ig}^{-1}}\mu_A(w_{ig}^{-1})_{ig}.
\end{aligned}$$

On the other hand, using the work above and the definition of ${}^g\phi_i$ we have

$$\begin{aligned}
& \mu_B(g)_i \circ {}^g\phi_i \\
&= \mu_B(w_i)_i \circ \bar{m}(w_i^{-1})^* \mu_F(k_{i,g}) \circ \bar{m}(g^{-1})^* {}^{w_{ig}}\mu_B(w_{ig}^{-1})_{ig} \\
&\quad \circ \bar{m}(g^{-1})^* \left(\mu_B(w_{ig})_{ig} \circ {}^{w_{ig}}\mu_A(w_{ig}^{-1})_{ig} \right) \\
&= \mu_B(w_i)_i \circ \bar{m}(w_i^{-1})^* \mu_F(k_{i,g}) \\
&\quad \circ \bar{m}(g^{-1})^* \left({}^{w_{ig}}\mu_B(w_{ig}^{-1})_{ig} \circ \mu_B(w_{ig})_{ig} \circ {}^{w_{ig}}\mu_A(w_{ig}^{-1})_{ig} \right) \\
&= \mu_B(w_i)_i \circ \bar{m}(w_i^{-1})^* \mu_F(k_{i,g}) \circ \bar{m}(g^{-1})^* {}^{w_{ig}}\mu_A(w_{ig}^{-1})_{ig}.
\end{aligned}$$

Thus,

$$\phi_i \circ \mu_A(g)_i = \mu_B(g)_i \circ {}^g\phi_i, \quad (42)$$

for all facets i of $I(G, \mathbb{K})$. This concludes the proof that $\phi : A \rightarrow B$ is a weakly-equivariant morphism in $\bar{C}G$. Since ϕ is clearly an isomorphism by Lemma 2.7, this concludes the proof of Proposition 2.9. \square

DEFINITION 2.10. For any vertex i of $I(G, \mathbb{K})$ and any cuspidal character sheaf F on \bar{G}_i , let $\text{cind}_i F$ denote the cuspidal object of $\bar{C}G$ given by Proposition 2.9.

THEOREM 2.11. *If C is a cuspidal coefficient system for G then there is a vertex i_0 of $I(G, \mathbb{K})$ and a cuspidal character sheaf F such that $C_{i_0} = F$ and $C_i = 0$ unless i is in the $G(\mathbb{K})$ -orbit of i_0 . Moreover, this vertex i_0 is unique up to $G(\mathbb{K})$ -conjugation and F is unique up to isomorphism in $D_c^b(\bar{G}_{i_0}; \bar{\mathbb{Q}}_\ell)$.*

Proof. If $C_i = 0$ for all vertices i of $I(G, \mathbb{K})$, then $C = 0$ because each $C_{i \leq j} : \text{res}_{i \leq j} C_i \rightarrow C_j$ is an isomorphism for each $i \leq j$ (see Definition 1.12). Since cuspidal coefficient systems are non-zero (see Definition 2.8), that is not the case. Thus, there is some vertex i of $I(G, \mathbb{K})$ such that $C_i \neq 0$. Using Definition 2.8(a) we write $C_i = \bigoplus_m C_{i,m}$ (finite direct sum) where each $C_{i,m}$ is a character sheaf. Let M be a proper Levi subgroup of \bar{G}_i . Using [Lan96] we can identify the star of i in $I(G, \mathbb{K})$ with the building for \bar{G}_i , so there is some facet j in the star of i such that $M = \bar{G}_j$. Since $M \subset \bar{G}_i$ is proper, $i \not\leq j$. Now, there is a parabolic subgroup P with Levi component L and a facet k of $I(L, \mathbb{K})$ such that $k_G = i$ and $k_P = j$. Note also that $k_G \leq k_P$. By Definition 2.8(d), $\text{res}_P^G C = 0$. Thus, $(\text{res}_P^G C)_k = 0$ so $\text{res}_{k_G \leq k_P} C_{k_G} = 0$ so $\text{res}_{i \leq j} C_i = 0$ so $\text{res}_M^{\bar{G}_i} C_i = 0$. Thus, $\text{res}_M^{\bar{G}_i} C_{i,m} = 0$, for each m above. Since this argument

applies to any proper levi subgroup M of \bar{G}_i , and since character sheaves are equivariant, it follows that $C_{i,m}$ is a cuspidal character sheaf. Thus we have shown that if i is a vertex then C_i is 0 or is a finite direct sum of cuspidal character sheaves for \bar{G}_i . Note that it follows immediately that $C_i = 0$ unless i is a vertex.

Now, let $\{i_0, i_1, \dots, i_d\}$ be the vertices of a fundamental $G(\mathbb{K})$ -domain in $I(G, \mathbb{K})$; thus, the convex hull of $\{i_0, i_1, \dots, i_d\}$ is a chamber in $I(G, \mathbb{K})$. For each such vertex i_n , we write $C_{i_n} = \bigoplus_m C_{i_n,m}$, where $C_{i_n,m}$ is a cuspidal character sheaf for \bar{G}_{i_n} . Consider

$$A := \bigoplus_{n,m} \text{cind}_{i_n} C_{i_n,m},$$

where $0 \leq n \leq d$ and $0 \leq m \leq d_n$ runs over an index set corresponding to the irreducible summands of C_{i_n} in the category of perverse sheaves on \bar{G}_{i_n} . Notice that $A_{i_n} = \bigoplus_m C_{i_n,m} = C_{i_n} = C_{i_n}$ for each vertex i_n above. Thus, $A_{i_n} = C_{i_n}$ for each vertex i_n above and $A_i = C_i = 0$ unless i is a vertex. Note also that A and C are both weakly-equivariant.

We now show that $A \cong C$. Observe that A and C are weakly-equivariant and $A_{i_n} = C_{i_n}$ for every vertex $i_n \in \{i_0, \dots, i_n\}$. For each such vertex i_n , let $i \mapsto w(n)_i$ denote a normalized section of $g \mapsto gi_n$, as in the proof of Proposition 2.9; thus, $i = w(n)_i i_n$ for each vertex i in the $G(\mathbb{K})$ -orbit of i_n . Since the set of vertices in $I(G, \mathbb{K})$ is partitioned into $G(\mathbb{K})$ -orbits, for each vertex i there is a unique vertex i_n such that $i = w(n)_i i_n$. Let i be any vertex and define $\phi_i : A_i \rightarrow C_i$ by

$$\phi_i = \mu_C(w(n)_i)_i \circ \mu_A(w(n)_i)_i^{-1}.$$

This composition is defined since the codomain of $\mu_A(w(n)_i)_i^{-1}$ is the domain of $\mu_C(w(n)_i)_i$ which is ${}^{w(n)_i}A_i$, and by Proposition 2.4,

$$\begin{aligned} {}^{w(n)_i}A_i &= \bar{m}(w(n)_i^{-1})_i^* A_{w(n)_i^{-1}i} \\ &= \bar{m}(w(n)_i^{-1})_i^* A_{i_n} \\ &= \bar{m}(w(n)_i^{-1})_i^* B_{i_n} \\ &= \bar{m}(w(n)_i^{-1})_i^* B_{w(n)_i^{-1}i} \\ &= {}^{w(n)_i}B_i, \end{aligned}$$

which is the domain of $\mu_C(w(n)_i)_i$. Observe also that the domain of ϕ_i is indeed A_i since the codomain of $\mu_A(w(n)_i)_i$ is A_i ; likewise, the codomain of ϕ_i is indeed B_i since the codomain of $\mu_C(w(n)_i)_i$ is C_i . If i is not a vertex, define $\phi_i = 0$ (Recall that $A_i = C_i = 0$ unless i is a vertex.) To see that this defines a morphism in $\bar{D}G$ is it necessary to see that the diagramme appearing in Definition 1.12(mor) commutes for all $i \leq j$. But the case $i = j$ is trivial since $A_{i \leq j} = C_{i \leq j} = 1$ if $i = j$; the case $i \leq j$ is equally trivial, since $A_{i \leq j} = C_{i \leq j} = 0$ if $i \not\leq j$; Thus, ϕ is a morphism in $\bar{D}G$. In fact, from the definition of each ϕ_i is

is also clear that ϕ is an isomorphism in $\bar{D}G$. Since $\bar{C}G$ is a full subcategory of $\bar{D}G$ and since the domain and codomain of ϕ are objects of $\bar{C}G$ it follows that ϕ is an isomorphism in $\bar{C}G$. (In fact, ϕ is weakly-equivariant.)

Since $A \cong C$ and C is cuspidal, it follows that A is also cuspidal. Since each $\text{cind}_{i_n} C_{i_n, m}$ is weakly-equivariant by Proposition 2.9, it follows from Definition 2.8(c) that $\text{cind}_{i_n} C_{i_n, m} = 0$ for all but one vertex i_n and for all but one index m . Let i_0 be that vertex, set $m = 0$ and let $F = C_{i_0, 0}$. Then $A = \text{cind}_{i_0} F$. Now, $A \cong C$ implies $C \cong \text{cind}_{i_0} F$, which completes the proof of Theorem 2.11. \square

COROLLARY 2.12. *Let i_0, i_1, \dots, i_d be a set of representatives for the $G(\mathbb{K})$ -orbits of vertices in $I(G, \mathbb{K})$. The isomorphism classes in $\bar{A}^{(0)}G$ (see Definition 2.10) are parameterized by*

$$\left\{ (i_n, F) \mid 0 \leq n \leq d, F \in \hat{G}_{i_n}^{(0)} \right\},$$

where $\hat{G}_{i_n}^{(0)}$ denotes a set of representative for the isomorphism classes of cuspidal character sheaves for \bar{G}_{i_n} (cf. [Lus85, 3.10]).

DEFINITION 2.13. A Levi subgroup $L \subseteq G$ is said to be a *cuspidal Levi subgroup* if there is a cuspidal coefficient system for L ; in other words, L is a cuspidal Levi subgroup of G if $\bar{A}^{(0)}L$ is non-empty (cf. Definition 2.8).

EXAMPLE 2.14. The algebraic group $GL(n)_{\mathbb{K}}$ admits only one cuspidal Levi subgroup, up to conjugacy, and that is the split torus. The split torus $GL(1)_{\mathbb{K}}$ in $SL(2)_{\mathbb{K}}$ is also a cuspidal Levi subgroup, as is $SL(2)_{\mathbb{K}}$ itself, as we shall see in Section 6. The cuspidal Levi subgroups of $Sp(4)$ are described in Section 6.

3 ADMISSIBLE COEFFICIENT SYSTEMS

Throughout Section 3 we assume \mathbb{K} is a maximal unramified closure of a local field with finite residue field. We note that such a field is strictly henselian and that the residue field is an algebraic closure of the finite field. Let G be a connected reductive linear algebraic group over \mathbb{K} satisfying the conditions of Section 1.3.

Let $\sigma : P \rightarrow G$ be a parabolic subgroup with Levi component L . In Section 3.1 we define a parabolic induction function ind_P^G taking weakly-equivariant objects of $\bar{C}L$ to weakly-equivariant objects of $\bar{C}G$. It should be noted that, in contrast to parabolic restriction ind_P^G (Section 2.1), conjugation $\bar{m}(g)^*$ (Section 2.2), and Frobenius fr^* (Section 4 below), parabolic induction ind_P^G is a function, not a functor. This is because we can only apply our definition to weakly-equivariant objects in $\bar{C}L$, and we have not constructed a category of weakly-equivariant objects in $\bar{C}L$. This is not to say that such a construction is not possible. Note that we took pains in Section 1.8 to treat $\text{ind}_{i \leq j}$ as a functor on the category of equivariant perverse sheaves, not just a function, and we will

use that improvement below. However, the definition of weakly-equivariant objects, while built upon that of equivariant perverse sheaves, is considerably less sophisticated and almost certainly not the correct definition upon which to try to build a well-behaved category. Nevertheless, our parabolic induction $\text{ind}_P^{\tilde{G}}$, as a function, is all that is needed to define admissible coefficient systems (see Definition 3.7 below) for the same reason that Lusztig's parabolic induction – also a function, not a functor – suffices to define character sheaves.

3.1 PARABOLIC SUBGROUPS

Let $G \rightarrow G/P$ be the cokernel of $\sigma : P \rightarrow G$, let $\pi : P \rightarrow L$ be canonical quotient map and let $U \hookrightarrow P$ be the unipotent radical of P . For any facet j of $I(G, \mathbb{K})$, let P_j (resp. L^j, U_j) be the schematic closure of P (resp. L, U) in G_j . Then P_j (resp. L^j, U_j) is a smooth integral model for P (resp. L^j, U_j). Since $\sigma(P_j(\mathfrak{o}_{\mathbb{K}})) \subseteq G_j(\mathfrak{o}_{\mathbb{K}})$ and $\pi(P_j(\mathfrak{o}_{\mathbb{K}})) \subseteq L_j(\mathfrak{o}_{\mathbb{K}})$, it follows from the Extension Principle (cf. [BT84, 1.7]) that $\sigma : P \rightarrow G$ and $\pi : P \rightarrow L$ and $U \hookrightarrow P$ extend uniquely to $\mathfrak{o}_{\mathbb{K}}$ -scheme morphisms $\sigma_j : P_j \rightarrow G_j$ and $\pi_j : P_j \rightarrow L^j$ and $U_j \rightarrow P_j$ such that the squares commute in the following diagramme.

$$\begin{array}{ccccc} G & \xleftarrow{\sigma} & P & \xrightarrow{\pi} & L \\ \downarrow & & \downarrow & & \downarrow \\ L^j & \xleftarrow{\sigma_j} & P_j & \xrightarrow{\pi_j} & G_j \end{array} \quad (43)$$

DEFINITION 3.1. Let i be an arbitrary facet of $I(G, \mathbb{K})$. Let $D_i(G, P, \mathbb{K})$ denote the elements of the double coset space $G_i(\mathfrak{o}_{\mathbb{K}}) \backslash G(\mathbb{K}) / L(\mathbb{K})$ represented by $g \in G(\mathbb{K})$ with the following properties, where L^{ig} denotes the integral closure of L in G_{ig} : L^{ig} is a smooth integral model for L , $L^{ig}(\mathfrak{o}_{\mathbb{K}})$ is a parahoric subgroup of $L(\mathbb{K})$, and \bar{L}^{ig} is a levi subgroup of \bar{G}_{ig} . It follows from [Lan96, 9.22] that there is a unique facet i_P^g in the star of ig such that $\bar{G}_{i_P^g} = \bar{L}^{ig}$ and $\bar{G}_{ig \leq i_P^g}$ coincides with the schematic intersection of P with \bar{G}_{ig} . Let i_L^g denote the facet of $I(L, \mathbb{K})$ such that $L^{ig} = L_{i_L^g}$. Notice that if g satisfies this condition then so does hgl for all $h \in G_i(\mathfrak{o}_{\mathbb{K}})$ and $l \in L(\mathbb{K})$.

We begin by remarking that $D_i(G, P, \mathbb{K})$ is finite. First, notice that $G_i(\mathfrak{o}_{\mathbb{K}}) \backslash G(\mathbb{K}) / P(\mathbb{K})$ is finite. Thus, the image of $D_i(G, P, \mathbb{K})$ under the surjection

$$\begin{aligned} G_i(\mathfrak{o}_{\mathbb{K}}) \backslash G(\mathbb{K}) / L(\mathbb{K}) &\rightarrow G_i(\mathfrak{o}_{\mathbb{K}}) \backslash G(\mathbb{K}) / P(\mathbb{K}) \\ G_i(\mathfrak{o}_{\mathbb{K}}) g L(\mathbb{K}) &\mapsto G_i(\mathfrak{o}_{\mathbb{K}}) g P(\mathbb{K}) \end{aligned}$$

is finite. Suppose now that g represents a double coset in $D_i(G, P, \mathbb{K})$. Then each element in the pre-image of $G_i(\mathfrak{o}_{\mathbb{K}}) g P(\mathbb{K})$ under the map above is represented by gu , for some $u \in U(\mathbb{K})$. If gu represents an element of $D_i(G, P, \mathbb{K})$ then g and gu have the properties listed in Definition 3.1. This implies

$u \in U_{ig}(\mathfrak{o}_{\mathbb{K}})$, in which case $gu = hg$ for some $h \in G_i(\mathfrak{o}_{\mathbb{K}})$. Thus, the intersection of the pre-image of $G_i(\mathfrak{o}_{\mathbb{K}})gP(\mathbb{K})$ with $D_i(G, P, \mathbb{K})$ is a singleton. It follows that $D_i(G, P, \mathbb{K})$ is finite.

In order to define parabolic induction functors, we now begin the process of picking specific representatives in $G(\mathbb{K})$ for the double cosets appearing in Definition 3.1. We begin by recalling some basic notions and establishing some notation. Let T be a maximal \mathbb{K} -split torus of G and let $A(G, T, \mathbb{K})$ be the apartment for T . Let T_0 be the Néron model for T and let $W(G, T, \mathbb{K})$ be the associated affine Weyl group associated to the pair $(G(\mathbb{K}), T_0(\mathfrak{o}_{\mathbb{K}}))$; that is, $W(G, T, \mathbb{K}) = N(\mathbb{K})/T_0(\mathfrak{o}_{\mathbb{K}})$, where N is the normalizer of T in G . If i is a facet of $A(G, T, \mathbb{K})$ let $W_i(G, T, \mathbb{K})$ denote the stabiliser of i in $W(G, T, \mathbb{K})$. Let $\dot{W}(G, T, \mathbb{K})$ be a set of representatives for $W(G, T, \mathbb{K})$ contained in $N(\mathbb{K})$ and chosen so that vw is represented by $\dot{v}\dot{w}$ when the length of vw equals the length of v plus the length of w (cf. [Mor93, 5.2]). Let $\dot{W}_i(G, T, \mathbb{K})$ denote the subset corresponding to $W_i(G, T, \mathbb{K})$.

Suppose i and j are facets of $I(G, \mathbb{K})$ such that $i \leq j$. Let $d_i(G, P, \mathbb{K})$ (resp. $d_j(G, P, \mathbb{K})$) be a set of representatives for the double coset space $D_i(G, P, \mathbb{K})$ (resp. $D_j(G, P, \mathbb{K})$). The map $f_{i \leq j} : G_j \rightarrow G_i$ (cf. Section 1.4) defines an inclusion of points $G_j(\mathfrak{o}_{\mathbb{K}}) \subseteq G_i(\mathfrak{o}_{\mathbb{K}})$ which in turn defines a surjection

$$\begin{aligned} G_j(\mathfrak{o}_{\mathbb{K}}) \backslash G(\mathbb{K}) / L(\mathbb{K}) &\rightarrow G_i(\mathfrak{o}_{\mathbb{K}}) \backslash G(\mathbb{K}) / L(\mathbb{K}) \\ G_j(\mathfrak{o}_{\mathbb{K}}) y L(\mathbb{K}) &\mapsto G_i(\mathfrak{o}_{\mathbb{K}}) y L(\mathbb{K}). \end{aligned}$$

If $y \in d_j(G, P, \mathbb{K})$ then y satisfies the conditions of Definition 3.1; in particular, $P_{jy} \hookrightarrow G_{jy}$ is projective. Now $f_{iy \leq jy}$ induces $P_{jy} \hookrightarrow P_{iy}$ so $P_{iy} \hookrightarrow G_{iy}$ is projective. The other conditions appearing in Definition 3.1 are also satisfied, so y represents a double coset in $D_i(G, P, \mathbb{K})$. Therefore, the surjection above restricts to a surjection

$$D_j(G, P, \mathbb{K}) \rightarrow D_i(G, P, \mathbb{K}) \tag{44}$$

which in turn defines

$$d_j(G, P, \mathbb{K}) \rightarrow d_i(G, P, \mathbb{K}). \tag{45}$$

Let x be the image of y under Equation 45. The fibre of Equation 44 above the double coset represented by x is

$$\{G_j(\mathfrak{o}_{\mathbb{K}}) u x L(\mathbb{K}) \mid u \in G_{jx}(\mathfrak{o}_{\mathbb{K}}) \backslash G_{ix}(\mathfrak{o}_{\mathbb{K}}) / (G_{ix}(\mathfrak{o}_{\mathbb{K}}) \cap L(\mathbb{K}))\}, \tag{46}$$

which we now denote $D_{i \leq j}^x(G, P, \mathbb{K})$. Observe that

$$G_{ix}(\mathfrak{o}_{\mathbb{K}}) \cap L(\mathbb{K}) = L^{ix}(\mathfrak{o}_{\mathbb{K}}) = L_{i_{\mathbb{Z}}}^x(\mathfrak{o}_{\mathbb{K}}) \tag{47}$$

and this is a parahoric subgroup of $L(\mathbb{K})$. Recall also that $ix \leq i_P^x$ (see Definition 3.1). Using the affine Bruhat Decomposition (cf. [Mor93, 3.22]) we find $D_{i \leq j}^x(G, P, \mathbb{K})$ is in bijection with

$$W_{jx}(G, T, \mathbb{K}) \backslash W_{ix}(G, T, \mathbb{K}) / W_{i_{\mathbb{Z}}}^x(L, T, \mathbb{K}). \tag{48}$$

Following [Vig03, C.1.1] we find a unique element of minimal length, called a distinguished element, in each double coset above, and let $d_{i \leq j}^x(G, P, \mathbb{K})$ be the elements of $\dot{W}(G, T, \mathbb{K})$ corresponding to distinguished elements representing the double coset space above. Thus, there is a unique $u \in d_{i \leq j}^x(G, P, \mathbb{K})_x$ such that $ux = hyl$, for some $l \in L(\mathbb{K})$ and some $h \in G_j(\mathfrak{o}_{\mathbb{K}})$. In fact, we can say much more, as Lemma 3.2 shows.

LEMMA 3.2. *Keep notation as above. There exists a collection of sets $d_i(G, P, \mathbb{K})$, as i ranges over all facets of $I(G, \mathbb{K})$, consisting of representatives for the double coset spaces $D_i(G, P, \mathbb{K})$, with the following properties:*

- (a) *Let i and j be facets of $I(G, \mathbb{K})$ with $i \leq j$. If $g \in d_j(G, P, \mathbb{K})$ then $ux = gl$, where x is the image of g under the map of Equation 45 and u is an element of $d_{i \leq j}^x(G, P, \mathbb{K})$. Moreover, u and l (and x) are determined uniquely by g .*
- (b) *Let i be a facet of $I(G, \mathbb{K})$ and let g be an element of $G(\mathbb{K})$. If y is an element of $d_{i_g}(G, P, \mathbb{K})$ then there is unique x in $d_i(G, P, \mathbb{K})$ and h in $G_i(\mathfrak{o}_{\mathbb{K}})$ such that $gy = hx$.*

Proof. Let i be a facet of $A(G, T, \mathbb{K})$. Let $d_i(G, P, \mathbb{K})$ be a set representatives for $D_i(G, P, \mathbb{K})$ chosen from $\dot{W}(G, T, \mathbb{K})$. Suppose i and j are facets of $A(G, T, \mathbb{K})$ such that $i \leq j$. Now, then x, g and u are elements of $\dot{W}(G, T, \mathbb{K})$ and $g^{-1}ux$ is contained in $\dot{W}(L, T, \mathbb{K})$. Set $l = g^{-1}ux$; we now have

$$ux = gl, \tag{49}$$

as promised. Now, let i be any facet of $I(G, \mathbb{K})$. There is some z in $G(\mathbb{K})/N(\mathbb{K})$ such that i is contained in the apartment $A(G, T^z, \mathbb{K})$. Since zi is a facet of $A(G, T, \mathbb{K})$ we set

$$d_i(G, P, \mathbb{K}) = z^{-1}d_{zi}(G, P, \mathbb{K}). \tag{50}$$

If $i' \leq j'$ and j' is a facet of $A(G, T^z, \mathbb{K})$ then i' is a facet of $A(G, T^z, \mathbb{K})$ also. Suppose $x' \in d_{j'}(G, P, \mathbb{K})$ and g' is the image of x' under the map to $d_{i'}(G, P, \mathbb{K})$. Set $i = zi'$, $j = zj'$, $x = zx'$ and $g = zg'$. Then $z^{-1}u'z$ is an element of $d_{i \leq j}(G, P, \mathbb{K})$ so set $u = zu'z^{-1}$. Then $ux = gl$ by Equation 49, so $(zu'z^{-1})(zx') = (zg')l$, so $u'x' = g'l$, as desired.

The proof of part (b) is omitted. \square

3.2 LOCAL PARABOLIC INDUCTION

DEFINITION 3.3. Let P be a parabolic subgroup of G with Levi component L . Let i be any facet of $I(G, \mathbb{K})$. Let g be an element of $G(\mathbb{K})$ such that i_g satisfies the conditions appearing in Definition 3.1. Let $\text{ind}_{\bar{L}^{i_g}}^{\bar{G}^{i_g}}$ denote the functor from the category of equivariant perverse sheaves whose objects are finite direct sums of character sheaves on \bar{L}^{i_g} to the category of equivariant perverse sheaves whose objects are finite direct sums of character sheaves on

\bar{G}_{ig} given by $\text{ind}_{ig \leq i_P^g}$. For any $B \in \text{obj} \bar{\mathcal{C}}L$ we will write B^{ig} for $B_{i_L^g}$. Thus, $\text{ind}_{\bar{L}^{ig}} \bar{G}_{ig} B^{ig} = \text{ind}_{ig \leq i_P^g} B_{i_L^g}$. (This is not a typo; observe that $\bar{G}_{i_P^g} = \bar{L}_{i_L^g}$.) Likewise for morphisms in $\bar{\mathcal{C}}L$.

For j a facet of $I(G, \mathbb{K})$, let Σ_j be the set of affine roots vanishing on j . Let l be a second facet of $I(G, \mathbb{K})$. We will denote by $j \wedge l$ the facet of $I(G, \mathbb{K})$ defined by $\Sigma_{j \wedge l} = \Sigma_j \cap \Sigma_l$.

LEMMA 3.4. *Let i, j and l be facets of $I(G, \mathbb{K})$ such that $i \leq j$ and $i \leq l$. There is a choice of representatives $d_{j,l}^i$ for $G_{i \leq j}(\mathfrak{o}_{\mathbb{K}}) \backslash G_i(\mathfrak{o}_{\mathbb{K}}) / G_{i \leq l}(\mathfrak{o}_{\mathbb{K}})$ such that*

$$\text{res}_{i \leq j} \text{ind}_{i \leq l} F \cong \sum_{x \in d_{j,l}^i} \bar{m}(x^{-1})^* \text{ind}_{jx \leq jx \wedge l} \text{res}_{l \leq l \wedge jx} F$$

and the diagramme

$$\begin{array}{ccc} \text{res}_{j \leq k} \text{res}_{i \leq j} \text{ind}_{i \leq l} F & \longrightarrow & \text{res}_{j \leq k} \sum_{x \in d_{j,l}^i} \bar{m}(x^{-1})^* \text{ind}_{jx \leq jx \wedge l} \text{res}_{l \leq l \wedge jx} F \\ \downarrow \text{res}_{i \leq j \leq k} \text{ind}_{i \leq l} F & & \downarrow \\ \text{res}_{i \leq k} \text{ind}_{i \leq l} F & \longrightarrow & \sum_{x \in d_{k,l}^i} \bar{m}(x^{-1})^* \text{ind}_{kx \leq kx \wedge l} \text{res}_{l \leq l \wedge kx} F \end{array}$$

commutes, for any character sheaf F on \bar{G}_l .

Proof. Recall that \mathbb{K} is an unramified closure of a p -adic field. We choose a uniformizer and, as in [Vig03, C.1.1], we find a unique element of minimal length, called a distinguished element, in each double coset above, and let $d_{k,l}^i$ be the set of distinguished elements representing the double coset space above. Then use [MS89, Prop 10.1.2] (which gives considerably more information concerning the isomorphism above than [Lus85, Prop 15.2]). Note that the proofs of [MS89, Prop 10.1.2] and [Lus85, Prop 15.2], while different, both depend on the fact that \mathbb{k} is the algebraic closure of a finite field. \square

3.3 PARABOLIC INDUCTION

PROPOSITION 3.5. *Let P be a parabolic subgroup G with Levi component L . If B is a weakly-equivariant admissible object of $\bar{\mathcal{C}}L$ then there is a weakly-equivariant object $\text{ind}_P^G B$ of $\bar{\mathcal{C}}G$ such that*

$$(\text{ind}_P^G B)_i := \sum_{g \in d_i(G, P, \mathbb{K})} \bar{m}(g^{-1})_i^* \text{ind}_{\bar{L}^{ig}} \bar{G}_{ig} B^{ig}.$$

for each facet i of $I(G, \mathbb{K})$.

Proof. We must begin by defining $(\text{ind}_P^G B)_{i \leq j}$ for every $i \leq j$ in $I(G, \mathbb{K})$. To that end, we fix one such pair of facets and consider $\text{res}_{i \leq j}(\text{ind}_P^G B)_i$.

$$\begin{aligned} \text{res}_{i \leq j}(\text{ind}_P^G B)_i &= \text{res}_{i \leq j} \sum_{y \in d_i(G, P, \mathbb{K})} \bar{m}(g^{-1})_i^* \text{ind}_{\bar{L}^{ig}}^{\bar{G}^{ig}} B^{ig} \\ &= \sum_{g \in d_i(G, P, \mathbb{K})} \text{res}_{\bar{G}_j}^{\bar{G}_i} \bar{m}(g^{-1})_i^* \text{ind}_{\bar{L}^{ig}}^{\bar{G}^{ig}} B^{ig}. \end{aligned}$$

Let

$$\begin{aligned} \sum_{g \in d_i(G, P, \mathbb{K})} \text{res}_{\bar{G}_j}^{\bar{G}_i} \bar{m}(g^{-1})_i^* \text{ind}_{\bar{L}^{ig}}^{\bar{G}^{ig}} B^{ig} & \quad (51) \\ \downarrow 1. & \\ \sum_{g \in d_i(G, P, \mathbb{K})} \bar{m}(g^{-1})_i^* \text{res}_{\bar{G}_{jg}}^{\bar{G}_{ig}} \text{ind}_{\bar{L}^{ig}}^{\bar{G}^{ig}} B^{ig} & \end{aligned}$$

be the isomorphism in $\mathcal{M}_{\bar{G}_j, \bar{G}_i}$ determined by the natural isomorphism

$$\text{res}_{\bar{G}_j}^{\bar{G}_i} \bar{m}(g^{-1})_i^* \cong \bar{m}(g^{-1})_i^* \text{res}_{\bar{G}_{jg}}^{\bar{G}_{ig}} \quad (52)$$

of Lemma 1.8. Now, since B is an object in $\bar{C}L$ it follows that $B^{ig} = B_{i_L}^{ig}$ is a finite direct sum of character sheaves. Recall that $\bar{L}^{ig} = \bar{G}_{i_P}^{ig}$. By Lemma 3.4 (MacKey's formula for character sheaves) we have an isomorphism

$$\text{res}_{\bar{G}_{jg}}^{\bar{G}_{ig}} \text{ind}_{\bar{G}_{i_P}^{ig}}^{\bar{G}^{ig}} B^{ig} \cong \sum_{u \in d_{i \leq j}^g(G, P, \mathbb{K})} \bar{m}(u^{-1})^* \text{ind}_{\bar{G}_{j_P}^{gu}}^{\bar{G}_{jgu}} \text{res}_{\bar{G}_{i_P}^{gu}}^{\bar{G}_{i_P}^g} B^{ig}. \quad (53)$$

which defines

$$\begin{aligned} \sum_{g \in d_i(G, P, \mathbb{K})} \bar{m}(g^{-1})_i^* \text{res}_{ig \leq jg} \text{ind}_{\bar{L}^{ig}}^{\bar{G}^{ig}} B^{ig} & \quad (54) \\ \downarrow 2. & \\ \sum_{g \in d_i(G, P, \mathbb{K})} \bar{m}(g^{-1})_i^* \sum_{u \in d_{i \leq j}^g(G, P, \mathbb{K})} \bar{m}(u^{-1})_j^* \text{ind}_{\bar{G}_{j_P}^{gu}}^{\bar{G}_{jgu}} \text{res}_{\bar{G}_{i_P}^{gu}}^{\bar{G}_{i_P}^g} B^{ig}. & \end{aligned}$$

(Here we have used the notation of Lemma 3.2.) Let

$$\begin{aligned} \sum_{g \in d_i(G, P, \mathbb{K})} \bar{m}(g^{-1})_i^* \sum_{u \in d_{i \leq j}^g(G, P, \mathbb{K})} \bar{m}(u^{-1})_j^* \text{ind}_{\bar{G}_{j_P}^{gu}}^{\bar{G}_{jgu}} \text{res}_{\bar{G}_{i_P}^{gu}}^{\bar{G}_{i_P}^g} B^{ig} & \quad (55) \\ \downarrow 3. & \\ \sum_{\substack{g \in d_i(G, P, \mathbb{K}) \\ u \in d_{i \leq j}^g(G, P, \mathbb{K})}} \bar{m}((gu)^{-1})_j^* \text{ind}_{\bar{G}_{j_P}^{gu}}^{\bar{G}_{jgu}} \text{res}_{\bar{G}_{i_P}^{gu}}^{\bar{G}_{i_P}^g} B^{ig}. & \end{aligned}$$

be the isomorphism in determined by the natural isomorphism

$$\bar{m}(g^{-1})^* \bar{m}(u^{-1})^* \cong \bar{m}((gu)^{-1})^*. \quad (56)$$

Since $u \in G_{jj}(\mathfrak{o}_{\mathbb{K}})$ we have $G_{jj} = G_{jgu}$; it follows that the integral closure of L in G_{jj} is the integral closure of L in G_{jgu} , which we denote L_{jP}^{gu} . Moreover, since $i \leq j$ we have $G_i \supseteq G_j$ and $G_{ig} \supseteq G_{jj}$. Also, $G_{ig} = G_j^{gu}$, so it follows that the integral closure of L in G_i^g is the integral closure of L in G_i^{gu} . Therefore,

$$\begin{aligned} & \sum_{\substack{g \in d_i(G, P, \mathbb{K}) \\ u \in d_i^g(G, P, \mathbb{K})}} \bar{m}((gu)^{-1})_j^* \text{ind}_{\bar{G}_{jP}^{gu}} \bar{G}_{jP}^{gu} \text{res}_{\bar{G}_{jP}^{gu}} B^{ig} \\ &= \sum_{\substack{g \in d_i(G, P, \mathbb{K}) \\ u \in d_i^g(G, P, \mathbb{K})}} \bar{m}((gu)^{-1})_j^* \text{ind}_{\bar{L}_{jgu}} \bar{G}_{jgu} \text{res}_{\bar{L}_{jgu}} \bar{L}_{jgu}^{igu} B^{igu}. \end{aligned}$$

By Lemma 3.2, for each g and u as above there is a unique $h \in d_j(G, P, \mathbb{K})$ and $l = l_j^i(h)$ such that $gu = hl$. Thus,

$$\begin{aligned} & \sum_{\substack{g \in d_i(G, P, \mathbb{K}) \\ u \in d_i^g(G, P, \mathbb{K})}} \bar{m}((gu)^{-1})_j^* \text{ind}_{\bar{L}_{jgu}} \bar{G}_{jgu} \text{res}_{\bar{L}_{jgu}} \bar{L}_{jgu}^{igu} B^{igu} \\ &= \sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}((hl)^{-1})_j^* \text{ind}_{\bar{L}_{jhl}} \bar{G}_{jhl} \text{res}_{\bar{L}_{jhl}} \bar{L}_{jhl}^{ihl} B^{ihl}. \end{aligned}$$

Now, the natural isomorphism

$$\bar{m}((hl)^{-1})^* \cong \bar{m}(h^{-1})^* \bar{m}(l^{-1})^* \quad (57)$$

defines

$$\sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}((hl)^{-1})_j^* \text{ind}_{\bar{L}_{jhl}} \bar{G}_{jhl} \text{res}_{\bar{L}_{jhl}} \bar{L}_{jhl}^{ihl} B^{ihl} \quad (58)$$

↓ 4.

$$\sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}(h^{-1})^* \bar{m}(l^{-1})^* \text{ind}_{\bar{L}_{jhl}} \bar{G}_{jhl} \text{res}_{\bar{L}_{jhl}} \bar{L}_{jhl}^{ihl} B^{ihl}.$$

Lemma 1.11 gives

$$\bar{m}(l^{-1})^* \text{ind}_{\bar{L}_{jhl}} \bar{G}_{jhl} \cong \text{ind}_{\bar{L}_{jh}} \bar{G}_{jh} \bar{m}(l^{-1})^* \quad (59)$$

and therefore defines

$$\sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}(h^{-1})^* \bar{m}(l^{-1})^* \text{ind}_{\bar{L}_{jhl}} \bar{G}_{jhl} \text{res}_{\bar{L}_{jhl}} \bar{L}_{jhl}^{ihl} B^{ihl} \quad (60)$$

↓ 5.

$$\sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}(h^{-1})^* \text{ind}_{\bar{L}_{jh}} \bar{G}_{jh} \bar{m}(l^{-1})^* \text{res}_{\bar{L}_{jhl}} \bar{L}_{jhl}^{ihl} B^{ihl}.$$

Since B is an object of $\bar{\mathcal{D}}L$ (see Definition 1.12(obj)) we have the isomorphism

$$\operatorname{res}_{\bar{L}^{jhl}}^{\bar{L}^{ihl}} B^{ihl} \cong B^{jhl} \quad (61)$$

which defines

$$\begin{array}{c} \sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}(h^{-1})^* \operatorname{ind}_{\bar{L}^{jh}}^{\bar{G}^{jh}} \bar{m}(l^{-1})^* \operatorname{res}_{\bar{L}^{jhl}}^{\bar{L}^{ihl}} B^{ihl} \\ \downarrow 6. \\ \sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}(h^{-1})^* \operatorname{ind}_{\bar{L}^{jh}}^{\bar{G}^{jh}} \bar{m}(l^{-1})^* B^{jhl} \end{array} \quad (62)$$

Since B is weakly-equivariant (see Definition 2.6) and $l \in L(\mathbb{K})$, we have the isomorphism

$$B^{ihl} \cong B^{ih} \quad (63)$$

which defines

$$\begin{array}{c} \sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}(h^{-1})^* \operatorname{ind}_{\bar{L}^{jh}}^{\bar{G}^{jh}} \bar{m}(l^{-1})^* \operatorname{res}_{\bar{L}^{jhl}}^{\bar{L}^{ihl}} B^{jhl} \\ \downarrow 7. \\ \sum_{h \in d_j(G, P, \mathbb{K})} \bar{m}(h^{-1})^* \operatorname{ind}_{\bar{L}^{jh}}^{\bar{G}^{jh}} B^{jh}. \end{array} \quad (64)$$

Since this last expression is precisely, $(\operatorname{ind}_P^G B)_j$, composing the isomorphisms of Equations 51, 54, 55, 58, 60, 62, 64 defines an isomorphism

$$(\operatorname{ind}_P^G B)_{i \leq j} : \operatorname{res}_{i \leq j}(\operatorname{ind}_P^G B)_i \rightarrow (\operatorname{ind}_P^G B)_j. \quad (65)$$

Now we must show that the family $((\operatorname{ind}_P^G B)_i, (\operatorname{ind}_P^G B)_{i \leq j})_{i, j}$ satisfies the condition of Definition 1.12(obj). Inspecting the isomorphisms appearing in the definition of $(\operatorname{ind}_P^G B)_{i \leq j}$, is clear that $(\operatorname{ind}_P^G B)_{i \leq i} = \operatorname{id}_{(\operatorname{ind}_P^G B)_i}$, so we turn now to the second condition in Definition 1.12(obj). Let i, j and k be facets of $I(G, \mathbb{K})$ and consider the following diagramme.

$$\begin{array}{ccc} \operatorname{res}_{j \leq k} \operatorname{res}_{i \leq j}(\operatorname{ind}_P^G B)_i & \xrightarrow{\operatorname{res}_{j \leq k}(\operatorname{ind}_P^G B)_{i \leq j}} & \operatorname{res}_{j \leq k}(\operatorname{ind}_P^G B)_j \\ \downarrow \operatorname{res}_{i \leq j \leq k}(\operatorname{ind}_P^G B)_i & & \downarrow (\operatorname{ind}_P^G B)_{j \leq k} \\ \operatorname{res}_{i \leq k}(\operatorname{ind}_P^G B)_k & \xrightarrow{(\operatorname{ind}_P^G B)_{i \leq k}} & (\operatorname{ind}_P^G B)_k \end{array} \quad (66)$$

In order to see that this diagramme commutes, it is sufficient to observe that each isomorphism appearing in the definition of $(\operatorname{ind}_P^G B)_{i \leq j}$ (and $(\operatorname{ind}_P^G B)_{j \leq k}$

and $(\text{ind}_P^G B)_{i \leq k}$ is compatible with the restriction functors of Section 1.4. Fortunately, this work has already been done in various lemmata, in anticipation of this need: for isomorphism 1. of Equation 51 use Lemma 1.8; for isomorphism 2. of Equation 54 use Lemma 3.4 and Lemma 3.2 (to see that $d_{i \leq j}^g = d_{j_P^g \geq j g \leq i_P^g}$); for isomorphism 3. of Equation 55 use Lemma 1.8; for isomorphism 4. of Equation 58 use Lemma 1.8; for isomorphism 5. of Equation 60 use Lemma 1.11; for isomorphism 6. of Equation 62 use Definition 1.12(obj); for isomorphism 7. of Equation 64 use Definition 2.6. Each of these arguments makes use of Proposition 1.6! \square

3.4 TRANSITIVE PARABOLIC INDUCTION

Recall the notation of Proposition 2.2.

PROPOSITION 3.6. *Let P be a parabolic subgroup of G with Levi component L . Let Q be a parabolic subgroup of L with Levi component M . Let R be a parabolic subgroup of G with Levi component M . If $C \in \text{obj } \bar{C}G$ is weakly-equivariant then*

$$\text{ind}_P^G \text{ind}_Q^L C \cong \text{ind}_R^G C. \quad (67)$$

Proof. To simplify notation, let $B = \text{ind}_Q^L C$ and let $A = \text{ind}_P^G B$. We fix a facet i of $I(G, \mathbb{K})$. Then

$$\begin{aligned} A_i &= (\text{ind}_P^G B)_i \\ &= \sum_{x \in d_i(G, P, \mathbb{K})} \bar{m}(x^{-1})_i^* \text{ind}_{\bar{P}_{i_P^x}}^{\bar{G}_{i_P^x}} B_{i_P^x} \\ &= \sum_{x \in d_i(G, P, \mathbb{K})} \bar{m}(x^{-1})_i^* \text{ind}_{\bar{P}_{i_P^x}}^{\bar{G}_{i_P^x}} \sum_{y \in d_{i_P^x}(L, Q, \mathbb{K})} \bar{m}(y^{-1})_{i_P^x}^* \text{ind}_{\bar{Q}_{(i_P^x)_Q^y}}^{\bar{L}_{i_P^x y}} C_{(i_P^x)_Q^y}. \end{aligned}$$

Since $L_{i_P^x} = L^{ix}$ and $y \in L(\mathbb{K})$, it follows that $L_{i_P^x y} = L^{ixy} = L_{i_P^{xy}}$. Since $T \subseteq M \subseteq L$, the schematic closure of M in $L_{i_P^{xy}}$ is equal to the schematic closure of M in $G_{i_P^{xy}}$, and we have $L_{i_P^x y} = L_{i_P^{xy}}$ and $M_{(i_P^x)_Q^y} = M_{i_P^{xy}}$. Using Lemma 1.11 we now have

$$\begin{aligned} &\sum_{x \in d_i(G, P, \mathbb{K})} \bar{m}(x^{-1})_i^* \text{ind}_{\bar{P}_{i_P^x}}^{\bar{G}_{i_P^x}} \sum_{y \in d_{i_P^x}(L, Q, \mathbb{K})} \bar{m}(y^{-1})_{i_P^x}^* \text{ind}_{\bar{Q}_{(i_P^x)_Q^y}}^{\bar{L}_{i_P^x y}} C_{(i_P^x)_Q^y} \\ &= \sum_{\substack{x \in d_i(G, P, \mathbb{K}) \\ y \in d_{i_P^x}(L, Q, \mathbb{K})}} \bar{m}(x^{-1})_i^* \text{ind}_{\bar{P}_{i_P^x}}^{\bar{G}_{i_P^x}} \bar{m}(y^{-1})_{i_P^x}^* \text{ind}_{\bar{Q}_{i_P^{xy}}}^{\bar{L}_{i_P^{xy}}} C_{i_P^{xy}} \\ &= \sum_{\substack{x \in d_i(G, P, \mathbb{K}) \\ y \in d_{i_P^x}(L, Q, \mathbb{K})}} \bar{m}(x^{-1})_i^* \bar{m}(y^{-1})_{i_P^x}^* \text{ind}_{\bar{P}_{i_P^{xy}}}^{\bar{G}_{i_P^{xy}}} \text{ind}_{\bar{Q}_{i_P^{xy}}}^{\bar{L}_{i_P^{xy}}} C_{i_P^{xy}} \\ &= \sum_{\substack{x \in d_i(G, P, \mathbb{K}) \\ y \in d_{i_P^x}(L, Q, \mathbb{K})}} \bar{m}((xy)^{-1})_i^* \text{ind}_{\bar{P}_{i_P^{xy}}}^{\bar{G}_{i_P^{xy}}} \text{ind}_{\bar{Q}_{i_P^{xy}}}^{\bar{L}_{i_P^{xy}}} C_{i_P^{xy}}. \end{aligned}$$

Using Propositions [Lus85, 4.2], [Lus85, 4.8(b)] and [Lus85, 2.18(a)] we have

$$\operatorname{ind}_{\bar{P}_i^{xy}} \bar{G}^{i,xy} \operatorname{ind}_{\bar{Q}_i^{xy}} \bar{L}_i^{xy} C_i^{xy} = \operatorname{ind}_{\bar{R}_i^{xy}} \bar{G}^{i,xy} C_i^{xy}. \quad (68)$$

Since $i_Q^{xy} = i_R^{xy}$, we have

$$(\operatorname{ind}_P^G \operatorname{ind}_Q^L C)_i = \sum_{\substack{x \in d_i(G, P, \mathbb{K}) \\ y \in d_i^x(L, Q, \mathbb{K})}} \bar{m}((xy)^{-1})_i^* \operatorname{ind}_{\bar{R}_i^{xy}} \bar{G}^{i,xy} C_i^{xy}. \quad (69)$$

Now, the canonical surjection from $D_i(G, R, \mathbb{K}) \rightarrow D_i(G, P, \mathbb{K})$ defines a surjection $d_i(G, R, \mathbb{K}) \rightarrow d_i(G, P, \mathbb{K})$; the pre-image of x in $d_i(G, P, \mathbb{K})$ is exactly $d_{i\bar{p}}(L, Q, \mathbb{K})$. For each pair (x, y) in Equation 69 there is a unique z in $d_i(G, R, \mathbb{K})$ such that $xy = hzm$ for h in $G_i(\mathfrak{o}_{\mathbb{K}})$ and $m \in M(\mathbb{K})$. Thus, $i_R^{xy} = i_R^z m$ and

$$\begin{aligned} & \sum_{\substack{x \in d_i(G, P, \mathbb{K}) \\ y \in d_i^x(L, Q, \mathbb{K})}} \bar{m}((xy)^{-1})_i^* \operatorname{ind}_{\bar{R}_i^{xy}} \bar{G}^{i,xy} C_i^{xy} \\ &= \sum_{z \in d_i(G, R, \mathbb{K})} \bar{m}((hzm)^{-1})_i^* \operatorname{ind}_{\bar{R}_i^z m} \bar{G}^{i, hzm} C_i^{hzm} \\ &= \sum_{z \in d_i(G, R, \mathbb{K})} \bar{m}(h^{-1})^* \bar{m}(z^{-1})^* \bar{m}(m^{-1})^* \operatorname{ind}_{\bar{R}_i^z m} \bar{G}^{i, hzm} C_i^{hzm} \\ &= \sum_{z \in d_i(G, R, \mathbb{K})} \bar{m}(h^{-1})^* \bar{m}(z^{-1})^* \operatorname{ind}_{\bar{R}_i^z} \bar{G}^{i, z} {}^m C_i^z. \end{aligned}$$

Since $C \in \operatorname{obj} \bar{\mathcal{D}}M$ and $h \in G_i(\mathfrak{o}_{\mathbb{K}})$ we have

$$\begin{aligned} & \sum_{z \in d_i(G, R, \mathbb{K})} \bar{m}(h^{-1})^* \bar{m}(z^{-1})^* \operatorname{ind}_{\bar{R}_i^z} \bar{G}^{i, z} {}^m C_i^z \\ & \cong \sum_{z \in d_i(G, R, \mathbb{K})} \bar{m}(z^{-1})^* \operatorname{ind}_{\bar{R}_i^z} \bar{G}^{i, z} C_i^z \\ &= (\operatorname{ind}_R^G C)_i. \end{aligned}$$

In summary, we have shown that

$$(\operatorname{ind}_P^G \operatorname{ind}_Q^L C)_i \cong (\operatorname{ind}_R^G C)_i. \quad (70)$$

Now, as i ranges over all facets of $I(G, \mathbb{K})$ these isomorphisms define an isomorphism in $\bar{\mathcal{C}}G$ (details omitted). \square

3.5 ADMISSIBLE COEFFICIENT SYSTEMS

We now come to the main definition of Section 3.

DEFINITION 3.7. An *irreducible admissible coefficient system for G* is a simple object of $\bar{\mathcal{C}}G$ which is a summand of $\text{ind}_P^G C$ in $\bar{\mathcal{C}}G$ for some parabolic subgroup $P \subseteq G$ with Levi component L and some cuspidal coefficient system C for L (see Definition 2.8). We write $\bar{\mathcal{A}}G$ for the set of irreducible admissible objects in $\bar{\mathcal{C}}G$. An *admissible coefficient system for G* is an object of $\bar{\mathcal{C}}G$ which is a finite direct sum of irreducible admissible coefficient systems.

Remark 3.8. Observe that any cuspidal coefficient system (see Definition 2.8) is an irreducible admissible coefficient system; thus, $\bar{\mathcal{A}}^{(0)}G \subseteq \bar{\mathcal{A}}G$.

The adjective ‘admissible’ is surely one of the most over-used in mathematics, and our use of it here suggests a lack of imagination. However, as we shall show in Section 5, since our admissible coefficient systems form a bridge between certain admissible perverse sheaves and certain admissible representations, we have elected to use the adjective here also. Nevertheless, a word of caution is in order: admissible perverse sheaves are, by definition, irreducible, while our admissible coefficient systems are not.

Examples of admissible coefficient systems are provided in Section 6.

4 ENTERS FROBENIUS

Let \mathbb{K}_1 be a p -adic field, let $\mathfrak{o}_{\mathbb{K}_1}$ be the ring of integers of \mathbb{K}_1 and let \mathbb{F}_q denote the residue field for \mathbb{K}_1 . Let \mathbb{K}_1^{nr} be a maximal unramified extension of \mathbb{K}_1 and let $\bar{\mathbb{F}}_q$ be the residue field for \mathbb{K}_1^{nr} . As in Section 3 we note that \mathbb{K}_1^{nr} is strictly henselian and that $\bar{\mathbb{F}}_q$ is an algebraic closure of \mathbb{F}_q , which is a finite field. Fix an isomorphism $\text{Gal}(\mathbb{K}_1^{nr}/\mathbb{K}_1) \cong \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$; this determines a ‘lift’ $\text{fr}_{\mathbb{K}_1} \in \text{Gal}(\mathbb{K}_1^{nr}/\mathbb{K}_1)$ of the geometric Frobenius $\text{Fr}_{\mathbb{F}_q} \in \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Let $G_{\mathbb{K}_1}$ be a connected reductive linear algebraic group over \mathbb{K}_1 such that

$$G := G_{\mathbb{K}_1} \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr}) \quad (71)$$

satisfies the conditions of Section 1. Then G is a connected reductive split linear algebraic group over \mathbb{K}_1^{nr} which is defined over \mathbb{K}_1 . In particular, since G is split it follows that $G_{\mathbb{K}_1}$ splits over an unramified extension of \mathbb{K}_1 . Let $G(\mathbb{K}_1)$ denote the group of \mathbb{K}_1 -rational points on G .

Since G is defined over \mathbb{K}_1 , the Galois group $\text{Gal}(\mathbb{K}_1^{nr}/\mathbb{K}_1)$ acts on $I(G, \mathbb{K}_1^{nr})$, and

$$I(G, \mathbb{K}_1) = I(G, \mathbb{K}_1^{nr})^{\text{Gal}(\mathbb{K}_1^{nr}/\mathbb{K}_1)}, \quad (72)$$

where $I(G, \mathbb{K}_1)$ is the Bruhat-Tits building for $G(\mathbb{K}_1)$.

4.1 FROBENIUS-STABLE COEFFICIENT SYSTEMS

Let i be any facet of $I(G, \mathbb{K}_1^{nr})$ and let G_i be the associated $\mathfrak{o}_{\mathbb{K}_1^{nr}}$ -scheme, as in Section 1.2, and let $G_{\text{fr}(i)}$ be the $\mathfrak{o}_{\mathbb{K}_1^{nr}}$ -scheme associated to $\text{fr}(i)$. The geometric Frobenius $\text{fr}_{\mathbb{K}_1} : \mathbb{K}_1^{nr} \rightarrow \mathbb{K}_1^{nr}$ defines a isomorphism $\text{fr}_i : G_i \rightarrow G_{\text{fr}(i)}$ of \mathbb{K}_1^{nr} -schemes which restricts to an isomorphism of special fibres, and factors

through the reduction quotient maps to an isomorphism of reductive quotients $\mathrm{Fr}_i : \bar{G}_i \rightarrow \bar{G}_{\mathrm{fr}(i)}$. Let $\mathrm{Fr}_i^* : D_c^b(\bar{G}_{\mathrm{fr}(i)}; \mathbb{Q}_\ell) \rightarrow D_c^b(\bar{G}_i; \mathbb{Q}_\ell)$ be the derived functor.

LEMMA 4.1. *Let i, j and k be facets of $I(G, \mathbb{K}_1^{nr})$ with $i \leq j \leq k$. Then there is an isomorphism of functors*

$$\mathrm{res}_{i \leq j}^{\mathrm{fr}} : \mathrm{res}_{i \leq j} \mathrm{Fr}_i^* \rightarrow \mathrm{Fr}_j^* \mathrm{res}_{\mathrm{fr}(i) \leq \mathrm{fr}(j)}$$

such that

$$\begin{array}{ccc} \mathrm{res}_{j \leq k} \mathrm{res}_{i \leq j} \mathrm{Fr}_i^* & \xrightarrow{\mathrm{res}_{i \leq j \leq k} \mathrm{Fr}_i^*} & \mathrm{res}_{i \leq k} \mathrm{Fr}_i^* \\ \downarrow \mathrm{res}_{j \leq k} \mathrm{res}_{i \leq j}^{\mathrm{fr}} & & \downarrow \mathrm{res}_{i \leq k}^{\mathrm{fr}} \\ \mathrm{res}_{j \leq k} \mathrm{Fr}_j^* \mathrm{res}_{\mathrm{fr}(i) \leq \mathrm{fr}(j)} & & \mathrm{Fr}_k^* \mathrm{res}_{\mathrm{fr}(i) \leq \mathrm{fr}(k)} \\ \downarrow \mathrm{res}_{j \leq k}^{\mathrm{fr}} \mathrm{res}_{\mathrm{fr}(i) \leq \mathrm{fr}(j)} & & \downarrow \\ \mathrm{Fr}_k^* \mathrm{res}_{\mathrm{fr}(j) \leq \mathrm{fr}(k)} \mathrm{res}_{\mathrm{fr}(i) \leq \mathrm{fr}(j)} & \xrightarrow{\mathrm{Fr}_k^* \mathrm{res}_{\mathrm{fr}(i) \leq \mathrm{fr}(j) \leq \mathrm{fr}(k)}} & \mathrm{Fr}_k^* \mathrm{res}_{\mathrm{fr}(i) \leq \mathrm{fr}(k)} \end{array}$$

commutes.

Proof. The proof of Lemma 4.1 follows the lines of the proofs of Lemmas 1.7 and 1.8. In particular, the isomorphism $\mathrm{res}_{i \leq j}^{\mathrm{fr}}$ is defined by the base-change homomorphism and the natural isomorphisms resulting from the construction of the derived functors Fr_i^* , $s_{i \leq j}^*$ and $r_{i \leq j}!$. The proof of the lemma then follows from [SGA4, Exposé XVII, §5.2]. \square

PROPOSITION 4.2. *There is a unique canonical functor $\mathrm{fr}^* : \bar{\mathcal{D}}G \rightarrow \bar{\mathcal{D}}G$ such that $(\mathrm{fr}^* A)_i = \mathrm{Fr}_i^* A_{\mathrm{fr}(i)}$ for each object A of $\bar{\mathcal{C}}G$ and for each facet i of $I(G, \mathbb{K})$.*

Proof. The promised functor is given as follows. For $A \in \mathrm{obj} \bar{\mathcal{D}}G$ and $\phi \in \mathrm{mor} \bar{\mathcal{D}}G$, and for any facet i and j of $I(G, \mathbb{K}_1^{nr})$ such that $i \leq j$, define

$$(\mathrm{fr}^* A)_i := \mathrm{Fr}_i^* A_{\mathrm{fr}(i)},$$

define

$$(\mathrm{fr}^* A)_{i \leq j} := \mathrm{Fr}_j^* A_{\mathrm{fr}(i) \leq \mathrm{fr}(j)} \circ \mathrm{res}_{i \leq j}^{\mathrm{fr}} A_{\mathrm{fr}(i)}$$

and define

$$(\mathrm{fr}^* \phi)_i := \mathrm{Fr}_i^* \phi_{\mathrm{fr}(i)}.$$

To show that $\mathrm{fr}^* A$ is an object of $\bar{\mathcal{D}}G$ when A is an object of $\bar{\mathcal{D}}G$, we must turn again to Definition 1.12(obj). From the definition of $\mathrm{res}_{i \leq j}^{\mathrm{fr}}$ in the proof of Lemma 4.1 (and Remark 1.5) it is clear that $\mathrm{res}_{i \leq i}^{\mathrm{fr}} A_{\mathrm{fr}(i)} = \mathrm{id}_{\mathrm{Fr}(i)^* A_{\mathrm{fr}(i)}}$. Thus,

$$\begin{aligned} (\mathrm{fr}^* A)_{i \leq i} &= \mathrm{Fr}_i^* A_{\mathrm{fr}(i) \leq \mathrm{fr}(i)} \circ \mathrm{res}_{i \leq i}^{\mathrm{fr}} \mathrm{Fr}_i^* A_{\mathrm{fr}(i)} \\ &= \mathrm{Fr}_i^* \mathrm{id}_{A_{\mathrm{fr}(i)}} \circ \mathrm{id}_{\mathrm{Fr}_i^* A_{\mathrm{fr}(i)}} \\ &= \mathrm{id}_{\mathrm{Fr}_i^* A_{\mathrm{fr}(i)}} \circ \mathrm{id}_{\mathrm{Fr}_i^* A_{\mathrm{fr}(i)}} \\ &= \mathrm{id}_{(\mathrm{fr}^* A)_i}. \end{aligned}$$

Having shown that fr^*A satisfies the first condition set out in Definition 1.12(obj), we now turn to the second part of Definition 1.12(obj). Suppose i, j and k are facets of $I(G, \mathbb{K})$ with $i \leq j \leq k$. We must now show that the following diagramme commutes.

$$\begin{array}{ccc}
 \text{res}_{j \leq k} \text{res}_{i \leq j}(\text{fr}^*A)_i & \xrightarrow{\text{res}_{j \leq k}(\text{fr}^*A)_{i \leq j}} & \text{res}_{j \leq k}(\text{fr}^*A)_j \\
 \downarrow \text{res}_{i \leq j \leq k}(\text{fr}^*A)_{i \leq j} & & \downarrow (\text{fr}^*A)_{j \leq k} \\
 \text{res}_{i \leq k}(\text{fr}^*A)_i & \xrightarrow{(\text{fr}^*A)_{i \leq k}} & (\text{fr}^*A)_k
 \end{array} \quad (73)$$

To that end, consider the diagramme below, in which the outer square is Diagramme 73. (To save space we temporarily write $R_{\text{fr}(j)}^{\text{fr}(i)}$ for $\text{res}_{\text{fr}(i) \leq \text{fr}(j)}$ and $A_{\text{fr}(j)}^{\text{fr}(i)}$ for $A_{\text{fr}(i) \leq \text{fr}(j)}$, etc...)

$$\begin{array}{ccc}
 R_k^j R_j^i(\text{fr}^*A)_i & \xrightarrow{\hspace{10em}} & R_k^j(\text{fr}^*A)_j \\
 \downarrow & \searrow 4. & \swarrow 2. \\
 & \text{Fr}_k^* R_{\text{fr}(k)}^{\text{fr}(j)} R_{\text{fr}(j)}^{\text{fr}(i)} A_{\text{fr}(i)} & \longrightarrow & \text{Fr}_k^* R_{\text{fr}(k)}^{\text{fr}(j)} A_{\text{fr}(j)} \\
 & \downarrow & & \downarrow \\
 & \text{Fr}_k^* R_{\text{fr}(k)}^{\text{fr}(i)} A_{\text{fr}(i)} & \longrightarrow & \text{Fr}_k^* A_{\text{fr}(k)} \\
 \downarrow & \swarrow 3. & & \nwarrow 1. \\
 R_k^i(\text{fr}^*A)_i & \xrightarrow{\hspace{10em}} & (\text{fr}^*A)_k
 \end{array}$$

To show that fr^*A satisfies the second condition appearing in Definition 1.12(obj) we must show that the outer square commutes in Diagramme 73. The inner square is the result of applying the functor Fr_k^* to the commuting square appearing in Definition 1.12(obj) applied to $A_{\text{fr}(i)}$, and is therefore commutative. The arrow marked 1. is the identity. The arrow marked 2. is $\text{res}_{j \leq k}^{\text{fr}} A_{\text{fr}(j)}$, so the right-hand square commutes by virtue of the definition of $(\text{fr}^*A)_{j \leq k}$; likewise, the arrow marked 3. is $\text{res}_{i \leq k}^{\text{fr}} A_{\text{fr}(i)}$, so the bottom square commutes by virtue of the definition of $(\text{fr}^*A)_{j \leq k}$. The arrow marked 4. is $\text{res}_{j \leq k}^{\text{fr}} \text{res}_{\text{fr}(i) \leq \text{fr}(j)} A_{\text{fr}(i)} \circ \text{res}_{j \leq k} \text{res}_{i \leq j}^{\text{fr}} A_{\text{fr}(i)}$ and the top and left-hand squares commute by Lemma 4.4. This concludes the demonstration that fr^*A is an object in $\bar{D}G$.

Suppose $\phi : A \rightarrow B$ is a morphism in $\bar{D}G$. In order to show that $\text{fr}^*\phi$ is a

morphism in $\bar{\mathcal{D}}G$ we must show that the following diagramme commutes.

$$\begin{array}{ccc}
 \text{res}_{i \leq j}(\text{fr}^* A)_i & \xrightarrow{\text{res}_{i \leq j}(\text{fr}^* \phi)_i} & \text{res}_{i \leq j}(\text{fr}^* B)_i \\
 (\text{fr}^* A)_{i \leq j} \downarrow & & \downarrow (\text{fr}^* B)_{i \leq j} \\
 (\text{fr}^* A)_j & \xrightarrow{(\text{fr}^* \phi)_j} & (\text{fr}^* B)_j
 \end{array} \quad (74)$$

Consider the following diagramme, in which the outer square is Diagramme 74. (To save space we temporarily write R_j^i for $\text{res}_{i \leq j}$ and $R_{\text{fr}(j)}^{\text{fr}(i)}$ for $\text{res}_{\text{fr}(i) \leq \text{fr}(j)}$ etc... , as above.)

$$\begin{array}{ccccc}
 R_j^i \text{Fr}_i^* A_{\text{fr}(i)} & \xrightarrow{\hspace{10em}} & R_j^i \text{Fr}_i^* B_{\text{fr}(i)} & & \\
 \downarrow & \searrow 3. & & \swarrow 4. & \downarrow \\
 & \text{Fr}_j^* R_{\text{fr}(j)}^{\text{fr}(i)} A_{\text{fr}(i)} & \xrightarrow{0.} & \text{Fr}_j^* R_{\text{fr}(j)}^{\text{fr}(i)} B_{\text{fr}(i)} & \\
 & \swarrow 2. & & \searrow 1. & \\
 \text{Fr}_j^* A_{\text{fr}(j)} & \xrightarrow{\hspace{10em}} & \text{Fr}_j^* B_{\text{fr}(j)} & &
 \end{array}$$

The arrow marked 0. is $\text{Fr}_j^* \text{res}_{\text{fr}(i) \leq \text{fr}(j)} \phi_{\text{fr}(i)}$, the arrow marked 1. is $\text{Fr}_j^* B_{\text{fr}(i) \leq \text{fr}(j)}$ and the arrow marked 2. is $\text{Fr}_j^* A_{\text{fr}(i) \leq \text{fr}(j)}$; thus, the bottom square is the result of applying the functor Fr_j^* to the relevant form of the commuting square appearing in Definition 1.12(mor), and therefore commutes since ϕ is a morphism in $\bar{\mathcal{D}}G$. The arrow marked 3. is $\text{res}_{i \leq j}^{\text{fr}} A_{\text{fr}(i)}$ and the arrow marked 3. is $\text{res}_{i \leq j}^{\text{fr}} B_{\text{fr}(i)}$, so the upper square commutes because $\text{res}_{i \leq j}^{\text{fr}}$ is a natural transformation. The left-hand triangle commutes by virtue of the definition of $(\text{fr}^* A)_{i \leq j}$ and likewise the right-hand triangle commutes by virtue of the definition of $(\text{fr}^* B)_{i \leq j}$. Therefore, the outer square commutes. This concludes the demonstration that $\text{fr}^* \phi$ is a morphism in $\bar{\mathcal{D}}G$. \square

DEFINITION 4.3. An object A of the category $\bar{\mathcal{D}}G$ is *frobenius-stable* if there is an isomorphism $\alpha : \text{fr}^* A \rightarrow A$ in category $\bar{\mathcal{D}}G$.

LEMMA 4.4. Let i and j be facets of $I(G, \mathbb{K}_1^{nr})$ with $i \leq j$. Let g be an element of $G(\mathbb{K}_1^{nr})$. Then

$$\text{ind}_{i \leq j} \text{Fr}_j^* \cong \text{Fr}_i^* \text{ind}_{\text{fr}(i) \leq \text{fr}(j)},$$

and

$$\bar{m}(g)_i^* \text{Fr}_{gi}^* \cong \text{Fr}_i^* \bar{m}(\text{fr}(g))_{\text{fr}(i)}^*.$$

4.2 PARABOLIC RESTRICTION AND FROBENIUS

DEFINITION 4.5. Let $L_{\mathbb{K}_1} \subseteq G_{\mathbb{K}_1}$ be a subgroup. Here, $L_{\mathbb{K}_1}$, $G_{\mathbb{K}_1}$ and the inclusion are all over \mathbb{K}_1 . We say that $L_{\mathbb{K}_1} \subseteq G_{\mathbb{K}_1}$ is an *unramified twisted-Levi subgroup* if there is a finite extension $\mathbb{K}_1' : \mathbb{K}_1$ contained in \mathbb{K}_1^{nr} such that

$L_{\mathbb{K}_1'} \subseteq G_{\mathbb{K}_1'}$ is a Levi subgroup. Note that this implies that there is a parabolic subgroup P defined over \mathbb{K}_1' such that $L_{\mathbb{K}_1'}$ is the maximal reductive quotient of $P_{\mathbb{K}_1'}$.

Note that an unramified twisted-Levi subgroup is not, in general, a Levi subgroup; rather, an unramified twisted-Levi subgroup is a *form* of a Levi subgroup.

PROPOSITION 4.6. *Let P be a parabolic subgroup of G with Levi component L . Suppose L is defined over \mathbb{K}_1 and $L_{\mathbb{K}_1} \subseteq G_{\mathbb{K}_1}$ is an unramified twisted-Levi subgroup. Let A be an object of $\bar{D}G$. If A is Frobenius-stable then $\text{res}_P^G A$ is also Frobenius-stable.*

Proof. Let $\alpha : \text{fr}^* A \rightarrow A$ be an isomorphism in category $\bar{D}G$. Recall the definition of $\text{res}_P^G \phi$ (see Proposition 2.1). Lemma 4.4 shows that $\text{res}_P^G \alpha$ is an isomorphism and therefore that $\text{res}_P^G A$ is Frobenius-stable. \square

4.3 CUSPIDAL COEFFICIENT SYSTEMS REVISITED

In this section we briefly revisit Section 2.3 and adapt Definition 2.8 and Theorem 2.11 to the present context.

Let i_0 be a vertex of $I(G, \mathbb{K}_1)$ and let F be a cuspidal character sheaf for \bar{G}_{i_0} equipped with an isomorphism $\varphi : \text{Fr}_{i_0}^* F \rightarrow F$ in $D_c^b(\bar{G}_{i_0}; \bar{\mathbb{Q}}_\ell)$. Then there is an object B and isomorphism $\beta : \text{fr}_G^* B \rightarrow B$ in $\bar{C}G$ such that $B_{i_0} = F$, $\beta_{i_0} = \varphi$. The proof follows the lines of the proof of Proposition 2.9. For any vertex i of $I(G, \mathbb{K}_1)$, cuspidal character sheaf F on \bar{G}_i and isomorphism $\varphi : \text{Fr}_i^* F \rightarrow F$, we will write $\text{cind}_{\bar{G}_i}^G F$ and $\text{cind}_{\bar{G}_i}^G \varphi$ for the object and morphism in $\bar{C}G$ promised above. Suppose C is a cuspidal coefficient system for G and Frobenius-stable. Then there is a vertex i_0 of $I(G, \mathbb{K}_1)$ and a Frobenius-stable cuspidal character sheaf F for \bar{G}_{i_0} such that $C \cong \text{cind}_{\bar{G}_{i_0}}^G F$. The proof follows the lines of the proof of Theorem 2.11.

4.4 PARABOLIC INDUCTION AND FROBENIUS

PROPOSITION 4.7. *Let P be a parabolic subgroup of G with Levi component L . Suppose L is defined over \mathbb{K}_1 and $L_{\mathbb{K}_1} \subseteq G_{\mathbb{K}_1}$ is an unramified-twisted Levi subgroup. If B is a cuspidal coefficient system for L then*

$$\text{ind}_P^G \text{fr}_L^* B \cong \text{fr}_G^* \text{ind}_P^G B. \quad (75)$$

Proof. Let i be a facet of $I(G, \mathbb{K}_1^{nr})$ and consider $(\text{ind}_P^G \text{fr}_L^* B)_i$. Using Propo-

sition 3.5 and Proposition 4.2 we have

$$\begin{aligned}
 (\operatorname{ind}_P^G \operatorname{fr}_L^* B)_i &= \sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \bar{m}(g^{-1})^* \operatorname{ind}_{\bar{P}_{i_P^g}}^{\bar{G}_{ig}} (\operatorname{fr}_L^* B)_{i_P^g} \\
 &= \sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \bar{m}(g^{-1})^* \operatorname{ind}_{\bar{P}_{i_P^g}}^{\bar{G}_{ig}} (\operatorname{fr}_L^* B)_{\operatorname{fr}(i_P^g)} \\
 &= \sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \bar{m}(g^{-1})^* \operatorname{ind}_{\bar{P}_{i_P^g}}^{\bar{G}_{ig}} \operatorname{Fr}_{i_P^g}^* B_{\operatorname{fr}(i)^{\operatorname{fr}(g)}},
 \end{aligned}$$

since L is defined over \mathbb{K}_1 . Now, using Lemma 4.4 we have

$$\begin{aligned}
 &\sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \bar{m}(g^{-1})^* \operatorname{ind}_{\bar{P}_{i_P^g}}^{\bar{G}_{ig}} \operatorname{Fr}_{i_P^g}^* B_{\operatorname{fr}(i)^{\operatorname{fr}(g)}} \\
 &= \sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \bar{m}(g^{-1})^* \operatorname{Fr}_{i_P^g}^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{\operatorname{fr}(g)}}}^{\bar{G}_{\operatorname{fr}(ig)}} B_{\operatorname{fr}(i)^{\operatorname{fr}(g)}} \\
 &= \sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \operatorname{Fr}_i^* \bar{m}(\operatorname{fr}(g)^{-1})^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{\operatorname{fr}(g)}}}^{\bar{G}_{\operatorname{fr}(ig)}} B_{\operatorname{fr}(i)^{\operatorname{fr}(g)}} \\
 &= \operatorname{Fr}_i^* \sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \bar{m}(\operatorname{fr}(g)^{-1})^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{\operatorname{fr}(g)}}}^{\bar{G}_{\operatorname{fr}(ig)}} B_{\operatorname{fr}(i)^{\operatorname{fr}(g)}}.
 \end{aligned}$$

If $P_{ig} \subset G_{ig}$ is projective then $P_{\operatorname{fr}(i)\operatorname{fr}(g)} \subset G_{\operatorname{fr}(i)\operatorname{fr}(g)}$ is projective, since L is a \mathbb{K}_1 -scheme, so $\operatorname{fr}(g)$ represents an element of $D_{\operatorname{fr}(i)}(G, P, \mathbb{K}_1^{nr})$ (see Definition 3.1). Accordingly, there is some $h \in G_{\operatorname{fr}(i)}(\mathfrak{o}_{\mathbb{K}_1^{nr}})$ and $l \in L(\mathbb{K}_1^{nr})$ such that $\operatorname{fr}(g) = hg'l$, for a unique $g' \in d_{\operatorname{fr}(i)}(G, P, \mathbb{K}_1^{nr})$ (see Definition 3.1 again). Thus,

$$\begin{aligned}
 &\operatorname{Fr}_i^* \sum_{g \in d_i(G, P, \mathbb{K}_1^{nr})} \bar{m}(\operatorname{fr}(g)^{-1})^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{\operatorname{fr}(g)}}}^{\bar{G}_{\operatorname{fr}(ig)}} B_{\operatorname{fr}(i)^{\operatorname{fr}(g)}} \\
 &= \operatorname{Fr}_i^* \sum_{g' \in d_{\operatorname{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}((hg'l)^{-1})^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{hg'l}}}^{\bar{G}_{\operatorname{fr}(i)hg'l}} B_{\operatorname{fr}(i)^{hg'l}} \\
 &= \operatorname{Fr}_i^* \sum_{g' \in d_{\operatorname{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}(h^{-1})^* \bar{m}(g'^{-1})^* \bar{m}(l^{-1})^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{g'l}}}^{\bar{G}_{\operatorname{fr}(i)xl}} B_{\operatorname{fr}(i)^{g'l}} \\
 &= \operatorname{Fr}_i^* \sum_{g' \in d_{\operatorname{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}(h^{-1})^* \bar{m}(g'^{-1})^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{g'l}}}^{\bar{G}_{\operatorname{fr}(i)g'l}} \bar{m}(l^{-1})^* B_{\operatorname{fr}(i)^{g'l}} \\
 &= \operatorname{Fr}_i^* \sum_{g' \in d_{\operatorname{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}(h^{-1})^* \bar{m}(g'^{-1})^* \operatorname{ind}_{\bar{P}_{\operatorname{fr}(i)^{g'l}}}^{\bar{G}_{\operatorname{fr}(i)g'l}} {}^l B_{\operatorname{fr}(i)^{g'l}}.
 \end{aligned}$$

Since B is weakly-equivariant (cf. Definition 2.6) we have

$$\begin{aligned} & \mathrm{Fr}_i^* \sum_{g' \in d_{\mathrm{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}(h^{-1})^* \bar{m}(g'^{-1})^* \mathrm{ind}_{\bar{L}_{\mathrm{fr}(i)}^{g'}}^{\bar{G}_{\mathrm{fr}(i)}^{g'}} {}^l B_{\mathrm{fr}(i)_P^{g'}} \\ & \cong \mathrm{Fr}_i^* \sum_{g' \in d_{\mathrm{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}(h^{-1})^* \bar{m}(g'^{-1})^* \mathrm{ind}_{\bar{P}_{\mathrm{fr}(i)}^{g'}}^{\bar{G}_{\mathrm{fr}(i)}^{g'}} B_{\mathrm{fr}(i)_P^{g'}}. \end{aligned}$$

Since B is cuspidal (cf. Definition 2.8) we have

$$\begin{aligned} & \mathrm{Fr}_i^* \sum_{g' \in d_{\mathrm{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}(h^{-1})^* \bar{m}(g'^{-1})^* \mathrm{ind}_{\bar{P}_{\mathrm{fr}(i)}^{g'}}^{\bar{G}_{\mathrm{fr}(i)}^{g'}} B_{\mathrm{fr}(i)_P^{g'}} \\ & \cong \mathrm{Fr}_i^* \sum_{g' \in d_{\mathrm{fr}(i)}(G, P, \mathbb{K}_1^{nr})} \bar{m}(g'^{-1})^* \mathrm{ind}_{\bar{P}_{\mathrm{fr}(i)}^{g'}}^{\bar{G}_{\mathrm{fr}(i)}^{g'}} B_{\mathrm{fr}(i)_P^{g'}} \\ & = \mathrm{Fr}_i^* (\mathrm{ind}_P^G B)_{\mathrm{fr}(i)} \\ & = (\mathrm{fr}^* \mathrm{ind}_P^G B)_i. \end{aligned}$$

The isomorphism of Proposition 4.7 is now found by arguments similar to those employed in the proof of Proposition 3.5. \square

COROLLARY 4.8. *Let P be a parabolic subgroup of G with Levi component L . Suppose L is defined over \mathbb{K}_1 and $L_{\mathbb{K}_1} \subseteq G_{\mathbb{K}_1}$ is an unramified twisted-Levi subgroup. Let B be an weakly-equivariant object equipped with an isomorphism $\beta : \mathrm{fr}_L^* B \rightarrow B$ in $\bar{C}L$. Then $\mathrm{ind}_L^G \beta$ defines an isomorphism $\mathrm{fr}_G^* \mathrm{ind}_P^G B \rightarrow \mathrm{ind}_P^G B$ in $\bar{C}G$.*

Proof. Let $A = \mathrm{ind}_P^G B$ and let $\alpha = \mathrm{ind}_P^G \beta$. Since $\beta : \mathrm{fr}_L^* B \rightarrow B$ is an isomorphism in $\bar{C}L$, and since parabolic induction is a functor, it follows that α is an isomorphism. However, the domain of α is $\mathrm{ind}_P^G \mathrm{fr}_L^* B$, rather than $\mathrm{fr}_G^* \mathrm{ind}_P^G B$. Corollary 4.8 now follows directly from Proposition 4.7. \square

5 SUPERCUSPIDAL DEPTH-ZERO REPRESENTATIONS

Let the field extensions $\mathbb{K}_1^{nr} : \mathbb{K}_1$ and $\bar{\mathbb{F}}_q : \mathbb{F}_q$ be as in Section 4; likewise, let $G_{\mathbb{K}_1}$ be a connected unramified linear algebraic group over \mathbb{K}_1 and let G denoted the group scheme over \mathbb{K}_1^{nr} obtained by extension of scalars, as in Section 4. Suppose now that i is fixed by the Galois action on the building. Using the principle of étale descent we see that there is a smooth group scheme $G_{i/\mathfrak{o}_{\mathbb{K}_1}}$ over $\mathfrak{o}_{\mathbb{K}_1}$ such that: $G_{i/\mathfrak{o}_{\mathbb{K}_1}}$ is an integral model of $G_{\mathbb{K}_1}$ equipped with a $\mathfrak{o}_{\mathbb{K}_1}$ -rational structure, compatible with the isomorphism of generic fibres, and such that $G_{i/\mathfrak{o}_{\mathbb{K}_1}}(\mathfrak{o}_{\mathbb{K}_1}) = G(\mathbb{K}_1)_i$ (cf. [Lan96, 10.10]). The special fibre $\tilde{G}_{i/\mathbb{F}_q}$ of $G_{i/\mathfrak{o}_{\mathbb{K}_1}}$ is a linear algebraic group and it defines a \mathbb{F}_q -rational structure for \tilde{G}_i . Moreover, there is a maximal reductive quotient $\nu_{i/\mathbb{F}_q} : \tilde{G}_{i/\mathbb{F}_q} \rightarrow \tilde{G}_{i/\mathbb{F}_q}$ (cf. Section 1.2) and it defines a \mathbb{F}_q -rational structure for \tilde{G}_i . Let $\rho_{i/\mathbb{K}_1} : G_i(\mathfrak{o}_{\mathbb{K}_1}) \rightarrow \tilde{G}_i(\mathbb{F}_q)$ be the canonical quotient (cf. Section 1.2).

5.1 CHARACTERISTIC FUNCTIONS

Let A be a frobenius-stable coefficient system for G and let $\alpha : \text{fr}_G^* A \rightarrow A$ be an isomorphism in $\bar{\mathcal{C}}G$. For each facet i of $I(G, \mathbb{K}_1^{nr})$, α defines an isomorphism $\alpha_i : \text{fr}_i^* A_{\text{fr}(i)} \rightarrow A_i$ in category $D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell)$. If i is actually a facet of $I(G, \mathbb{K}_1)$ (so $\text{fr}(i) = i$) then $A_i \in D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell)$ is frobenius-stable in the usual sense; in that case, let $\chi_{A_i, \alpha_i} : \bar{G}_i(\mathbb{F}_q) \rightarrow \bar{\mathbb{Q}}_\ell$ be the characteristic function associated to the pair (A_i, α_i) (cf. [Lus85, 8.4], for example).

PROPOSITION 5.1. *Let A be a weakly-equivariant coefficient system for G and let $\alpha : \text{fr}^* A \rightarrow A$ be an isomorphism in $\bar{\mathcal{C}}G$. If i is a facet of $I(G, \mathbb{K}_1)$ then*

$$\forall x \in G_i(\mathfrak{o}_{\mathbb{K}_1}), \quad \chi_{A_{g_i}, \alpha_{g_i}}(\rho_{g_i}(gxg^{-1})) = \chi_{A_i, \alpha_i}(\rho_i(x)), \quad (76)$$

for all $g \in G(\mathbb{K}_1)$.

Let $L_{\mathbb{K}_1} \subseteq G_{\mathbb{K}_1}$ be a twisted-Levi subgroup; so $L := L_{\mathbb{K}_1} \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr})$ is a Levi subgroup of $G := G_{\mathbb{K}_1} \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr})$. Let P be a parabolic subgroup of G with Levi component L . Let B be a frobenius-stable weakly-equivariant coefficient system for L equipped with $\beta : \text{fr}^* B \rightarrow B$ and let $A = \text{ind}_P^G B$ equipped with $\alpha : \text{fr}^* A \rightarrow A$ as in Corollary 4.8. For any facet i of $I(G, \mathbb{K}_1^{nr})$ and $g \in d_i(G, P, \mathbb{K}_1^{nr})$ let $A_i(g)$ denote the summand of A_i in $D_c^b(\bar{G}_i; \bar{\mathbb{Q}}_\ell)$ given by

$$A_i(g) := \bar{m}(g^{-1})_i^* \text{ind}_{L_i^g}^{\bar{G}_i} B_{i_L^g}. \quad (77)$$

We will need the following result concerning characteristic functions of induced objects in the proof of Theorem 5.4.

PROPOSITION 5.2. *With notation as above, suppose i is a facet of $I(G, \mathbb{K}_1)$ and let $d_i(G, L, \mathbb{K}_1)$ denote the set of $g \in d_i(G, P, \mathbb{K}_1^{nr})$ (cf. Definition 3.1) such that G_{ig} is defined over $\mathfrak{o}_{\mathbb{K}_1}$. Then*

$$\forall x \in \bar{G}_i(\mathbb{F}_q), \quad \chi_{A_i, \alpha_i}(x) = \sum_{g \in d_i(G, P, \mathbb{K}_1)} \chi_{A_i(g), \alpha_i(g)}(x), \quad (78)$$

where the right-hand side is trivial if $d_i(G, L, \mathbb{K}_1)$ is empty.

Proof. If g is an element of $d_i(G, P, \mathbb{K}_1)$ then $A_i(g)$ is itself frobenius-stable; more precisely, the restriction of α_i to $\text{Fr}_i^* A_i(g)$, which we denote $\alpha_i(g)$, is an isomorphism onto $A_i(g)$. It follows that

$$\sum_{g \in d_i(G, P, \mathbb{K}_1)} A_i(g) \quad (79)$$

is a frobenius-stable summand of A_i . For any $g \in d_i(G, P, \mathbb{K}_1^{nr})$ we have $\text{fr}(g) = hg'l$ for a unique $g' \in d_i(G, L, \mathbb{K}_1^{nr})$ with $h \in G_i(\mathfrak{o}_{\mathbb{K}_1^{nr}})$ and $l \in L(\mathbb{K}_1^{nr})$ (cf. proof of Proposition 4.7). Suppose $g \in d_i(G, P, \mathbb{K}_1^{nr})$ and $g \notin d_i(G, P, \mathbb{K}_1)$.

Then $\bar{G}_{i,g}(\mathbb{F}_q) = \emptyset$. There are two cases to consider: either $g' = g$ or $g' \neq g$. In the first case, $A_i(g)$ is a frobenius-stable summand of A_i with $\chi_{A_i(g), \alpha_i(g)} = 0$, with $\alpha_i(g)$ as above; it follows that the sum of such $A_i(g)$ is a frobenius-stable summand of A_i with trivial characteristic function. The sum of the objects $A_i(g)$ with g in the second case is also a frobenius-stable summand of A_i with trivial characteristic function. However, in this case the summands $A_i(g)$ are not themselves frobenius-stable. In summary, we have

$$\forall x \in \bar{G}_i(\mathbb{F}_q), \quad \chi_{A_i, \alpha_i}(x) = \sum_{g \in d_i(G, P, \mathbb{K}_1)} \chi_{A_i(g), \alpha_i(g)}(x), \quad (80)$$

where the right-hand side is trivial if $d_i(G, P, \mathbb{K}_1)$ is empty, as desired. \square

5.2 MODELS FOR REPRESENTATIONS

Let $\pi : G(\mathbb{K}_1) \rightarrow \text{End}_{\bar{\mathbb{Q}}_\ell}(V)$ be an admissible representation. For each facet i of the building $I(G, \mathbb{K}_1)$, let V_i denote the $\bar{\mathbb{Q}}_\ell$ -vector space consisting of all $v \in V$ for which $\pi(h)v = v$ for each $h \in G_i(\mathfrak{o}_{\mathbb{K}_1^r})$ such that $\rho_{i/\mathbb{K}_1}(h) = 1$. We let

$$\pi_i : G_i(\mathfrak{o}_{\mathbb{K}_1}) \rightarrow \text{End}_{\bar{\mathbb{Q}}_\ell}(V_i), \quad (81)$$

denote the *compact restriction* of π at i ; that is to say, we let π_i denote the representation of the group $G_i(\mathfrak{o}_{\mathbb{K}_1})$ on V_i defined by $\pi_i(h)v = \pi(h)v$ for $h \in G_i(\mathfrak{o}_{\mathbb{K}_1})$ and $v \in V_i$. We note that, for each $x \in i$, the group $G(\mathbb{K}_1)_i$ equals $G(\mathbb{K}_1)_{x,0}$, where the latter is defined in [MP96]. Moreover, the group of all $h \in G_i(\mathbb{K}_1)$ such that $\rho_{i/\mathbb{K}_1}(h) = 1$ is exactly the group $G(\mathbb{K}_1)_{x,0^+}$, for $x \in i$ (cf. [MP96] also). By [Vig97, Propn 1.1], $G(\mathbb{K}_1)_{x,0^+}$ equals $U_x^{(0)}$, where the latter is defined in [SS97]. Thus the set of $\bar{\mathbb{Q}}_\ell$ -vector spaces

$$\gamma_0(V) = \{V_i \mid i \text{ facet of } I(G, \mathbb{K}_1)\}, \quad (82)$$

equipped with inclusions $V_j \hookrightarrow V_i$ for $i \leq j$, forms a $G(\mathbb{K}_1)$ -equivariant coefficient system of $\bar{\mathbb{Q}}_\ell$ -vector spaces, in the sense of [SS97].

Now suppose π is a depth-zero supercuspidal representation. Then, for each facet i of $I(G, \mathbb{K}_1)$, the compact restriction π_i factors through ρ_{i/\mathbb{K}_1} to a representation

$$\bar{\pi}_i : \bar{G}_i(\mathbb{F}_q) \rightarrow \text{End}_{\bar{\mathbb{Q}}_\ell}(V_i); \quad (83)$$

that is, $\bar{\pi}_i = \pi_i \circ \rho_{i/\mathbb{K}_1}$. If π is non-trivial, then, by the definition of depth-zero representations, the coefficient system $\gamma_0(V)$ is non-trivial; in fact there is some vertex i_0 of $I(G, \mathbb{K}_1)$ such that $V_{i_0} \neq 0$.

DEFINITION 5.3. Let $\bar{K}G$ denote the subgroup of the Grothendieck group for $\bar{D}G$ generated by admissible coefficient systems for G (cf. Definition 3.7). Let $\pi : G(\mathbb{K}_1) \rightarrow \text{Aut}_{\bar{\mathbb{Q}}_\ell}(V)$ be a depth-zero admissible representation. A *model* for π is an element $\sum_n a_n [A^n]$ of $\bar{K}G \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_\ell$ where each A^n is equipped with an isomorphism $\alpha^n : \text{fr}^* A^n \rightarrow A^n$ (so A^n is frobenius-stable) such that

$$\forall x \in \bar{G}_i(\mathbb{F}_q), \quad \sum_n a_n \chi_{A_i^n, \alpha_i^n}(x) = \text{trace } \bar{\pi}_i(x), \quad (84)$$

for each facet i of $I(G, \mathbb{K}_1)$.

THEOREM 5.4. *Supercuspidal depth-zero representations admit models; that is, for each supercuspidal depth-zero representation π of $G(\mathbb{K}_1)$ there is a some $\sum_n a_n [A^n] \in \bar{K}G \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_\ell$ and isomorphisms $\alpha^n : \text{fr}^* A^n \rightarrow A^n$ such that*

$$\forall x \in \bar{G}_i(\mathbb{F}_q), \quad \sum_n a_n \chi_{A_i^n, \alpha_i^n}(x) = \text{trace } \bar{\pi}_i(x), \quad (85)$$

for each facet i of $I(G, \mathbb{K}_1)$.

Proof. Let $\pi : G(\mathbb{K}_1) \rightarrow \text{Aut}_{\bar{\mathbb{Q}}_\ell}(V)$ be an irreducible supercuspidal representation. Since π has depth-zero, there is a vertex i_0 of $I(G, \mathbb{K}_1)$ such that $(G_{i_0}(\mathfrak{o}_{\mathbb{K}_1}), \pi_{i_0})$ is a type of π . Moreover, since π is irreducible, we have $\pi_i = 0$ unless i is contained in the $G(\mathbb{K}_1)$ -orbit of i_0 in $I(G, \mathbb{K}_1)$. By [Lus85, 25.1] and [Lus85, (10.4.5)], we may write

$$\text{trace } \bar{\pi}_{i_0} = \sum_n a_n \chi_{\text{ind}_{M_n}^{\bar{G}_{i_0}} F_n, \text{ind}_{M_n}^{\bar{G}_{i_0}} \varphi_n}, \quad (86)$$

where the sum is taken over cuspidal pairs (M_n, F_n) for \bar{G}_{i_0} (as defined in [Lus84, 2.4]) such that M_n is defined over \mathbb{F}_q and where F_n is equipped with a fixed isomorphism $\varphi_n : \text{Fr}_{M_n}^* F_n \rightarrow F_n$. Since $\bar{\pi}_{i_0}$ is cuspidal, the scalars a_n are zero except when M_n is anisotropic over \mathbb{F}_q (cf. [Lus85, (15.2.1)]). We define $E_n = \text{ind}_{M_n}^{\bar{G}_{i_0}} F_n$ and $\varepsilon_n = \text{ind}_{M_n}^{\bar{G}_{i_0}} \varphi_n$.

Let T^n be a maximal \mathbb{K}_1^{nr} -split torus in G , defined over \mathbb{K}_1 , such that the associated apartment contains i_0 , and such that the image of $T^n(\mathbb{K}_1^{nr}) \cap G_{i_0}(\mathfrak{o}_{\mathbb{K}_1})$ in M_n is the group of \mathbb{F}_q -rational points of an elliptic torus of M_n (the existence of T^n is guaranteed by [BT84, end of the proof of prop. 5.1.10]). The torus T^n is elliptic. Thus, the centralizer L^n of T^n in G is an elliptic Levi subgroup defined over \mathbb{K}_1 such that the image of $I(L^n, \mathbb{K}_1)$ under $I(L^n, \mathbb{K}_1^{nr}) \rightarrow I(G, \mathbb{K}_1^{nr})$ is $\{i_0\}$, and $\bar{L}_{i_0}^n = M_n$. Now $L_{\mathbb{K}_1}^n$ is an elliptic unramified twisted-Levi subgroup of $G_{\mathbb{K}_1}$. Let P^n be a parabolic subgroup of G with Levi component $L^n = L_{\mathbb{K}_1}^n \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr})$.

Using Section 4.3, set

$$B^n = \text{cind}_{L_{i_0}^n}^{L^n} F_n \quad \text{and} \quad \beta^n = \text{cind}_{L_{i_0}^n}^{L^n} \varphi_n. \quad (87)$$

Then

$$B_{i_0}^n = F_n \quad \text{and} \quad \beta_{i_0}^n = \varphi_n. \quad (88)$$

Define

$$A^n = \text{ind}_P^G B^n. \quad (89)$$

Let

$$\alpha^n : \text{fr}^* A^n \rightarrow A^n \quad (90)$$

be the isomorphism given by Corollary 4.8. In order to prove Theorem 5.4 we now show that

$$\forall x \in \bar{G}_i(\mathbb{F}_q), \quad \sum_n a_n \chi_{A_i^n, \alpha_i^n}(x) = \text{trace } \bar{\pi}_i(x), \quad (91)$$

for each facet i of $I(G, \mathbb{K}_1)$.

Consider the case $i = i_0$. Notice that since i_0 is contained in the image of the map from $I(L^n, \mathbb{K}_1^{nr})$ to $I(G, \mathbb{K}_1^{nr})$, it follows that the schematic closure of L^n in G_{i_0} is $L_{i_0}^n$. If $g \in d_{i_0}(G, P^n, \mathbb{K}_1)$ then G_{i_0g} is defined over \mathbb{K}_1 , so the schematic closure of L^n in G_{i_0g} is a parahoric subgroup of L^n defined over \mathbb{K}_1 . Since $L_{\mathbb{K}_1}^n$ is elliptic there is exactly one such parahoric subgroup, namely $L_{i_0}^n$, and $d_{i_0}(G, P^n, \mathbb{K}_1)$ is a singleton. Thus, g and 1 represent the same double coset in $D_{i_0}(G, L^n, \mathbb{K}_1^{nr})$ (cf. Definition 3.1), so $g = hl$ for some $h \in G_i(\mathfrak{o}_{\mathbb{K}_1^{nr}})$ and $l \in L^n(\mathbb{K}_1^{nr})$. Using Lemma 4.4 we have

$$\begin{aligned} A_{i_0g}^n &= \bar{m}(g^{-1})_{i_0}^* \text{ind}_{(\bar{P}^n)_{(i_0)P^n}^g}^{\bar{G}_{i_0g}} B_{(i_0)P^n}^g \\ &= \bar{m}((hl)^{-1})_{i_0}^* \text{ind}_{(\bar{P}^n)_{(i_0)P^n}^{(hl)}}^{\bar{G}_{i_0hl}} B_{(i_0)P^n}^{hl} \\ &= \bar{m}(h^{-1})^* \bar{m}(l^{-1})_{i_0}^* \text{ind}_{(\bar{P}^n)_{i_0}^l}^{\bar{G}_{i_0l}} B_{i_0l} \\ &= \bar{m}(h^{-1})^* \text{ind}_{\bar{P}_{i_0}^n}^{\bar{G}_{i_0}} \bar{m}(l^{-1})_{i_0}^* B_{i_0l} \\ &= \bar{m}(h^{-1})^* \text{ind}_{\bar{P}_{i_0}^n}^{\bar{G}_{i_0}} {}^l B_{i_0}. \end{aligned}$$

Moreover, since B is a weakly-equivariant object of $\bar{C}L^n$ and $h \in G_{i_0}(\mathfrak{o}_{\mathbb{K}_1^{nr}})$ we have canonical isomorphisms

$$\bar{m}(h^{-1})^* \text{ind}_{\bar{P}_{i_0}^n}^{\bar{G}_{i_0}} {}^l B_{i_0} \cong \bar{m}(h)_{i_0}^* \text{ind}_{\bar{P}_{i_0}^n}^{\bar{G}_{i_0}} B_{i_0}^n \cong \text{ind}_{\bar{P}_{i_0}^n}^{\bar{G}_{i_0}} B_{i_0}^n.$$

Finally, since

$$\bar{P}_{i_0}^n = \bar{L}_{i_0}^n = M_n, \quad (92)$$

we have

$$\text{ind}_{\bar{L}_{i_0}^n}^{\bar{G}_{i_0}} B_{i_0}^n = \text{ind}_{M_n}^{\bar{G}_{i_0}} F_n = E_n,$$

by Equation 88. Since the isomorphisms just used are exactly those appearing in the definition of $\alpha_{i_0}^n$ (cf. Corollary 4.8), together with Proposition 5.2 we have

$$\forall x \in \bar{G}_{i_0}(\mathbb{F}_q), \quad \chi_{A_{i_0}^n, \alpha_{i_0}^n}(x) = \chi_{E_n, \varepsilon_n}(x). \quad (93)$$

By Equation 86 we now have

$$\forall x \in \bar{G}_{i_0}(\mathbb{F}_q), \quad \sum_n a_n \chi_{A_{i_0}^n, \alpha_{i_0}^n}(x) = \text{trace } \bar{\pi}_{i_0}(x). \quad (94)$$

This verifies Equation 91 in this case. Proposition 5.1 extends Equation 94, *mut. mut.*, to all i in the $G(\mathbb{K}_1)$ -orbit of i_0 .

If i is a facet of $I(G, \mathbb{K}_1)$ which does not lie in the $G(\mathbb{K}_1)$ -orbit of i_0 then $d_i(G, P^n, \mathbb{K}_1)$ is empty, again since $L_{\mathbb{K}_1}^n$ is an elliptic twisted-Levi subgroup of $G_{\mathbb{K}_1}$, so Proposition 5.2 gives

$$\forall x \in \bar{G}_i(\mathbb{F}_q), \quad \chi_{A_i^n, \alpha_i^n}(x) = 0. \quad (95)$$

Gathering Equations 86, 94 and 95 gives Equation 91 in this case also.

Finally, consider the $\bar{\mathbb{Q}}_\ell$ -vector space formed by taking the tensor product of the subgroup $\mathcal{K}_0(\bar{G}_{i_0})$ of the Grothendieck group of perverse sheaves on \bar{G}_{i_0} generated by character sheaves of \bar{G}_{i_0} with $\bar{\mathbb{Q}}_\ell$ (*cf.* [Lus85, 14.10]). Then $\sum_n a_n[E_n]$ (summation as in Equation 86) is an element of $\mathcal{K}_0(\bar{G}_{i_0}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_\ell$. Now, Equation 91 shows that $\sum_n a_n[A^n]$, equipped with the isomorphisms α^n , is a model for π , which completes the proof of Theorem 5.4. \square

5.3 DISTRIBUTIONS ASSOCIATED TO ADMISSIBLE COEFFICIENT SYSTEMS

DEFINITION 5.5. Let $G(\mathbb{K}_1)_{\text{er}}$ denote the set of elliptic regular elements of $G(\mathbb{K}_1)$. Let A be an admissible coefficient system for G which is Frobenius-stable with respect to α and weakly-equivariant. Let $\chi_{A, \alpha} : G(\mathbb{K}_1)_{\text{er}} \rightarrow \bar{\mathbb{Q}}_\ell$ be the function defined by

$$\chi_{A, \alpha}(g) := \sum_{\{i \in I(G, \mathbb{K}_1) \mid g \in G_i(\mathfrak{o}_{\mathbb{K}_1})\}} (-1)^{\dim i} \chi_{A_i, \alpha_i}(\rho_i(g)). \quad (96)$$

We will refer to $\chi_{A, \alpha}$ as the *character of A* .

THEOREM 5.6. *Let $\pi : G(\mathbb{K}_1) \rightarrow \text{Aut}_{\bar{\mathbb{Q}}_\ell}(V)$ be a depth-zero admissible representation and let Θ_π be the character of π in the sense of Harish-Chandra. Let A be a model for π . Then*

$$\Theta_\pi(g) = \chi_{A, \alpha}(g), \quad (97)$$

for all $g \in G(\mathbb{K}_1)_{\text{er}}$.

Proof. Since π is a depth-zero representation, there is a vertex i_0 of $I(G, \mathbb{K}_1)$ such that the $G(\mathbb{K}_1)$ -algebra V is generated by V_{i_0} . Thus, V is an object in category $\text{Alg}^0 G(\mathbb{K}_1^{nr})$ (*cf.* [SS97]). Now [SS97, III.4.10] and [SS97, III.4.16] extend to any quasi-split reductive linear algebraic group (*cf.* [Cou03, Cor.3.33]) so

$$\Theta_\pi(g) = \sum_{i \in I(G, \mathbb{K}_1), g \in G_i(\mathfrak{o}_{\mathbb{K}_1})} (-1)^{\dim i} \text{trace}(g, V_i). \quad (98)$$

For $g \in G_i(\mathfrak{o}_{\mathbb{K}_1})$, the trace $\text{trace}(g, V_i)$ is the character of $\bar{\pi}_i$. By Definition 5.3 we have $\text{trace}(g, V_i) = \chi_{A_i, \alpha_i}(\rho_i(g))$, which concludes the demonstration. \square

6 EXAMPLES

Let \mathbb{K}_1 and \mathbb{K}_1^{nr} be as in Section 5.

6.1 $SL(2)$

Let $G_{\mathbb{K}_1} = SL(2)_{\mathbb{K}_1}$. We now describe all Frobenius-stable cuspidal coefficient systems for each unramified twisted-Levi subgroup of $G_{\mathbb{K}_1}$. Table 1 records models for all irreducible depth-zero supercuspidal representations of $SL(2, \mathbb{K}_1)$; in this section we describe the terms used in Table 1.

Table 1: Models for depth-zero supercuspidal representations of $SL(2, \mathbb{K}_1)$

Representation	Model
π_θ	$-B_\theta$
π_+	$-\frac{1}{2}B_{\text{sgn}} + \frac{1}{2}(C^+ - C^-)$
π_-	$-\frac{1}{2}B_{\text{sgn}} - \frac{1}{2}(C^+ - C^-)$
π'_+	$-\frac{1}{2}B'_{\text{sgn}} + \frac{1}{2}(D^+ - D^-)$
π'_-	$-\frac{1}{2}B'_{\text{sgn}} - \frac{1}{2}(D^+ - D^-)$
π'_θ	$-B'_\theta$

Up to conjugation over \mathbb{K}_1 , there are four unramified twisted-Levi subgroups of $G_{\mathbb{K}_1}$: the split torus $S_{\mathbb{K}_1}$, two unramified elliptic tori $T_{\mathbb{K}_1}$ and $T'_{\mathbb{K}_1}$, and the group $G_{\mathbb{K}_1}$ itself.

- Consider the split torus

$$S_{\mathbb{K}_1} = \left\{ \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \mid s_1 s_2 = 1 \right\}.$$

Set $S = S_{\mathbb{K}_1} \otimes \mathbb{K}_1^{nr}$. All Frobenius-stable cuspidal coefficient systems for S take the form $\text{cind}_{\bar{S}(0)}^S \mathcal{L}_\theta[1]$, where \mathcal{L}_θ indicates a Kummer system on $\bar{S}(0) = GL(1)_{\mathbb{F}_q}$ equipped with an isomorphism $\text{Fr}^* \mathcal{L}_\theta \rightarrow \mathcal{L}_\theta$ such that the characteristic function of \mathcal{L}_θ equals the character θ of $GL(1, \mathbb{F}_q)$. Define

$$A(\theta) := \text{ind}_S^G \text{cind}_{\bar{S}(0)}^S \mathcal{L}_\theta[1].$$

Using Definition 2.8 and Proposition 3.5 we see that

$$\begin{aligned} A(\theta)_{(0)} &= \text{ind}_{\bar{T}(0)}^{\bar{G}(0)} \mathcal{L}_\theta[1] \\ A(\theta)_{(01)} &= \mathcal{L}_\theta[1] \oplus \bar{m}(s_{(1)})_{(01')}^* \mathcal{L}_\theta[1] \\ A(\theta)_{(1)} &= \text{ind}_{\bar{T}(0)}^{\bar{G}(1)} \mathcal{L}_\theta[1] \end{aligned}$$

Observe that $\bar{G}_{(01)} = \bar{G}_{(01')} = \bar{T}_{(0)}$. The object $A(\theta)$ is simple in $\bar{C}G$ unless $\theta^2 = 1$; thus, in that case, $A(\theta)$ is an admissible coefficient system.

- All frobenius-stable cuspidal coefficient systems for

$$T_{\mathbb{K}_1} = \left\{ \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix} \mid x^2 - \varepsilon y^2 = 1 \right\},$$

where $\varepsilon \in \mathfrak{o}_{\mathbb{K}_1}$ is a fixed quadratic residue and a unit in $\mathfrak{o}_{\mathbb{K}_1}$, take the form $\text{cind}_{\bar{T}(0)}^T \mathcal{L}_\theta[1]$, where θ is a character of the finite group $\bar{T}(0)(\mathbb{F}_q)$. Define

$$B(\theta) := \text{ind}_{\bar{T}}^G \text{cind}_{\bar{T}(0)}^T \mathcal{L}_\theta[1].$$

The object $B(\theta)$, equipped with an isomorphism $\beta(\theta) : \text{fr}^* B(\theta) \rightarrow B(\theta)$ induced from $\phi_\theta : \text{Fr}^* \mathcal{L}_\theta \rightarrow \mathcal{L}_\theta$, provides our first non-trivial example of the phenomenon described in Proposition 5.2. From Definition 2.8 and the definition of parabolic induction on objects (*cf.* Proposition 3.5) we have

$$\begin{aligned} B(\theta)_{(0)} &= \text{ind}_{\bar{T}(0)}^{\bar{G}(0)} \mathcal{L}_\theta[1] \\ B(\theta)_{(01)} &= \bar{m}(z^{-1})_{(01)}^* \mathcal{L}_\theta[1] \oplus \bar{m}(z^{-1}w)_{(01)}^* \mathcal{L}_\theta[1] \\ B(\theta)_{(1)} &= \bar{m}(z^{-1})_{(1)}^* \text{ind}_{\bar{T}(1)z}^{\bar{G}(1)z} \mathcal{L}_\theta[1], \end{aligned}$$

where $T_{\mathbb{K}_1} = S_{\mathbb{K}_1}^z$ and where w is a representative for the non-trivial element in the Weyl group for S . (Recall that S is split.) Here, z is an element of $G(\mathbb{K}_1')$, where \mathbb{K}_1' is a quadratic extension of \mathbb{K}_1 , *not* an element of $G(\mathbb{K}_1)$, which represents a class in $H^1(S_{\mathbb{K}_1}, \text{Gal}(\mathbb{K}_1^{nr}/\mathbb{K}_1))$; in other words, we view $T_{\mathbb{K}_1}$ as a twist of S and represent that twist by conjugation by z^{-1} . Now, from Proposition 4.2 and the definition of parabolic induction on maps (*cf.* Proposition 3.5) we have

$$\begin{aligned} \beta(\theta)_{(0)} &= \text{ind}_{\bar{T}(0)}^{\bar{G}(0)} \phi_\theta[1] \\ \beta(\theta)_{(01)} &= \bar{m}(z^{-1})_{(01)}^* \phi_\theta[1] \oplus \bar{m}(z^{-1}w)_{(01)}^* \phi_\theta[1] \\ \beta(\theta)_{(1)} &= \bar{m}(z^{-1})_{(1)}^* \text{ind}_{\bar{T}(0)}^{\bar{G}(0)} \phi_\theta[1]. \end{aligned}$$

Therefore

$$\begin{aligned} \chi_{B(\theta)_{(0)}, \beta(\theta)_{(0)}} &= \chi_{\text{ind}_{\bar{T}(0)}^{\bar{G}(0)} \mathcal{L}_\theta[1], \text{ind}_{\bar{T}(0)}^{\bar{G}(0)} \phi_\theta[1]} \\ \chi_{B(\theta)_{(01)}, \beta(\theta)_{(01)}} &= 0 \\ \chi_{B(\theta)_{(1)}, \beta(\theta)_{(1)}} &= 0. \end{aligned}$$

- Likewise, all frobenius-stable cuspidal coefficient systems for

$$T'_{\mathbb{K}_1} = \left\{ \begin{pmatrix} x & \varpi^{-1}y \\ \varepsilon\varpi y & x \end{pmatrix} \mid x^2 - \varepsilon y^2 = 1 \right\},$$

where $\varpi \in \mathfrak{o}_{\mathbb{K}_1}$ is a fixed uniformizer for \mathbb{K}_1 , take the form $\text{cind}_{\bar{T}'_{(1)}}^T \mathcal{L}_\theta[1]$, where θ is a character of the finite group $\bar{T}'_{(1)}(\mathbb{F}_q)$. Define

$$B(\theta)' := \text{ind}_{T'}^G \text{cind}_{\bar{T}'_{(1)}}^T \mathcal{L}_\theta[1].$$

The object $B(\theta)'$, equipped with an isomorphism $\beta(\theta)' : \text{fr}^* B(\theta) \rightarrow B(\theta)'$ induced from $\phi_\theta : \text{Fr}^* \mathcal{L}_\theta \rightarrow \mathcal{L}_\theta$ provides another example of the phenomenon described in Proposition 5.2. From Definition 2.8 and the definition of parabolic induction on objects (*cf.* Proposition 3.5) we have

$$\begin{aligned} B(\theta)'_{(0)} &= \bar{m}(v^{-1})_{(0)}^* \text{ind}_{T'_{(0)v}}^{\bar{G}_{(0)v}} \mathcal{L}_\theta[1] \\ B(\theta)'_{(01)} &= \bar{m}(v^{-1})_{(01)}^* \mathcal{L}_\theta[1] \oplus \bar{m}(v^{-1}w)_{(01)}^* \mathcal{L}_\theta[1] \\ B(\theta)'_{(1)} &= \text{ind}_{\bar{T}'_{(1)}}^{\bar{G}_{(1)}} \mathcal{L}_\theta[1], \end{aligned}$$

where $T'_{\mathbb{K}_1} = S_{\mathbb{K}_1}^v$ and where w is a representative for the non-trivial element in the Weyl group for S as above. Note that v is an element of $G(\mathbb{K}_1^{nr})$, *not* an element of $G(\mathbb{K}_1)$, which represents a class in $H^1(S_{\mathbb{K}_1}, \text{Gal}(\mathbb{K}_1^{nr}/\mathbb{K}_1))$. As above, from Proposition 4.2 and the definition of parabolic induction on maps (*cf.* Proposition 3.5) it follows that

$$\begin{aligned} \beta(\theta)'_{(0)} &= \bar{m}(v^{-1})_{(0)}^* \text{ind}_{\bar{T}'_{(1)}}^{\bar{G}_{(1)}} \phi_\theta[1] \\ \beta(\theta)'_{(01)} &= \bar{m}(v^{-1})_{(01)}^* \phi_\theta[1] \oplus \bar{m}(v^{-1}w)_{(01)}^* \phi_\theta[1] \\ \beta(\theta)'_{(1)} &= \text{ind}_{\bar{T}'_{(1)}}^{\bar{G}_{(1)}} \phi_\theta[1]. \end{aligned}$$

Therefore,

$$\begin{aligned} \chi_{B(\theta)'_{(0)}, \beta(\theta)'_{(0)}} &= 0 \\ \chi_{B(\theta)'_{(01)}, \beta(\theta)'_{(01)}} &= 0 \\ \chi_{B(\theta)'_{(1)}, \beta(\theta)'_{(1)}} &= \chi_{\text{ind}_{\bar{T}'_{(1)}}^{\bar{G}_{(1)}} \mathcal{L}_\theta[1], \text{ind}_{\bar{T}'_{(1)}}^{\bar{G}_{(1)}} \phi_\theta[1]}. \end{aligned}$$

- Finally, there are exactly four frobenius-stable cuspidal coefficient systems for

$$G_{\mathbb{K}_1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\};$$

they are

$$\begin{aligned} C^\pm &:= \text{cind}_{G_{(0)}}^G F_\pm \\ D^\pm &:= \text{cind}_{G_{(1)}}^G F_\pm. \end{aligned}$$

The following facts are simple consequences of Definition 2.8:

$$\begin{aligned} C_{(0)}^\pm &= F_\pm \\ C_{(01)}^\pm &= 0 \\ C_{(1)}^\pm &= 0, \end{aligned}$$

and

$$\begin{aligned} D_{(0)}^\pm &= 0 \\ D_{(01)}^\pm &= 0 \\ D_{(1)}^\pm &= F_\pm. \end{aligned}$$

We now describe all depth-zero supercuspidal representations of $SL(2, \mathbb{K}_1)$. Let π_θ denote the representation obtained by compact induction from the Deligne-Lusztig representation $-R_{\bar{T}_{(0)}}^{\bar{G}_{(0)}}(\theta)$ of $\bar{G}_{(0)}(\mathbb{F}_q)$, where θ is a character of $\bar{T}_{(0)}(\mathbb{F}_q)$ in general position; likewise, let π'_θ denote the representation obtained by compact induction from the Deligne-Lusztig representation $-R_{\bar{T}'_{(1)}}^{\bar{G}_{(1)}}(\theta)$ of $\bar{G}_{(1)}(\mathbb{F}_q)$, where θ is a character of $\bar{T}'_{(1)}(\mathbb{F}_q)$ in general position. Next, let χ_0^\pm denote the two cuspidal representations appearing in the Lusztig series for $R_{SU(1)}^{SL(2)}(\text{sgn})$, where sgn is the sign character of $SU(1, \mathbb{F}_q)$; let π_\pm denote the representation obtained by compact induction from χ_0^\pm on $\bar{G}_{(0)}(\mathbb{F}_q)$ and likewise let π'_\pm denote the representation obtained by compact induction from χ_0^\pm on $\bar{G}_{(1)}(\mathbb{F}_q)$. Although $-B(\theta)$ is a model for the depth-zero supercuspidal representation π_θ (see Table 1) this example illustrates how *not* to find models for representations. Consider the coefficient system \underline{V} for π_θ in the sense of [SS97]. Even though $V_{(01)}$ and $V_{(1)}$ are both zero, it does not follow that $B(\theta)_{(01)}$ and $B(\theta)_{(1)}$ are zero. As one sees from the proof of Theorem 5.4, to make a model for a representation it is necessary to identify a type for that representation. Observe that $(G_{(0)}(\mathfrak{o}_{\mathbb{K}_1}), -R_{\bar{T}_{(0)}}^{\bar{G}_{(0)}}(\theta))$ is a type for π_θ and that

$$-\text{trace } R_{\bar{T}_{(0)}}^{\bar{G}_{(0)}}(\theta) = \chi_{\text{ind}_{\bar{T}_{(0)}, \mathcal{L}_\theta[1]}^{\bar{G}_{(0)}}, \text{ind}_{\bar{T}_{(0)}}^{\bar{G}_{(0)}} \phi_\theta[1]}. \quad (99)$$

We remark that $\{\pi_+, \pi'_+, \pi_-, \pi'_-\}$ is an L-packet and that $\{\pi(\theta), \pi'(\theta)\}$ is also an L-packet when θ is in general position.

6.2 $Sp(4)$

Let $G_{\mathbb{K}_1} = Sp(4)_{\mathbb{K}_1}$. We now describe all frobenius-stable cuspidal coefficient systems for each unramified twisted-Levi subgroup of $G_{\mathbb{K}_1}$, as in Example 6.1. Using [Sri68] and [Sri94] we produce models for all supercuspidal depth-zero representations $Sp(4, \mathbb{K}_1)$ and present the results in Tables 2, 3 and 4, in which

Table 2: Models for representations induced from $G_{(1)}(\mathfrak{o}_{\mathbb{K}_1})$

Representation	Model
π_0	$A_4(\theta_1, \theta_2)$
π_1	$\frac{1}{2}A_4(\theta, \text{sgn}) - \frac{1}{2}(B_2^+(\theta) - B_2^-(\theta))$
π_2	$\frac{1}{2}A_4(\theta, \text{sgn}) + \frac{1}{2}(B_2^+(\theta) - B_2^-(\theta))$
π_3	$\frac{1}{4}A_4(\text{sgn}, \text{sgn}) - \frac{1}{4}(B_2^+(\text{sgn}) - B_2^-(\text{sgn})) - \frac{1}{4}(B_1^+(\text{sgn}) - B_1^-(\text{sgn})) + \frac{1}{4}(D^{++} - D^{+-} - D^{-+} + D^{--})$
π_4	$\frac{1}{4}A_4(\text{sgn}, \text{sgn}) + \frac{1}{4}(B_2^+(\text{sgn}) - B_2^-(\text{sgn})) - \frac{1}{4}(B_1^+(\text{sgn}) - B_1^-(\text{sgn})) - \frac{1}{4}(D^{++} - D^{+-} - D^{-+} + D^{--})$
π_5	$\frac{1}{4}A_4(\text{sgn}, \text{sgn}) + \frac{1}{4}(B_2^+(\text{sgn}) - B_2^-(\text{sgn})) + \frac{1}{4}(B_1^+(\text{sgn}) - B_1^-(\text{sgn})) + \frac{1}{4}(C_{++} - C_{+-} - C_{-+} + C_{--})$

Table 3: Models for representations induced from $G_{(0)}(\mathfrak{o}_{\mathbb{K}_1})$

Representation	Model
π_6	$A_7(\theta_1, \theta_2)$
π_7	$A_3(\theta_1, \theta_2)$
π_8	$\frac{1}{2}A_3(\theta, \text{sgn}) - \frac{1}{2}\zeta(B_1^+(\theta) - B_1^-(\theta))$
π_9	$\frac{1}{2}A_3(\theta, \text{sgn}) + \frac{1}{2}\zeta(B_1^+(\theta) - B_1^-(\theta))$
π_{10}	$-\frac{1}{4}A_7(1) - \frac{1}{4}A_3(1) - \zeta\frac{1}{2}C$

Table 4: Models for representations induced from $G_{(2)}(\mathfrak{o}_{\mathbb{K}_1})$

Representation	Model
π'_6	$A_8(\theta_1, \theta_2)$
π'_7	$A_5(\theta_1, \theta_2)$
π'_8	$\frac{1}{2}A_5(\theta, \text{sgn}) - \frac{1}{2}\zeta(B_2^+(\theta)' - B_2^-(\theta)')$
π'_9	$\frac{1}{2}A_5(\theta, \text{sgn}) + \frac{1}{2}\zeta(B_2^+(\theta)' - B_2^-(\theta)')$
π'_{10}	$-\frac{1}{4}A_8(1) - \frac{1}{4}A_5(1) - \zeta\frac{1}{2}C'$

$\zeta \in \bar{\mathbb{Q}}_\ell$ is a primitive fourth root of unity such that ζ^2 equals the quadratic residue of -1 in \mathbb{F}_q . We now explain the other terms appearing in these tables. To begin, we fix a representation of $Sp(4)$. Let $\theta : GL(4) \rightarrow GL(4)$ be the involution defined by $\theta(g) = J^{-1} {}^t g^{-1} J$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (100)$$

and let $G = Sp(4)$ be the subvariety fixed by θ .

We now consider cuspidal unramified twisted-Levi subgroups of $G_{\mathbb{K}_1}$. Up to conjugacy, the group $G_{\mathbb{K}_1}$ contains thirteen cuspidal unramified twisted-Levi subgroups: nine inner forms of the torus $GL(1)_{\mathbb{K}_1} \times GL(1)_{\mathbb{K}_1}$; six inner forms of the Levi subgroup $GL(1)_{\mathbb{K}_1} \times SL(2)_{\mathbb{K}_1}$; and $G_{\mathbb{K}_1}$ itself. The Levi subgroup $GL(2)_{\mathbb{K}_1}$ is not cuspidal since it does not admit any cuspidal coefficient systems.

- We begin with tori in G . Consider the split torus

$$T_{\mathbb{K}_1} = \left\{ \left(\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} \mid \begin{array}{l} t_1 t_4 = 1 \\ t_2 t_3 = 1 \end{array} \right) \right\}.$$

Let $T = T_{\mathbb{K}_1} \otimes \mathbb{K}_1^{nr}$. Each Frobenius-stable cuspidal coefficient system for $T_{\mathbb{K}_1}$ takes the form

$$C_0(\theta_1, \theta_2) := \text{cind}_{\bar{T}(0)}^S \mathcal{L}_{\theta_1} \boxtimes \mathcal{L}_{\theta_2}[2],$$

where θ_1 and θ_2 are characters of $\bar{T}(0)(\mathbb{F}_q)$. Here, \mathcal{L}_θ is the Frobenius-stable Kummer local system equipped with an isomorphism $\text{Fr}^* \mathcal{L}_\theta \rightarrow \mathcal{L}_\theta$ such that the characteristic function $\chi_{\mathcal{L}_\theta}$ of \mathcal{L} equals θ . Define

$$A_0(\theta_1, \theta_2) := \text{ind}_T^G C_0(\theta_1, \theta_2).$$

Thus, $A_0(\theta_1, \theta_2)$ is an object $\bar{\mathcal{C}}G(\mathbb{K}_1)$. If θ_1 and θ_2 are in general position, then $A_0(\theta_1, \theta_2)$ is simple in $\bar{\mathcal{C}}G$, in which case $A_0(\theta_1, \theta_2)$ is admissible (see Definition 3.7); in other words, if θ_1 and θ_2 are in general position then $A_0(\theta_1, \theta_2)$ is an admissible coefficient system.

- Next, consider torus

$$T_{\mathbb{K}_1}^1 = \left\{ \left(\begin{pmatrix} x & 0 & 0 & y \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ \varepsilon y & 0 & 0 & x \end{pmatrix} \mid \begin{array}{l} x^2 - \varepsilon y^2 = 1 \\ t_2 t_3 = 1 \end{array} \right) \right\}.$$

Clearly, $T_{\mathbb{K}_1}^1$ splits over a quadratic unramified extension of \mathbb{K}_1 . Let $T^1 = T_{\mathbb{K}_1}^1 \otimes \mathbb{K}_1^{nr}$. Each Frobenius-stable cuspidal coefficient system for $T_{\mathbb{K}_1}^1$ takes the form

$$C_1(\theta_1, \theta_2) := \text{cind}_{T_{(0)}^1} \mathcal{L}_{\theta_1} \boxtimes \mathcal{L}_{\theta_2}[2],$$

where θ_1 is a character of $SU(1, \mathbb{F}_q)$ and θ_2 is a character of $GL(1, \mathbb{F}_q)$. Define

$$A_1(\theta_1, \theta_2) = \text{ind}_{T^1}^G C_1(\theta_1, \theta_2).$$

If θ_1 and θ_2 are in general position then $A_1(\theta_1, \theta_2)$ is an admissible coefficient system.

- Next, consider the torus

$$T_{\mathbb{K}_1}^2 = \left\{ \left(\begin{array}{cccc} x & 0 & 0 & \varpi^{-1}y \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ \varpi \varepsilon y & 0 & 0 & x \end{array} \right) \mid \begin{array}{l} x^2 - \varepsilon y^2 = 1 \\ t_2 t_3 = 1 \end{array} \right\}.$$

This also splits over a quadratic unramified extension. Let $T^2 = T_{\mathbb{K}_1}^2 \otimes \mathbb{K}_1^{nr}$. Frobenius-stable depth-zero cuspidal character sheaves for $T_{\mathbb{K}_1}^2$ take the form

$$C_2(\theta_1, \theta_2) := \text{cind}_{T_{(1)}^2} \mathcal{L}_{\theta_1} \boxtimes \mathcal{L}_{\theta_2}[2],$$

where θ_1 is a character of $SU(1, \mathbb{F}_q)$ and θ_2 is a character of $GL(1, \mathbb{F}_q)$. Define

$$A_2(\theta_1, \theta_2) = \text{ind}_{T^2}^G C_2(\theta_1, \theta_2).$$

If θ_1 and θ_2 are in general position then $A_2(\theta_1, \theta_2)$ is an admissible coefficient system.

- Next, consider the elliptic torus

$$T_{\mathbb{K}_1}^3 = \left\{ \left(\begin{array}{cccc} x_1 & 0 & 0 & y_1 \\ 0 & x_2 & y_2 & 0 \\ 0 & \varepsilon y_2 & x_2 & 0 \\ \varepsilon y_1 & 0 & 0 & x_1 \end{array} \right) \mid \begin{array}{l} x_1^2 - \varepsilon y_1^2 = 1 \\ x_2^2 - \varepsilon y_2^2 = 1 \end{array} \right\}.$$

Let $T^3 = T_{\mathbb{K}_1}^3 \otimes \mathbb{K}_1^{nr}$. The building for $T^3(\mathbb{K}_1^{nr})$ in $G(\mathbb{K}_1^{nr})$ is $\{(0)\}$ and the reductive quotient is $\bar{T}_{(0)}^3 \cong SU(1) \times SU(1)$. Accordingly, each Frobenius-stable depth-zero cuspidal character sheaf for T^3 takes the form

$$C_3(\theta_1, \theta_2) := \text{cind}_{T_{(0)}^3} \mathcal{L}_{\theta_1} \boxtimes \mathcal{L}_{\theta_2}[2],$$

where θ_1 and θ_2 are characters of $SU(1, \mathbb{F}_q)$. Define

$$A_3(\theta_1, \theta_2) = \text{ind}_{T^3}^G C_3(\theta_1, \theta_2).$$

If θ_1 and θ_2 are in general position then $A_3(\theta_1, \theta_2)$ is an admissible coefficient system.

- Likewise, consider the elliptic torus

$$T_{\mathbb{K}_1}^4 = \left\{ \left(\begin{array}{cccc} x_1 & 0 & 0 & \varpi^{-1}y_1 \\ 0 & x_2 & y_2 & 0 \\ 0 & \varepsilon y_2 & x_2 & 0 \\ \varpi \varepsilon y_1 & 0 & 0 & x_1 \end{array} \right) \mid \begin{array}{l} x_1^2 - \varepsilon y_1^2 = 1 \\ x_2^2 - \varepsilon y_2^2 = 1 \end{array} \right\}.$$

The building for $T^4(\mathbb{K}_1)$ in $G(\mathbb{K}_1)$ is $\{(1)\}$. Each frobenius-stable depth-zero cuspidal character sheaf for $T_{\mathbb{K}_1}^4$ takes the form

$$C_4(\theta_1, \theta_2) := \text{cind}_{T_{(1)}^4} \mathcal{L}_{\theta_1} \boxtimes \mathcal{L}_{\theta_2}[2],$$

where θ_1 and θ_2 are characters of $SU(1, \mathbb{F}_q)$. Define

$$A_4(\theta_1, \theta_2) = \text{ind}_{T^4}^G C_4(\theta_1, \theta_2).$$

If θ_1 and θ_2 are in general position then $A_4(\theta_1, \theta_2)$ is an admissible coefficient system.

- We also have the elliptic torus

$$T_{\mathbb{K}_1}^5 = \left\{ \left(\begin{array}{cccc} x_1 & 0 & 0 & \varpi^{-1}y_1 \\ 0 & x_2 & \varpi^{-1}y_2 & 0 \\ 0 & \varpi \varepsilon y_2 & x_2 & 0 \\ \varpi \varepsilon y_1 & 0 & 0 & x_1 \end{array} \right) \mid \begin{array}{l} x_1^2 - \varepsilon y_1^2 = 1 \\ x_2^2 - \varepsilon y_2^2 = 1 \end{array} \right\}.$$

The building for $T^5(\mathbb{K}_1)$ in $G(\mathbb{K}_1)$ is $\{(2)\}$ and $\bar{T}_{(2)/\mathbb{F}_q}^5 \cong SU(1)_{\mathbb{F}_q} \times SU(1)_{\mathbb{F}_q}$. Accordingly, each frobenius-stable depth-zero cuspidal character sheaf for $T_{\mathbb{K}_1}^5$ takes the form

$$C_5(\theta_1, \theta_2) := \text{cind}_{\bar{T}_{(2)}^5} \mathcal{L}_{\theta_1} \boxtimes \mathcal{L}_{\theta_2}[2],$$

where θ_1 and θ_2 are characters of $SU(1, \mathbb{F}_q)$. Define

$$A_5(\theta_1, \theta_2) = \text{ind}_{T^5}^G C_5(\theta_1, \theta_2).$$

If θ_1 and θ_2 are in general position then $A_5(\theta_1, \theta_2)$ is an admissible coefficient system.

- Next, consider the torus

$$T_{\mathbb{K}_1}^6 = \left\{ \left(\begin{array}{cccc} x & y & 0 & 0 \\ \varepsilon y & x & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & \varepsilon v & u \end{array} \right) \mid \begin{array}{l} xu - \varepsilon yv = 1 \\ xv - yu = 0 \end{array} \right\}.$$

Frobenius-stable depth-zero cuspidal character sheaves for T^6 take the form

$$C_6(\theta) := \text{cind}_{T_{(1)}^6} \mathcal{L}_{\theta}[2],$$

where θ is a character of $\bar{T}_{(0)}^6(\mathbb{F}_q)$. Define

$$A_6(\theta) = \text{ind}_{T^6}^G C_6(\theta).$$

If θ is in general position then $A_6(\theta)$ is an admissible coefficient system.

- Next consider the unramified elliptic torus

$$T_{\mathbb{K}_1}^7 = \left\{ \left(\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \varepsilon x_4 & x_1 & x_2 & x_3 \\ \varepsilon x_3 & \varepsilon x_4 & x_1 & x_2 \\ \varepsilon x_2 & \varepsilon x_3 & \varepsilon x_4 & x_1 \end{array} \right) \mid \begin{array}{l} x_1^2 - 2\varepsilon x_2 x_4 + \varepsilon x_3^2 = 1 \\ x_2^2 - 2\varepsilon x_1 x_3 + \varepsilon x_4^2 = 0 \end{array} \right\}.$$

The image of $I(T^7, \mathbb{K}_1^{nr}) \hookrightarrow I(G, \mathbb{K}_1^{nr})$ is $\{(0)\}$. Frobenius-stable admissible coefficient systems for $T_{\mathbb{K}_1}^7$ take the form

$$C_7(\theta) := \text{cind}_{T_{(0)}^7}^{T^7} \mathcal{L}_\theta[2],$$

where θ is a character of $\bar{T}_{(0)}^7(\mathbb{F}_q)$. Define

$$A_7(\theta) = \text{ind}_{T^7}^G C_7(\theta).$$

If θ is in general position then $A_7(\theta)$ is an admissible coefficient system.

- Finally, consider the unramified elliptic torus

$$T_{\mathbb{K}_1}^8 = \left\{ \left(\begin{array}{cccc} x_1 & x_2 & \frac{x_3}{\varpi} & \frac{x_4}{\varpi} \\ \varepsilon x_4 & x_1 & \frac{x_2}{\varpi} & \frac{x_3}{\varpi} \\ \varpi \varepsilon x_3 & \varpi \varepsilon x_4 & x_1 & x_2 \\ \varpi \varepsilon x_2 & \varpi \varepsilon x_3 & \varepsilon x_4 & x_1 \end{array} \right) \mid \begin{array}{l} x_1^2 - 2\varepsilon x_2 x_4 + \varepsilon x_3^2 = 1 \\ x_2^2 - 2\varepsilon x_1 x_3 + \varepsilon x_4^2 = 0 \end{array} \right\}.$$

The image of $I(T^8, \mathbb{K}_1^{nr}) \hookrightarrow I(G, \mathbb{K}_1^{nr})$ is $\{(1)\}$. Frobenius-stable admissible coefficient systems for $T_{\mathbb{K}_1}^8$ take the form

$$C_8(\theta) := \text{cind}_{T_{(1)}^8}^{T^8} \mathcal{L}_\theta[2],$$

where θ is a character of $\bar{T}_{(1)}^8(\mathbb{F}_q)$. Define

$$A_8(\theta) = \text{ind}_{T^8}^G C_8(\theta).$$

If θ is in general position then $A_8(\theta)$ is an admissible coefficient system.

- Let

$$L_{\mathbb{K}_1} = \left\{ \left(\begin{array}{cccc} t_1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & t_4 \end{array} \right) \mid \begin{array}{l} t_1 t_4 = 1 \\ ad - bc = 1 \end{array} \right\}.$$

Up to conjugation, the building for $L(\mathbb{K}_1^{nr})$ has two polyvertices; these may be identified with (0) and (1) in $I(G, \mathbb{K}_1)$. Now $\bar{L}_{(0)}$ and $\bar{L}_{(1)}$ are each isomorphic to $GL(1) \times SL(2)$, and the Frobenius-stable cuspidal character sheaves on $GL(1) \times SL(2)$ take the form $\mathcal{L}_\theta \boxtimes K_\pm[1]$, where θ is a character of $GL(1, \mathbb{F}_q)$ and K_\pm is a cuspidal character sheaf on $SL(2)_{\mathbb{F}_q}$. (See Example 6.1.) Frobenius-stable cuspidal character sheaves for $L_{\mathbb{K}_1}$ take the form

$$\begin{aligned} C_0^\pm(\theta) &:= \text{cind}_{\bar{L}_{(0)}}^L \mathcal{L}_\theta \boxtimes K_\pm[1] \\ C_0^\pm(\theta)' &:= \text{cind}_{\bar{L}_{(1)}}^L \mathcal{L}_\theta \boxtimes K_\pm[1], \end{aligned}$$

where θ is a character of $GL(1, \mathbb{F}_q)$. Define

$$\begin{aligned} B_0^\pm(\theta) &:= \text{ind}_L^G C_0^\pm(\theta) \\ B_0^\pm(\theta)' &:= \text{ind}_L^G C_0^\pm(\theta)'. \end{aligned}$$

If θ is in general position, then $B_0^\pm(\theta)$ and $B_0^\pm(\theta)'$ are admissible coefficient systems.

- Next, consider the inner form

$$L_{\mathbb{K}_1}^1 = \left\{ \left(\begin{array}{cccc} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & \varepsilon y & x & 0 \\ c & 0 & 0 & d \end{array} \right) \mid \begin{array}{l} x^2 - \varepsilon y^2 = 1 \\ ad - bc = 1 \end{array} \right\}.$$

Observe that $L^1(\mathbb{K}_1) = SU(1, \mathbb{K}_1) \times SL(2, \mathbb{K}_1)$. Up to $L^1(\mathbb{K}_1)$ conjugation, the building for $L^1(\mathbb{K}_1)$ contains two polyvertices, which may be identified with (0) and (1) in $I(G, \mathbb{K}_1)$. Now $\bar{L}_{(0)/\mathbb{F}_q}^1$ and $\bar{L}_{(1)/\mathbb{F}_q}^1$ are each isomorphic to $SU(1)_{\mathbb{F}_q} \times SL(2)_{\mathbb{F}_q}$, so we define

$$\begin{aligned} C_1^\pm(\theta) &:= \text{cind}_{\bar{L}_{(0)}^1}^L \mathcal{L}_\theta \boxtimes F_\pm[1] \\ C_1^\pm(\theta)' &:= \text{cind}_{\bar{L}_{(1)}^1}^L \mathcal{L}_\alpha \boxtimes F_\pm[1], \end{aligned}$$

where α is a character of $SU(1, \mathbb{F}_q) \times SL(2, \mathbb{F}_q)$. Define

$$\begin{aligned} B_1^\pm(\theta) &:= \text{ind}_{L^1}^G C_1^\pm(\theta) \\ B_1^\pm(\theta)' &:= \text{ind}_{L^1}^G C_1^\pm(\theta)'. \end{aligned}$$

If θ is in general position, then $B_1^\pm(\theta)$ and $B_1^\pm(\theta)'$ are admissible coefficient systems.

- Likewise, consider the elliptic unramified-Levi subgroup

$$L_{\mathbb{K}_1}^2 = \left\{ \left(\begin{array}{cccc} x & 0 & 0 & y\varpi^{-1} \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ \varepsilon y\varpi & 0 & 0 & x \end{array} \right) \mid \begin{array}{l} x^2 - \varepsilon y^2 = 1 \\ ad - bc = 1 \end{array} \right\}.$$

Up to $L^2(\mathbb{K}_1)$ conjugation, the building for $L^2(\mathbb{K}_1)$ contains two polyvertices, which may be identified with (1) and (2) in $I(G, \mathbb{K}_1)$. Define

$$\begin{aligned} C_2^\pm(\theta) &:= \text{cind}_{L_{(1)}^2} L_\theta \boxtimes F_\pm[1] \\ C_2^\pm(\theta)' &:= \text{cind}_{L_{(2)}^2} L_\theta \boxtimes \mathcal{F}_\pm[1], \end{aligned}$$

where θ is a character of $\bar{L}_{(2)}^2(\mathbb{F}_q)$. Define

$$\begin{aligned} B_2^\pm(\theta) &:= \text{ind}_{L^2}^G C_2^\pm(\theta) \\ B_2^\pm(\theta)' &:= \text{ind}_{L^2}^G C_2^\pm(\theta)'. \end{aligned}$$

If θ is in general position, then $B_2^\pm(\theta)$ and $B_2^\pm(\theta)'$ are admissible coefficient systems.

- Finally, we turn to the most interesting cuspidal Levi subgroup of G , which is G itself. A fundamental $G(\mathbb{K}_1^{nr})$ -domain for $I(G, \mathbb{K}_1^{nr})$ has three polyvertices, which we denote (0), (1) and (2), with $\bar{G}_{(0)/\mathbb{F}_q} = Sp(4)_{\mathbb{F}_q}$, $\bar{G}_{(1)/\mathbb{F}_q} = SL(2)_{\mathbb{F}_q} \times SL(2)_{\mathbb{F}_q}$ and $\bar{G}_{(2)/\mathbb{F}_q} = Sp(4)_{\mathbb{F}_q}$. There is exactly one cuspidal character sheaf F_0 on $Sp(4)$ while $SL(2)_{\mathbb{F}_q} \times SL(2)_{\mathbb{F}_q}$ admits four cuspidal character sheaves, being $F_\pm \boxtimes F_\pm$ in the notation from Example 6.1; of these, only $F_+ \boxtimes F_+$ is a cuspidal unipotent character sheaf. Thus, by Corollary 2.12, the elements of $\bar{\mathcal{A}}^{(0)}G_{\mathbb{K}_1}$ are

$$\begin{aligned} C^0 &:= \text{cind}_{\bar{G}_{(0)}}^G F_0 \\ D^{++} &:= \text{cind}_{\bar{G}_{(1)}}^G F_+ \boxtimes F_+ \\ D^{+-} &:= \text{cind}_{\bar{G}_{(1)}}^G F_+ \boxtimes F_- \\ D^{-+} &:= \text{cind}_{\bar{G}_{(1)}}^G F_- \boxtimes F_+ \\ D^{--} &:= \text{cind}_{\bar{G}_{(1)}}^G F_- \boxtimes F_- \\ C^2 &:= \text{cind}_{\bar{G}_{(2)}}^G F_0. \end{aligned}$$

This completes the list of all frobenius-stable cuspidal coefficient systems for each cuspidal unramified twisted-Levi subgroup of $G_{\mathbb{K}_1}$. Every depth-zero character sheaf for $G_{\mathbb{K}_1}$ is a simple summand of an object of $\bar{\mathcal{C}}G_{\mathbb{K}_1}$ produced by parabolic induction from of the cuspidal coefficient systems listed above.

Each irreducible depth-zero supercuspidal representation of $G(\mathbb{K}_1)$ is equivalent to a representation obtained by compact induction from an irreducible cuspidal representation of $\bar{G}_{(0)}(\mathbb{F}_q)$ or $\bar{G}_{(1)}(\mathbb{F}_q)$ or $\bar{G}_{(2)}(\mathbb{F}_q)$, so we begin by listing all irreducible cuspidal representations of these finite groups. Consider the finite group $Sp(4, \mathbb{F}_q)$. Using notation from [Sri68] and [Sri94], each irreducible cuspidal representation of $Sp(4, \mathbb{F}_q)$ the appears in one of the following families:

- the Deligne-Lusztig representation $\chi_1 = R_{T_1}^{Sp(4)}(\theta)$ where θ is a character of $T_1(\mathbb{F}_q)$ in general position;
- the Deligne-Lusztig representation $\chi_4 = R_{T_4}^{Sp(4)}(\theta)$ where θ is a character of $T_4(\mathbb{F}_q)$ in general position;
- the irreducible constituent ξ'_{21} of the Deligne-Lusztig virtual representation $R_{T_4}^{Sp(4)}(\theta \times \text{sgn})$ where θ is a character of $SU(1, \mathbb{F}_q)$ in general position and sgn is the sign character of $SU(1, \mathbb{F}_q)$;
- the other irreducible constituent ξ'_{22} of the Deligne-Lusztig virtual representation $R_{T_4}^{Sp(4)}(\theta \times \text{sgn})$;
- the cuspidal unipotent representation θ_{10} .

With these conventions, and notation from Example 6.1, every irreducible depth-zero supercuspidal representation of $G(\mathbb{K}_1)$ is equivalent to one of the following:

$$\begin{aligned}
 \pi_0 &= \text{cInd}_{G_{(1)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_0 \times \chi_0) \\
 \pi_1 &= \text{cInd}_{G_{(1)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_0 \times \chi_0^+) \\
 \pi_2 &= \text{cInd}_{G_{(1)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_0 \times \chi_0^-) \\
 \pi_3 &= \text{cInd}_{G_{(1)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_0^+ \times \chi_0^+) \\
 \pi_4 &= \text{cInd}_{G_{(1)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_0^+ \times \chi_0^-) \\
 \pi_5 &= \text{cInd}_{G_{(1)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_0^- \times \chi_0^-).
 \end{aligned}$$

and

$$\begin{aligned}
 \pi_6 &= \text{cInd}_{G_{(0)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_1) & \pi'_6 &= \text{cInd}_{G_{(2)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_1) \\
 \pi_7 &= \text{cInd}_{G_{(0)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_4) & \pi'_7 &= \text{cInd}_{G_{(2)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\chi_4) \\
 \pi_8 &= \text{cInd}_{G_{(0)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\xi'_{22}) & \pi'_8 &= \text{cInd}_{G_{(2)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\xi'_{22}) \\
 \pi_9 &= \text{cInd}_{G_{(0)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\xi'_{21}) & \pi'_9 &= \text{cInd}_{G_{(2)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\xi'_{21}) \\
 \pi_{10} &= \text{cInd}_{G_{(0)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\theta_{10}) & \pi'_{10} &= \text{cInd}_{G_{(2)}(\mathfrak{o}_{\mathbb{K}_1})}^{G(\mathbb{K}_1)}(\theta_{10})
 \end{aligned} \tag{101}$$

6.3 $GL(N)$

Let $G = GL(n)$. By Definition 3.7, every irreducible admissible coefficient system is a summand of $\text{ind}_L^G C$, where L is a Levi subgroup of G and C is a cuspidal coefficient system (*cf.* Definition 2.8). Therefore, the first step in describing irreducible admissible coefficient systems for G is to enumerate all Levi subgroups and all cuspidal coefficient systems on those Levi subgroups. (*cf.* Definition 2.13.) Every Levi subgroup of G is $G(\mathbb{K}_1^{nr})$ -conjugate to $L^{\underline{m}}$, where $\underline{m} = [m_1, m_2, \dots, m_t]$ is a partition of n and $L^{\underline{m}} = \prod_{k=1}^t GL(m_k)_{\mathbb{K}_1^{nr}}$.

By Corollary 2.12, every cuspidal coefficient system on $L^{\underline{m}}$ is isomorphic to $\text{cind}_{L^{\underline{m}}} L^{\underline{m}} F$, where i is a vertex of the building $I(L^{\underline{m}}, \mathbb{K}_1^{nr})$ and F is a cuspidal character sheaf for $\bar{L}_i^{\underline{m}}$. Since the building for $L^{\underline{m}}$ is regular and all vertices are $L^{\underline{m}}(\mathbb{K}_1^{nr})$ -conjugate, we have $\bar{L}_i^{\underline{m}} = \prod_{k=1}^t GL(m_k)_{\mathbb{F}_q}$. Thus, cuspidal character sheaves on $\bar{L}_i^{\underline{m}}$ are all of the form $\boxtimes_{k=1}^t F_k$, where F_k is a cuspidal character sheaf on $GL(m_k)_{\mathbb{F}_q}$ (cf. [MS89, Lemma 5.4.1] for example). Since $GL(m_k)_{\mathbb{F}_q}$ admits cuspidal character sheaves if and only if $m_k = 1$, it follows that $L^{\underline{m}}$ admits cuspidal coefficient systems if and only if $\underline{m} = [1^m]$, whence $L^{\underline{m}}$ is a \mathbb{K}_1^{nr} -split torus. Let T be a \mathbb{K}_1^{nr} -split torus (there is exactly one in G , up to $G(\mathbb{K}_1^{nr})$ -conjugacy). From the discussion above we see that every cuspidal coefficient system for T takes the form

$$\text{cind}_{T(0)}^T \mathcal{L}_1 \boxtimes \mathcal{L}_2 \boxtimes \cdots \boxtimes \mathcal{L}_n[n], \quad (102)$$

where (0) is *the* vertex of $I(T, \mathbb{K}_1^{nr})$ and \mathcal{L}_k is a Kummer local system on $GL(1)_{\mathbb{F}_q}$ for each $k = 1, \dots, n$. Thus,

$$\text{ind}_T^G C = \text{ind}_T^G \text{cind}_{T(0)}^T \mathcal{L}_1 \boxtimes \mathcal{L}_2 \boxtimes \cdots \boxtimes \mathcal{L}_n[n]. \quad (103)$$

is an admissible coefficient system for G .

We now suppose $G = GL(n)_{\mathbb{K}_1} \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr})$, so $G_{\mathbb{K}_1} = GL(n)_{\mathbb{K}_1}$. Let $S_{\mathbb{K}_1}$ be the \mathbb{K}_1 -split torus in $G_{\mathbb{K}_1}$ and let $T_{\mathbb{K}_1}$ be an inner form of $S_{\mathbb{K}_1}$. (Following our conventions, we have $T = T_{\mathbb{K}_1} \times_{\text{Spec}(\mathbb{K}_1)} \text{Spec}(\mathbb{K}_1^{nr})$.) Then

$$T_{\mathbb{K}_1}^{\underline{n}} = \prod_{k=1}^t \text{Res}_{\mathbb{K}_{n_k}/\mathbb{K}_1} GL(1)_{\mathbb{K}_{n_k}}, \quad (104)$$

where $\underline{n} = [n_1, n_2, \dots, n_t]$ is a partition of n and \mathbb{K}_{n_k} is an unramified extension of \mathbb{K}_1 is degree n_k . For each $k = 1, \dots, t$, let

$$T_{\mathbb{K}_1}^{n_k} = \text{Res}_{\mathbb{K}_{n_k}/\mathbb{K}_1} GL(1)_{\mathbb{K}_{n_k}}.$$

Then

$$\bar{T}_{(0)/\mathbb{F}_q}^{n_k} = \text{Res}_{\mathbb{K}_{n_k}/\mathbb{F}_q} GL(1)_{\mathbb{K}_{n_k}}.$$

Let θ_k be a character of $\bar{T}_{(0)}^{n_k}(\mathbb{F}_q)$ and let \mathcal{L}_{θ_k} be the Kummer local system on $\bar{T}_{(0)}^{n_k}$ equipped with an isomorphism $\text{Fr}^* \mathcal{L}_{\theta_k} \rightarrow \mathcal{L}_{\theta_k}$ in $D_c^b(\bar{T}_{(0)}^{n_k}; \bar{\mathbb{Q}}_\ell)$ such that $\chi_{\mathcal{L}_{\theta_k}} = \theta_k$. Every frobenius-stable cuspidal coefficient system for $T^{\underline{n}}$ takes the form

$$C_{\theta}^{\underline{n}} := \text{cind}_{T(0)}^{T^{\underline{n}}} \boxtimes_{k=1}^t \mathcal{L}_{\theta_k}[n], \quad (105)$$

with $\theta = \otimes_{k=1}^t \theta_k$ a character of $\bar{T}_{(0)}^{\underline{n}}(\mathbb{F}_q)$, and every frobenius-stable irreducible admissible coefficient system for G is a summand of

$$A_{\theta}^{\underline{n}} := \text{ind}_{T^{\underline{n}}}^G C_{\theta}^{\underline{n}}. \quad (106)$$

If each character θ_k appearing in θ is in general position, in the sense of [DL76], then A_θ^n is irreducible, and therefore an irreducible admissible coefficient system itself.

One case is particularly important to us. If $\underline{n} = [n]$ we denote T^n (resp. C_θ^n , A_θ^n) by T^n (resp. C_θ^n , A_θ^n). In this case

$$T_{\mathbb{K}_1}^n = \text{Res}_{\mathbb{K}_n/\mathbb{K}_1} GL(1)_{\mathbb{K}_n} \tag{107}$$

is elliptic and θ is a character of $GL(1, \mathbb{K}_n)$. When θ is in general position, A_θ is an irreducible admissible coefficient system. This case is sufficient for a description of the models of all supercuspidal depth-zero representations, as one sees from the proof of Theorem 5.4.

Let π be an irreducible supercuspidal depth-zero representation of $G(\mathbb{K}_1)$. Then there is a non-trivial character χ of $ZG(\mathbb{K}_1)$ with conductor $ZG(\mathbb{K}_1)_0$ such that π is equivalent to

$$\pi_{\chi, \theta} := \text{cInd}_{ZG(\mathbb{K}_1)GL(n, \mathfrak{o}_{\mathbb{K}_1})}^{GL(n, \mathbb{K}_1)} (\chi \otimes \sigma), \tag{108}$$

where σ is a representation of $GL(n, \mathfrak{o}_{\mathbb{K}_1}) = G_{(0)}(\mathfrak{o}_{\mathbb{K}_1})$ produced by inflation from a cuspidal irreducible representation $\bar{\sigma}$ of $GL(n, \mathbb{F}_q) = \bar{G}_{(0)}(\mathbb{F}_q)$. Thus, there is a character θ of $\bar{T}_{(0)}^n(\mathbb{F}_q)$ in general position such that $\bar{\sigma}$ is equivalent to $(-1)^{n-1} R_{\bar{T}_{(0)}^n}^{\bar{G}_{(0)}} \theta$.

Using this notation, the models for depth-zero supercuspidal representations of $GL(n, \mathbb{K}_1)$ are presented in Table 5.

Table 5: Models for depth-zero supercuspidal representations of $GL(n, \mathbb{K}_1)$

Representation	Model
$\pi_{\chi, \theta}$	$(-1)^{n-1} \text{ind}_{T^n}^G \text{cind}_{T_{(0)}^n}^{T^n} \mathcal{L}_\theta[n]$

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