

# MOTIVIC PROOF OF A CHARACTER FORMULA FOR $SL(2)$

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ABSTRACT. We give a motivic proof of a character formula for depth zero supercuspidal representations of  $p$ -adic  $SL(2)$ . We begin by finding the virtual Chow motives for the character values of all depth zero supercuspidal representations of  $p$ -adic  $SL(2)$ , at topologically unipotent elements. Then we find the virtual Chow motives for the values of the Fourier transform of all regular elliptic orbital integrals with depth zero in their Cartan subalgebra, at topologically nilpotent elements. Finally, we prove a character formula for depth zero supercuspidal representations by showing that the formula corresponds to three identities in the ring of virtual Chow motives over  $\mathbb{Q}$ .

## INTRODUCTION

The representation theory of groups over the finite fields is stated entirely in terms of algebraic geometry, by the work of Deligne-Lusztig [DL76] and Lusztig [L85]. Naturally, it is desirable to “lift” the geometric constructions to the  $p$ -adic groups. Motivic integration offers an approach to this problem which is very different from all the previous ones; it is not aiming at complete understanding of the geometry underlying various objects of representation theory, but it could in principle provide an algorithm for computing them in each individual case. This paper is essentially an experiment illustrating this point.

We are working with the group  $G$  over a  $p$ -adic field  $\mathbb{K}$  for the simple reason that the geometric objects that arise as a result of computing the “motivic” volumes turn out to be easily computable by hand; in fact, all the ones that are non-trivial turn out to be conics. For bigger groups both the algorithms and the results are of course much more complicated, but we believe that still there are computable geometric objects responsible for the character values, and it will be the subject of a future article.

The virtual Chow motives we calculate appear in the context of the following expansion for the character of any depth zero supercuspidal representation of  $G$ . (The precise statement is Theorem 1.6.) Let  $p$  be an odd prime and let  $\mathbb{K}$  be a  $p$ -adic field with residue field  $\mathbb{F}_q$  (so  $q$  is a power of  $p$ ). There is a finite set of regular elliptic orbits (represented by elements  $X_z$  below) in the Lie algebra  $\mathfrak{g}$  with the following property. For each depth zero supercuspidal representation  $\pi$  of  $G$

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there are rational numbers  $c_z(\pi) \in \mathbb{Q}$  such that

$$(0.1) \quad \Theta_\pi(\exp(Y)) = \sum_z c_z(\pi) \hat{\Phi}_G(X_z, Y),$$

for all regular, topologically nilpotent elements  $Y$  of  $\mathfrak{g}$ . The sum is taken over the set of orbits promised above;  $X_z$  is an element of the orbit and  $X_z$  has depth zero in its Cartan subalgebra. Here,  $\Theta_\pi$  is the distribution character of  $\pi$  in the sense of Harish-Chandra and  $\hat{\Phi}_G(X_z, -)$  denotes the generalized function on  $\mathfrak{g}$  corresponding to the Fourier transform of the orbital integral at  $X_z$ . The coefficients  $c_z(\pi)$  are given in Table 4, from which one sees that  $c_z(\pi)$  is in fact a rational function in  $q$  (the cardinality of the residue field of  $\mathbb{K}$ ) with integer coefficients; of course, this is a meaningful observation only if one clarifies in what sense the left- and right-hand sides of equation (0.1) may be viewed as functions in  $q$ , as we shall do later in the paper. We shall refer to equation (0.1) as an *Elliptic Character Expansion*.

It is fair to say that this elliptic character expansion is well-known in one form or another, as is the fact that it extends, *mut. mut.*, to a much larger class of representations and groups. The novelty of this paper lies in our proof. We use motivic integration to “separate” the character and each of the orbital integrals into a linear combination of the values of the corresponding function defined over the residue field, with coefficients that are virtual Chow motives. Then we can see directly that on each side of the Elliptic Character Formula we have the combination of the same values with the same coefficients. This approach clearly shows two ways in which algebraic geometry appears in the values of the characters and orbital integrals: the finite field function is in fact the characteristic function of a character sheaf; and the “motivic” coefficients come from the process of inflation and induction that connects our representation with the representation of the group over the finite field.

As a consequence of this perspective, we find much more than the coefficients  $c_z(\pi)$  promised above. In fact, we produce expressions for the values of the characters of all depth zero supercuspidal representations on all topologically unipotent elements of  $G$ . Likewise, we produce expressions for the values of the Fourier transform of all regular elliptic orbital integrals having depth zero in their Cartan subalgebra, on all topologically nilpotent elements of  $\mathfrak{g}$ . Comparing these leads to Table 4 and the proof of equation (0.1).

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## 1. BASIC NOTIONS

Throughout this paper,  $\mathbb{K}$  denotes a  $p$ -adic field; by this we mean  $\mathbb{K}$  is a field equipped with a non-archimedean valuation such that the field is complete and locally compact with respect to the topology determined by the norm, and such that the residue field  $\mathbb{F}_q$  for  $\mathbb{K}$  is finite of characteristic  $p$ . Notice that we do not put any condition on the characteristic of  $\mathbb{K}$ . As is well-known, any such  $\mathbb{K}$  is a finite extension of  $\mathbb{Q}_p$  or of  $\mathbb{F}_p((T))$ . The ring of integers of  $\mathbb{K}$  will be denoted  $\mathcal{O}_{\mathbb{K}}$  and the maximal ideal in  $\mathcal{O}_{\mathbb{K}}$  will be denoted  $\mathfrak{p}_{\mathbb{K}}$ . The residue field will be denoted  $\mathbb{F}_q$ . We reserve  $q$  for the cardinality of  $\mathbb{F}_q$ , and  $p$  for the characteristic of  $\mathbb{F}_q$ , so that  $q$  is a power of  $p$ .

Next, we fix a prime  $\ell$  different from  $p$  and an algebraic closure  $\bar{\mathbb{Q}}_{\ell}$  of  $\mathbb{Q}_{\ell}$ . Henceforth, by ‘representation’ we mean a representation in a vector space over  $\bar{\mathbb{Q}}_{\ell}$ .

Let  $G$  denote the group  $G$ . In Sections 1.1 and 3.1 we make brief reference to the Bruhat-Tits building and Moy-Prasad filtrations for  $G$  (see [MP94]). In those sections of the paper, we will use the notation of Moy-Prasad, except that we denote points in the Bruhat-Tits building by  $i$  (or  $j$ ). In particular, for each pair  $(i, r)$ , where  $i$  is a point and  $r$  is a non-negative real number as above,  $G_{i,r}$  is a compact open subgroup of  $G$ ; when  $r = 0$ , this is often abbreviated to  $G_i$ . For each pair  $(i, r)$  as above, let  $\underline{G}_{i,r}$  be the integral models for  $G_{\mathbb{K}}$  introduced in [Y02] such that the set of integral points in  $\underline{G}_{i,r}$  coincides with  $G_{i,r}$ . We write  $\bar{G}_{i,r}$  for the reductive quotient of the special fibre of  $\underline{G}_{i,r}$ . Analogous notions apply to the Lie algebra  $\mathfrak{g}$ : for any point  $i$  and any real number  $r$ ,  $\mathfrak{g}_{i,r}$  denotes the Moy-Prasad lattice in  $\mathfrak{g}$  and  $\underline{\mathfrak{g}}_{i,r}$  denotes the corresponding integral model for  $\mathfrak{g}_{\mathbb{K}}$  while  $\bar{\mathfrak{g}}_{i,r}$  denotes the corresponding Lie algebra scheme over  $\mathbb{F}_q$ . When we use  $\mathfrak{g}_{i,r}$  and  $G_{i,r}$ , we will often write these down explicitly, with the hope that readers unfamiliar with Moy-Prasad filtrations should have little difficulty with the essential features of this paper.

Note that we have not yet chosen a uniformizer for  $\mathbb{K}$ , and that none of the constructions above require such a choice.

**1.1. Depth-zero representations.** In this section we remind the reader how to construct all depth zero supercuspidal irreducible representations, up to equivalence.

Let  $(\pi, V)$  be a representation of  $G$ . For each point  $i$  in the Bruhat-Tits building for  $G$ , let  $V_i$  denote the subspace consisting of those  $v \in V$  such that  $\pi(k)v = v$  for each  $k \in G_{i,0^+}$ . Then  $G_{i,0}$  acts on  $V_i$ , and the resulting representation is denoted  $\pi_i$ . By definition, the representation  $\pi$  has *depth zero* if  $V_i$  is non-trivial for some point  $i$  in the Bruhat-Tits building for  $G$ .

Now we choose a uniformizer  $\varpi$  for  $\mathbb{K}$ . This choice is necessary to specify a map  $\mathcal{O}_{\mathbb{K}} \rightarrow \mathbb{F}_q$  (which we indicate by  $x \mapsto \bar{x}$ ) and the reduction map  $\rho_{i,r} : \mathfrak{g}_{i,r} \rightarrow \bar{\mathfrak{g}}_{i,r}$  appearing in Moy-Prasad, for  $i$  and  $r$  as above.

For each depth zero  $\pi$  and each point  $i$  in the Bruhat-Tits building for  $G$ , the representation  $\pi_i$  factors through  $\rho_{i,0} : G_{i,0} \rightarrow \bar{G}_{i,0}$ . Let  $\bar{\pi}_i$  denote the representation of  $\bar{G}_{i,0}$  on  $V_i$  such that  $\pi_i(k) = (\bar{\pi}_i \circ \rho_{i,0})(k)$  for all  $k \in G_i$ ; then  $\bar{\pi}_i$  is called the representation of  $\bar{G}_{i,0}$  obtained by compact restriction and is also denoted  $\text{cRes}_{\bar{G}_{i,0}}^G(\pi, V)$ .

From the other direction, let  $i$  be a point in the Bruhat-Tits building for  $G$  and let  $(\sigma, W)$  be a representation of  $\bar{G}_i$ , and let  $(\rho_{i,0}^* \sigma, W)$  be the representation of  $G_i$  defined by  $(\rho_{i,0}^* \sigma)(k) = \sigma(\rho_{i,0}(k))$ , for each  $k \in G_i$ . The right-regular representation of  $G$  on the space of compactly supported functions  $f : G \rightarrow W$  such that  $f(kg) = (\rho_{i,0}^* \sigma)(k)f(g)$  for all  $g \in G$  and all  $k \in G_i$  is called the representation of  $G$  obtained from  $(\sigma, W)$  (or  $(\rho_{i,0}^* \sigma, W)$ ) by compact induction and denoted  $\text{cInd}_{G_i}^G(\sigma, W)$ .

Although  $(\text{cRes}_{\bar{G}_i}^G, \text{cInd}_{G_i}^G)$  is an adjoint pair of functors (see [V03]), one must be careful: even if  $\sigma$  is a cuspidal irreducible representation of  $\bar{G}_i$ , it does not follow in general that  $\text{cInd}_{G_i}^G \sigma$  is an admissible representation of  $G$ , let alone supercuspidal. To clarify matters somewhat, we have the following result, which we paraphrase from independent results by Lawrence Morris and Moy-Prasad. For each point  $i$  in the Bruhat-Tits building for  $G$ , the compact restriction functor  $\text{cRes}_{\bar{G}_i}^G$  restricts to a surjection from supercuspidal irreducible representations of  $G$  to cuspidal irreducible representations of  $\bar{G}_i$ . Moreover, if  $\pi$  is a depth zero supercuspidal representation of  $G$  then there is a vertex  $i$  in the Bruhat-Tits building for  $G$  such that  $\text{cInd}_{G_i}^G(\text{cRes}_{\bar{G}_i}^G \pi)$  is equivalent to  $\pi$ .

There are exactly two  $G$ -orbits of vertices in the Bruhat-Tits building for  $G$ ; let (0) and (1) be the adjacent vertices corresponding to the maximal compact subgroups below.

$$(1.1) \quad \begin{aligned} G_{(0)} &:= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathcal{O}_{\mathbb{K}}; ad - bc = 1 \right\} \\ G_{(1)} &:= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, d \in \mathcal{O}_{\mathbb{K}}, b \in \mathfrak{p}_{\mathbb{K}}^{-1}, c \in \mathfrak{p}_{\mathbb{K}}^1; ad - bc = 1 \right\}. \end{aligned}$$

The special fibres of the integral models  $\underline{G}_{(0)}$  and  $\underline{G}_{(1)}$  are both  $SL(2)_{\mathbb{F}_q}$ ; it follows that  $\bar{G}_{(0)}$  and  $\bar{G}_{(1)}$  are both  $SL(2, \mathbb{F}_q)$ . Therefore, up to equivalence, all depth zero supercuspidal irreducible representations of  $G$  are produced by compact induction from cuspidal irreducible representations of  $SL(2, \mathbb{F}_q)$ . Accordingly, to enumerate all depth zero supercuspidal irreducible representations of  $G$  it is necessary to recall the construction of cuspidal irreducible representations of  $SL(2, \mathbb{F}_q)$ . This we will do in Section 1.3. In the meantime, we record the result: each depth zero supercuspidal irreducible representation of  $G$  is equivalent to one in Table 1, where all terms are defined in Section 1.3.

*Remark 1.1.* The representations induced from  $G_{(0)}$  do not depend on the choice of uniformizer  $\varpi$ , because the reduction map  $\rho_{(0),0}$  is independent of this choice; however, this is not true for representations induced from  $G_{(1)}$ . From this we see that  $G_{(0)}$  and  $G_{(1)}$  are not entirely on equal footing.

 TABLE 1. The representations  $\pi$  appearing in Theorem 1.6

$\pi(0)_\theta := \text{cInd}_{G_{(0)}}^G(\sigma_\theta)$	$\pi(1)_\theta := \text{cInd}_{G_{(1)}}^G(\sigma_\theta)$
$\pi(0)_+ := \text{cInd}_{G_{(0)}}^G(\sigma_+)$	$\pi(1)_+ := \text{cInd}_{G_{(1)}}^G(\sigma_+)$
$\pi(0)_- := \text{cInd}_{G_{(0)}}^G(\sigma_-)$	$\pi(1)_- := \text{cInd}_{G_{(1)}}^G(\sigma_-)$

**1.2. Gauss sums.** Before moving on to a review of the cuspidal irreducible representations of  $SL(2, \mathbb{F}_q)$ , we say a few words about Gauss sums, which are the prototype for much that follows. Let  $\text{sgn} : \mathbb{F}_q \rightarrow \bar{\mathbb{Q}}_\ell$  be the quadratic character of  $\mathbb{F}_q^\times$  extended by zero to  $\mathbb{F}_q$ . *Arbitrarily but irrevocably*, we fix an additive character  $\bar{\psi} : \mathbb{F}_q \rightarrow \bar{\mathbb{Q}}_\ell$ . Let  $\widehat{\text{sgn}}$  denote the Fourier transform of  $\text{sgn}$  with respect to  $\bar{\psi}$  in the following sense:

$$\widehat{\text{sgn}}(a) = \sum_{x \in \mathbb{A}^1(\mathbb{F}_q)} \text{sgn}(x) \bar{\psi}(ax).$$

(Note that this Fourier transform is not unitary.) Consider the Gauss sums  $\gamma_\pm : \mathbb{F}_q \rightarrow \bar{\mathbb{Q}}_\ell$  defined by

$$(1.2) \quad \begin{aligned} \gamma_+(a) &:= \sum_{\{x \in \mathbb{A}^1(\mathbb{F}_q) \mid \text{sgn}(x) = +1\}} \bar{\psi}(xa) \\ \gamma_-(a) &:= \sum_{\{x \in \mathbb{A}^1(\mathbb{F}_q) \mid \text{sgn}(x) = -1\}} \bar{\psi}(xa). \end{aligned}$$

Then  $\gamma_+ - \gamma_- = \widehat{\text{sgn}}$ . Elementary arguments show that  $\gamma_+ + \gamma_- = -1$  and  $(\gamma_+ - \gamma_-)^2 = q \text{sgn}(-1)$ . Fix a square-root  $\sqrt{q}$  of  $q$  in  $\bar{\mathbb{Q}}_\ell$ . Then there is a unique square-root  $\zeta \in \bar{\mathbb{Q}}_\ell$  of  $\text{sgn}(-1)$  (determined by  $\bar{\psi}$ ) such that

$$(1.3) \quad \widehat{\text{sgn}} = \sqrt{q}\zeta \text{sgn}.$$

Observe that  $\text{sgn}$  is an eigenvector for the Fourier transform with eigenvalue  $\sqrt{q}\zeta$ .

*Remark 1.2.* We wish to emphasize here that  $\zeta \in \bar{\mathbb{Q}}_\ell$  is determined by two choices: the square-root  $\sqrt{q}$  of  $q$  in  $\bar{\mathbb{Q}}_\ell$  and the additive character  $\bar{\psi} : \mathbb{F}_q \rightarrow \bar{\mathbb{Q}}_\ell$ . Also, although  $\zeta$  is a fourth root-of-unity, it is not necessarily a primitive fourth root-of-unity; indeed,  $\zeta^2 = \text{sgn}(-1)$ .

**1.3. Cuspidal representations of  $SL(2, \mathbb{F}_q)$ .** Let  $T$  denote the anisotropic torus of  $SL(2)_{\mathbb{F}_q}$  with  $\mathbb{F}_q$ -rational points

$$(1.4) \quad T(\mathbb{F}_q) = \left\{ \begin{bmatrix} x & y \\ \epsilon y & x \end{bmatrix} \in SL(2, \mathbb{F}_q) \mid x^2 - y^2 \epsilon = 1 \right\},$$

where  $\epsilon$  is a non-square in  $\mathbb{F}_q^\times$ .

Let  $R_T^G$  denote the Deligne-Lusztig induction functor of [DL76]; this takes virtual representations of  $T(\mathbb{F}_q)$  to virtual representations of  $SL(2, \mathbb{F}_q)$ . If  $\theta$  is non-trivial and the order of  $\theta$  is not 2 (so  $\theta$  is in ‘general position’) then  $(-1)R_T^G(\theta)$  is an irreducible cuspidal representation of  $SL(2, \mathbb{F}_q)$ , which we henceforth denote  $\sigma_\theta$ ; in other words, as virtual representations,

$$(1.5) \quad R_T^G(\theta) = -\sigma_\theta.$$

Let  $Q_T$  denote the restriction of trace  $R_T^G(\theta)$  to the set of unipotent element of  $SL(2, \mathbb{F}_q)$ , where  $\theta$  is in general position; as the notation suggests,  $Q_T$  is independent of  $\theta$ . This is the (classical) Green’s polynomial for  $SL(2, \mathbb{F}_q)$ . For future reference:

$$(1.6) \quad Q_T \left( \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \right) = \begin{cases} 1, & xy + z^2 = 0 \wedge (x \neq 0 \vee y \neq 0) \\ 1 - q, & x = 0 \wedge y = 0 \wedge z = 0 \\ 0, & \text{otherwise.} \end{cases}$$

If  $\theta_0$  is the quadratic character of  $T(\mathbb{F}_q)$ , then the Deligne-Lusztig virtual representation  $R_T^G(\theta_0)$  contains two irreducible cuspidal representations, comprising the Lusztig series for  $(T, \theta_0)$ . It is well-known that the difference between the characters of these two representations is supported by the set of regular unipotent elements of  $SL(2, \mathbb{F}_q)$  (see Remark 1.4). We label the representations in the Lusztig series for  $(T, \theta_0)$  by  $\sigma_+$  and  $\sigma_-$  and define

$$(1.7) \quad Q_G := \text{trace } \sigma_+ - \text{trace } \sigma_-$$

in such a way that

$$(1.8) \quad Q_G \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) = \sqrt{q} \zeta^3.$$

Then,

$$(1.9) \quad Q_G \left( \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \right) = \begin{cases} \sqrt{q} \zeta^3 \text{sgn}(x), & xy + z^2 = 0 \wedge x \neq 0 \\ \sqrt{q} \zeta^3 \text{sgn}(y), & xy + z^2 = 0 \wedge y \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

For a variety of reasons, it is the characters  $\sigma_+$  and  $\sigma_-$  which are the most interesting. As is well-known,  $Q_G$  is supported by regular unipotent elements. (Readers familiar with the lovely table from the appendix to [DM91] must be cautioned that what they denote  $\sqrt{q \text{sgn}(-1)}$  is here denoted  $\sqrt{q} \zeta^3$ .)

Since  $Q_G$  and  $Q_T$  are supported by unipotent elements, then for all primes  $p$  except 2, we can and shall often abuse notation by considering these as functions on the nilpotent elements of  $\mathfrak{g}$ .

*Remark 1.3.* We wish to emphasize that the choice of  $\sqrt{q}$  and  $\bar{\psi}$  determined  $\zeta$ , and that we labelled the representations in the Lusztig series  $\{\sigma_+, \sigma_-\}$  for  $(T, \theta_0)$  so that equation (1.8) hold true.

We close this section by recalling a celebrated result of Lusztig (see [L85]). Let  $\sigma$  be any cuspidal representation of  $SL(2, \mathbb{F}_q)$  and let  $\chi_\sigma$  denote the restriction of trace  $\sigma$  to unipotent elements. Then there are unique  $a(\sigma, Q) \in \bar{\mathbb{Q}}_\ell$  such that

$$\chi_\sigma = \sum_Q a_Q(\sigma) Q,$$

where the sum is taken over the set  $\{Q_T, Q_G\}$  of (generalized) Green's polynomials. The values of  $a_Q(\sigma)$  are given in Table 2.

TABLE 2.  $\chi_\sigma = \sum_Q a_Q(\sigma) Q$ .

$a_Q(\sigma)$	$Q = Q_T$	$Q = Q_G$
$\sigma = \sigma_\theta$	-1	0
$\sigma = \sigma_\pm$	$-\frac{1}{2}$	$\mp \frac{1}{2}$

*Remark 1.4.* The characters of the representations  $\sigma_\theta$  and  $\sigma_+$  and  $\sigma_-$  introduced above are perhaps best understood in terms of characteristic functions of character sheaves. Let  $\theta$  be any character of  $T(\mathbb{F}_q)$  and let  $\mathcal{L}_\theta$  be the corresponding Frobenius-stable Kummer local system on the étale site of  $T_{\mathbb{F}_q}$ ; in this case, the characteristic function  $\chi_{\mathcal{L}_\theta} : T(\mathbb{F}_q) \rightarrow \bar{\mathbb{Q}}_\ell$  of  $\mathcal{L}_\theta$  coincides with  $\theta$  (see [L85, 8.4] for the definition of ‘characteristic function’). Let  $\text{ind}_T^G$  denote the cohomological induction functor of [L85]; this takes Frobenius-stable character sheaves for  $T_{\mathbb{F}_q}$  to Frobenius-stable perverse sheaves for  $G_{\mathbb{F}_q}$ . If  $\theta$  is non-trivial and not quadratic (*i.e.*, in general position for  $SL(2)$ ) then  $\text{ind}_T^G \mathcal{L}_\theta[1]$  is a Frobenius-stable character sheaf and the characteristic function of this perverse sheaf is the character of  $R_T^G(\theta)$ ; in other words,

$$\text{trace } R_T^G(\theta) = \chi_{\text{ind}_T^G \mathcal{L}_\theta[1]},$$

when  $\theta$  is in general position. (Note that  $\text{ind}_T^G \mathcal{L}_\theta[1]$  is not a *cuspidal* character sheaf, even though  $\text{trace } R_T^G(\theta)$  is a cuspidal function.) On the other hand,  $\text{ind}_T^G \mathcal{L}_{\theta_0}[1]$  is not a character sheaf; rather, it is a direct sum of character sheaves. The algebraic group  $G_{\mathbb{F}_q}$  admits two cuspidal character sheaves: one, denoted  $C_0$ , is unipotent (supported by the unipotent cone in  $SL(2)_{\mathbb{F}_q}$ ) while the other, denoted  $C_1$ , is supported by  $-1$  times the unipotent cone. Comparing with the notation above, we have  $Q_G = \chi_{C_0}$ . In this paper we are chiefly concerned with the characters of cuspidal representations of  $SL(2, \mathbb{F}_q)$  when restricted to unipotent elements in  $SL(2, \mathbb{F}_q)$ ; the vector space spanned by characters of cuspidal representations restricted to unipotent elements is two-dimensional and a basis for this space is given by  $Q_G$  (the restriction of  $\chi_{C_0}$  to the unipotent cone) and  $Q_T$  (the restriction of  $\chi_{C_\theta}$  to the unipotent cone) where  $\theta$  any fixed character in general position.

**1.4. The elements  $X_z$ .** Before specifying the orbits appearing in equation (0.1) we say a few words about Cartan subalgebras of  $\mathfrak{g}$ , or equivalently, about conjugacy classes of forms of  $GL(1)_{\mathbb{K}}$  in  $SL(2)_{\mathbb{K}}$ . (These are the proper twisted-Levi subgroups of  $G$ .) This is precisely the kernel of the map in Galois cohomology  $\alpha_{\mathbb{K}} : H^1(\mathbb{K}, N) \rightarrow H^1(\mathbb{K}, G)$  which is induced by the inclusion  $N \rightarrow G$ , where

$N$  denotes the normalizer of the split torus of diagonal matrices in  $G$ . To study this kernel, it is useful to first pass to the unramified closure  $\mathbb{K}_{nr}$  of  $\mathbb{K}$  in  $\bar{\mathbb{K}}$  and observe that  $\ker \alpha_{\mathbb{K}_{nr}}$  contains exactly one non-trivial element and this element corresponds to the conjugacy class of forms which split over a quadratic extension of  $\mathbb{K}_{nr}$ . (Note that this is a totally ramified extension.) Let  $\tau_0$  be the trivial element in  $\ker \alpha_{\mathbb{K}_{nr}}$  and let  $\tau_1$  be the non-trivial element in  $\ker \alpha_{\mathbb{K}_{nr}}$ . The inclusion  $\text{Gal}(\bar{\mathbb{K}}/\mathbb{K}_{nr}) \rightarrow \text{Gal}(\bar{\mathbb{K}}/\mathbb{K})$  induces a surjection in Galois cohomology which in turn restricts to a surjection  $\ker \alpha_{\mathbb{K}} \rightarrow \ker \alpha_{\mathbb{K}_{nr}}$ , as pictured below.

$$\begin{array}{ccc}
H^1(\mathbb{K}, G) & \twoheadrightarrow & H^1(\mathbb{K}_{nr}, G) \\
\uparrow \alpha_{\mathbb{K}} & & \uparrow \alpha_{\mathbb{K}_{nr}} \\
H^1(\mathbb{K}, N) & \twoheadrightarrow & H^1(\mathbb{K}_{nr}, N) \\
\uparrow & & \uparrow \\
\ker \alpha_{\mathbb{K}} & \twoheadrightarrow & \ker \alpha_{\mathbb{K}_{nr}}
\end{array}$$

The fibre above  $\tau_0$  consists of three classes in  $H^1(\mathbb{K}, N)$  and the fibre above  $\tau_1$  consists of four classes in  $H^1(\mathbb{K}, N)$  if  $\text{sgn}(-1) = 1$  and two classes in  $H^1(\mathbb{K}, N)$  if  $\text{sgn}(-1) = -1$ , where  $\text{sgn} : \mathbb{K} \rightarrow \bar{\mathbb{Q}}_\ell$  is the quadratic character of  $\mathcal{O}_{\mathbb{K}}^*$  extended by zero to all  $\mathbb{K}$ .

We now return to the orbits appearing in equation (0.1). In order to define these orbits, we now pick a non-square unit  $\varepsilon$  in  $\mathcal{O}_{\mathbb{K}}$ . For reasons which will be apparent later, we also now introduce a new parameter,  $v$ , taking values in  $\mathcal{O}_{\mathbb{K}}^*$  and define  $X_z(v)$  in Table 3. Representatives for the orbits appearing in equation (0.1) are given by  $X_z := X_z(1)$ . Notice that each  $X_z(v) \in \mathfrak{g}$  above is *good*, in the sense of [A98, §2.2] (see also [CH04, §1.1]).

TABLE 3. Elements  $X_z(v)$ . Orbits  $X_z := X_z(1)$  appear in Theorem 1.6.

$z$	$s_1$	$s_2$	$t_0$	$t_1$	$t_2$	$t_3$
$X_z(v)$	$\begin{bmatrix} 0 & v \\ -\varepsilon v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^{-1}v \\ -\varepsilon \varpi v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & v \\ -\varpi v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon v \\ -\varpi \varepsilon^{-1}v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & v \\ -\varpi \varepsilon v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon v \\ -\varpi v & 0 \end{bmatrix}$

Each of the  $X_z(v)$  determines a compact Cartan subalgebra  $\mathfrak{g}$  (independent of  $v \in \mathcal{O}_{\mathbb{K}}^*$ ), and therefore a *cocycle* representing a class in  $\ker \alpha_{\mathbb{K}}$ . Thus, the summation set in equation (0.1) is the set of cocycles  $\{s_1, s_2, t_0, t_1, t_2, t_3\} \subset Z^1(\mathbb{K}, N)$ . However, some of these cocycles represent the same cohomology class in  $H^1(\mathbb{K}, N)$ , depending on the sign of  $-1$  in  $\mathbb{K}$ ; specifically, if  $\text{sgn}(-1) = -1$  then  $t_0$  and  $t_1$  represent to same cohomology class, and  $t_2$  and  $t_3$  also represent the same class in  $H^1(\mathbb{K}, N)$ .

*Remark 1.5.* Observe that picking  $\varepsilon$  is exactly equivalent to picking representative cocycles for the cohomology classes in  $\ker \alpha_{\mathbb{K}}$ . In fact, our choice of  $s_1$  alone corresponds to our choice of  $\varepsilon$ .

**1.5. Normalization of the measures.** We choose the Haar measure on  $SL(2, \mathbb{K})$  that coincides with the Serre-Oesterlé measure on  $SL(2, \mathcal{O}_{\mathbb{K}})$ ; that is, our Haar measure for  $G$  is normalized so that the volume of the maximal compact subgroup  $SL(2, \mathcal{O}_{\mathbb{K}})$  is the cardinality of  $SL(2, \mathbb{F}_q)$ , which is  $q(q^2 - 1)$ . In this setting, all

the fibres of the projection  $SL(2, \mathcal{O}_{\mathbb{K}}) \rightarrow SL(2, \mathbb{F}_q)$  have volume 1. We denote this measure on  $G$  by  $m$ . This choice determines a Haar measure on  $\mathfrak{g}$  such that volume of the kernel of the map  $sl(2, \mathcal{O}_{\mathbb{K}}) \rightarrow sl(2, \mathbb{F}_q)$  is also 1; *i.e.*, the volume of  $sl(2, \mathcal{O}_{\mathbb{K}})$  equals the cardinality of  $sl(2, \mathbb{F}_q)$ . We denote this measure on  $\mathfrak{g}$  by  $\text{vol}$ . Observe that the volume of  $sl(2, \mathcal{O}_{\mathbb{K}})$  is  $\frac{q^2}{q^2-1}$  times that of  $SL(2, \mathcal{O}_{\mathbb{K}})$ .

**1.6. Statement of the character formula.** We now re-state the character formula appearing in the Introduction more carefully. Fix measures for  $G$  and  $\mathfrak{g}$  as specified in Section 1.5.

For any smooth representation  $\pi$  of  $G$ , let  $\Theta_\pi$  denote the generalized function corresponding to the distribution character  $\pi$  with respect to the Haar measure on  $G$  above. For any regular semi-simple  $X$  in  $\mathfrak{g}$  and any locally constant compactly supported  $f : \mathfrak{g}\mathbb{K} \rightarrow \mathbb{Q}_\ell$ , let  $\hat{\Phi}_G(X, f)$  denote the orbital integral of  $f$  at  $X$ . Fix an additive character  $\psi$  of  $\mathbb{K}$  with conductor  $\mathcal{O}_{\mathbb{K}}$  such that the induced additive character of the residue field  $\mathbb{F}_q$  is  $\bar{\psi}$  (see Section 1.2). For any  $f$  as above, let  $\hat{f}$  denote the Fourier transform of  $f$  taken with respect to the Killing form  $\langle Y, Z \rangle := \text{trace}(YZ)$ , character  $\psi$  and Haar measure on  $\mathfrak{g}$  specified in Section 1.5; thus,

$$\hat{f}(Y) = \int_{\mathfrak{g}} f(Z) \psi(\langle Y, Z \rangle) dZ.$$

Let  $\hat{\Phi}_G(X, f)$  denote the orbital integral of  $\hat{f}$  at  $X$ . We will also write  $\hat{\Phi}_G(X, )$  for the locally constant function representing the distribution  $f \mapsto \hat{\Phi}_G(X, f)$ .

**Theorem 1.6.** *Let  $\mathbb{K}$  be a  $p$ -adic field with  $p \neq 2$ . For each depth zero supercuspidal representation  $\pi$  of  $G$  and for each cocycle  $z \in \{s_1, s_2, t_0, t_1, t_2, t_3\}$  (see Section 1.4) there is a regular elliptic  $X_z \in \mathfrak{g}$  and  $c_z(\pi) \in \mathbb{Q}$  such that*

$$(1.10) \quad \Theta_\pi(\exp Y) = \sum_z c_z(\pi) \hat{\Phi}_G(X_z, Y) dX,$$

*for all regular topologically nilpotent elements  $Y \in \mathfrak{g}$ ; moreover, each  $c_z(\pi)$  is a rational function in  $q$  with integer coefficients.*

This paper is devoted to understanding Theorem 1.6 from the motivic perspective. To that end, we present a review of motivic integration in Section 1.7. Our proof of Theorem 1.6 is sketched in Section 1.8 and executed in Section 4; we will calculate the  $c_z(\pi)$  for each depth zero supercuspidal  $\pi$ , up to equivalence. The coefficients  $c_z(\pi)$  are given in Table 4.

**1.7. Motivic integration.** The goal of arithmetic motivic integration is essentially to reduce the calculation of  $p$ -adic volumes to the computation of the number of points on varieties over finite fields. In particular, suppose we are talking about subsets of an affine space defined over a  $p$ -adic field. Consider the measurable subsets that can be described in a language (of logic) that depends neither on  $p$  nor on the choice of the uniformizer of the valuation (one such language is the language of Denef-Pas that is described in the next subsection). Once a set is described by a logical formula in such a language, it is possible to associate a geometric object defined over  $\mathbb{Q}$  with it, in such a way that for almost all primes  $p$ , the volume of our set can be recovered from the number of points of the reduction of this object over  $\mathbb{F}_q$ , where  $q = p^r$  is the cardinality of the residue field of the given local field.

Motivic integration is based on the algorithm of elimination of quantifiers (there is also a more recent version [CL05] where in some parts quantifier elimination is

TABLE 4. The coefficients  $c_z(\pi)$  appearing in Theorem 1.6.

$c_z(\pi)$	$s_1$	$t_0$	$t_1$	$t_2$	$t_3$	$s_2$
$\pi(0)_\theta$	$q - 1$	0	0	0	0	0
$\pi(0)_+$	$\frac{q-1}{2}$	$-\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	0
$\pi(0)_-$	$\frac{q-1}{2}$	$+\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	0
$\pi(1)_+$	0	$-\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$\frac{q-1}{2}$
$\pi(1)_-$	0	$+\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$\frac{q-1}{2}$
$\pi(1)_\theta$	0	0	0	0	0	$q - 1$

replaced with cell decomposition, also an algorithmic procedure). Since at present motivic integration as an algorithm is not implemented, it has not previously been used for computing examples; nevertheless, it can be used to prove general existence results for the required geometric objects (which already has implications, as explored in [Ha05], [CH04] and [G04]).

We refer to the original papers [DL01], [CL05] and to the beautiful exposition [Hb05] for the detailed description the concept of motivic integration, and the statements of “comparison theorems” that relate the classical  $p$ -adic volumes with the motivic volumes. In the next few subsections we just give a list of the techniques we use.

1.7.1. *The language of rings and the language of Denef-Pas.* The language of rings is the language of logic such that its formulas can be interpreted in any ring with identity; thus, any such ring is a structure for this language. The language of rings has symbols for '0' and '1', symbols for countably many variables, and the symbols for the operations of addition '+' and multiplication '×'. A formula is a syntactically correct expression built from finitely many of these symbols, and also parentheses '()', quantifiers '∃', '∀', and symbols of conjunction '∧', disjunction '∨', and negation '¬'. We will usually use this language to work with the residue field of our valued field.

There is also a first-order language that is perfectly suited for defining subsets of non-archimedean valued fields: it is the language of Denef-Pas. Formulas in this language have variables of three *sorts*: the valued field sort, the residue field sort, and the value sort. There also are symbols for the function 'ord(·)' which takes variables of the valued field sort to the variables of the value sort, and the function

' $\text{ac}(\cdot)$ ' takes the variables of the valued field sort to the variables of the residue field sort. The formulas are built using algebraic operations on variables of the same sort, the symbols ' $\text{ord}(\cdot)$ ' and ' $\text{ac}(\cdot)$ ' (which can be applied to variables of the valued field sort), quantifiers, and the symbols for the logical operations of conjunction, disjunction and negation.

Every valued field with a choice of a uniformizer  $(\mathbb{K}, \varpi)$  is a structure for the language of Denef-Pas; then the functions ' $\text{ord}$ ' and ' $\text{ac}$ ' match the usual valuation and "angular component" maps, where the "angular component"  $\text{ac}(x)$  equals the first non-zero coefficient of the  $p$ -adic expansion of  $x$ , *i.e.*, if  $x$  is a unit,  $\text{ac}(x) = x \bmod (\varpi)$ ; if  $x$  is not a unit,  $\text{ac}(x) = x\varpi^{-\text{ord}(x)} \bmod (\varpi)$ .

A subset of an affine space over a local field  $\mathbb{K}$  is called *definable* if it can be defined by a formula in the language of Denef-Pas. For a formula  $\Phi$  with  $m$  free variables of the residue field sort and no other free variables, given a local field with the ring of integers  $\mathcal{O}_{\mathbb{K}}$  and a choice of the uniformizer, we will denote by  $Z(\Phi, \mathcal{O}_{\mathbb{K}})$  the subset of  $\mathcal{O}_{\mathbb{K}}^{\oplus m}$  defined by the formula  $\Phi$ .

1.7.2. *The ring of virtual Chow motives.* In all versions of motivic integration, the motivic volume takes values in some ring built from the Grothendieck ring of the category of algebraic varieties over the base field, which is  $\mathbb{Q}$  in our case.

Strictly speaking, the so-called ring of virtual Chow motives is a natural choice of the ring of values for the arithmetic motivic volume, for reasons described in [Hb05]. We will not define Chow motives here (see [S94] for a good introduction). As a first approximation, it is possible to think just of formal linear combinations of isomorphism classes of varieties with rational coefficients.

The motivic volume takes values in the ring  $\text{Mot}$ , which we will now define. Let  $\text{Mot}_{\mathbb{Q}}$  be the category of Chow motives over  $\mathbb{Q}$ . We take its Grothendieck ring  $K_0(\text{Mot}_{\mathbb{Q}})$ , *i.e.*, the ring of formal linear combinations of the isomorphism classes of the Chow motives, with natural relations (see [S94]), and the product coming from tensor product. This ring has a unit 1. For every smooth projective variety  $V$  there is a Chow motive that corresponds to  $V$  in a natural way. It is a deep theorem that this map from varieties to motives extends to all (not just smooth projective) varieties, and induces a homomorphism from the Grothendieck ring of the category of varieties  $K_0(\text{Var}_{\mathbb{Q}})$  to  $K_0(\text{Mot}_{\mathbb{Q}})$ . Let us denote the image of this map by  $K_0(\text{Mot}_{\mathbb{Q}})^v$ . The image of the class of the affine line under this map is usually denoted by  $\mathbb{L}$  (called the Lefschetz motive); the point maps to 1. Note that naturally, in  $K_0(\text{Mot}_{\mathbb{Q}})$  we have  $\mathbb{L} + 1 = [\mathbb{P}^1]$ , where  $[\mathbb{P}^1]$  is the class of the Chow motive corresponding to the projective line.

The ring  $\text{Mot}$  is obtained from  $K_0(\text{Mot}_{\mathbb{Q}})^v \otimes_{\mathbb{Z}} \mathbb{Q}$  by localizing at  $\mathbb{L}$  followed by localizing further at the set  $\{\mathbb{L}^{-i} - 1 \mid i > 0\}$  (*i.e.*, all these elements can be formally inverted).

Note that the localization at  $\{\mathbb{L}^{-i} - 1 \mid i > 0\}$  (which is equivalent to adding in the sums of geometric series with quotient  $\mathbb{L}^i$ ) replaces completion that was done in the original version [DL01].

Let us also adopt the following convention: for a polynomial  $F(q) \in \mathbb{Z}[q]$ , we will denote by  $[F(\mathbb{L})]$  the class of  $F(\mathbb{L})$  in the ring  $\text{Mot}$ .

For every prime power  $q$ , there is an action of  $\text{TrFrob}_q$  on the elements of  $\text{Mot}$  that comes from the Frobenius action on the Chow motives. It can be thought of as a generalization of counting points on a variety over  $\mathbb{F}_q$ . The trace of the Frobenius operator on a Chow motive is the alternating sum of the traces of Frobenius acting

on its  $\ell$ -adic cohomology groups, and it is, *a priori*, an element of  $\bar{\mathbb{Q}}_\ell$  (see [DL01, Section 3.3]), but in fact this number lies in  $\mathbb{Q}$  for all elements of  $\text{Mot}$  that arise from the motivic volumes (in particular, for us the choice of  $\ell$  doesn't matter). It is the trace of Frobenius action that allows us to relate the values of the motivic measure (elements of  $\text{Mot}$ ) with the usual  $p$ -adic volumes (rational numbers).

1.7.3. *The motivic volume.* We cannot describe the construction of the motivic volume here, but we only need very simple examples in this paper, so we list just a few main principles and introduce the notation. Following the pattern of [Hb05], we start by declaring that there is a map from formulas in the language of rings to the ring  $\text{Mot}$  defined above. We denote the image of a formula  $\phi$  under this map by  $[\phi]$ . We will use two properties of this map:

- The motivic volume of a formula in the language of rings with no quantifiers that defines a smooth algebraic variety is the class of that variety.
- If  $\phi_1$  and  $\phi_2$  are equivalent ring formulas, then they have equal motivic volumes. If  $\phi_1$  is an  $n$ -fold cover of  $\phi_2$  (both the equivalence, and the term “cover” are understood in the sense of [Hb05]), then

$$[\phi_2] = \frac{1}{n}[\phi_1].$$

Now let us say a few words about the construction of the motivic volume for definable sets that is compatible with the classical volume of  $p$ -adic sets. First, we need to talk about sets in a way that does not depend on the field – and that is done by considering *definable* sets, as described above. In order to pass back and forth between sets and formulas defining them, it is very convenient to use the notion of a *definable subassignment* introduced in [DL01].

Let  $h_{\mathbb{A}^m}$  be the functor of points of the affine space that takes fields to sets ( $K \mapsto \mathbb{A}^m(K)$ ). A *subassignment* of  $h_{\mathbb{A}^m}$  is simply a collection of subsets of  $\mathbb{A}^m(K)$ , one for each field  $K$ . A formula  $\Phi$  in the language of Denef-Pas with  $m$  free variables naturally defines a subassignment of  $h_{\mathbb{A}^m}$  on the category of local fields: for each local field  $K$ , take the subset of  $\mathbb{A}^m(K)$  on which the formula takes the value ‘true’. Such subassignments are called *definable*.

The *motivic volume* is defined on definable subassignments and takes values in the ring of virtual Chow motives  $\text{Mot}$ . We denote the motivic volume by  $\mu$ , so that  $\mu(\Phi)$  denotes the motivic volume of the subassignment defined by the formula  $\Phi$ . Sometimes, if we have a definable set  $H$  (defined by some formula  $\Phi$ ), we will write  $\mu(H)$ , meaning  $\mu(\Phi)$ .

The strategy of motivic integration is to replace the formula that defines our subassignment with an equivalent formula that has no quantifiers ranging over the valued field (the language of Denef-Pas admits quantifier elimination). The next step is to “approximate” the new formula by the formulas in the language of rings whose variables range only over the finite field. Finally, ring formulas can be mapped to the ring  $\text{Mot}$  defined above, as mentioned in the beginning of this subsection.

The main result that makes motivic integration relevant for us is the comparison theorem [DL01, Theorem 8.3.1, 8.3.2]. Let  $\Phi$  be a formula in the language of Denef-Pas. The comparison theorems (one of which deals with the fields of characteristic zero, and the other one – with function fields), state that for all but finitely many primes  $p$ , for any local field  $\mathbb{K}$  with the ring of integers  $\mathcal{O}_{\mathbb{K}}$  and the residue field  $\mathbb{F}_q$

of cardinality  $q = p^r$ , the volume of the set  $Z(\Phi, \mathcal{O}_{\mathbb{K}})$  equals the trace of Frobenius action on  $\mu(\Phi)$  (which is an element of  $\text{Mot}$ ).

In fact, it is possible to some extent to keep track of the “bad primes” where the comparison theorem fails, and we will do this as we compute the motivic volumes in our examples.

1.7.4. *Motivic integration with parameters.* Here we quote one more technical notation from [CL05] that will be used only in the proof of Lemma 2.3. Cluckers and Loeser [CL05] have a more general construction of the motivic volume: they define motivic volume on definable subassignments of the functor of points of  $\mathbb{A}_K^m \times A_k^n \times \mathbb{Z}^r$ , which is denoted by  $h[m, n, r]$  (here  $K$  stands for the valued field, and  $k$  for its residue field), which corresponds to considering families of definable subassignments of  $h_{A_K^m}$  with parameters in  $k$  and in  $\mathbb{Z}^r$ .

1.8. **Outline of the proof of Theorem 1.6.** Since the rest of the paper is devoted to the proof of Theorem 1.6, we describe the strategy before we get into the technicalities.

We need to prove an equality of two distributions represented by locally constant locally integrable functions on the set of regular topologically unipotent elements in the group  $G$ . Each of these functions takes countably many values. We prove the equality by partitioning the domain into sets on which the both sides are constant, and showing equality on each of these sets. Moreover, the problem is in fact not just countable, it is in some sense finite. Each one of the two functions is obtained by a Frobenius-like formula from a function which takes finitely many values (explicitly, three distinct values for the function on the left-hand side and five distinct values for the function on the right-hand side). Motivic integration allows us to isolate these values and the coefficients at each one of them that are acquired in the Frobenius formula. Those latter coefficients also turn out to be computable. It is in this sense that the proof is better suited for a computer than for a human – and that is the point of this paper, really.

In order to be more precise we must first say a few words about regular, topologically unipotent elements of  $G$ . To do this, we start by studying regular, topologically nilpotent elements in the Lie algebra  $\mathfrak{g}$ ; we will then invoke our assumption that  $p$  is odd, in which case these correspond exactly (via the exponential map) to regular, topologically unipotent elements of  $G$ .

What we need, ideally, is a partition of the set of regular topologically unipotent elements into *definable* sets such that all the functions involved (the characters and Fourier transforms of the orbital integrals) would be constant on them. We will not explicitly use any local constancy results (except in Section 5) instead, we will first make the partition, and then see that it is the right one.

Recall that at the moment,  $\varepsilon$  (a non-square unit) and  $\varpi$  (the uniformizer of the valuation) are fixed. One of the points of motivic integration approach is to do the calculation independently of these choices, but that requires some tricks (discussed below), since they are both very deeply ingrained in all the definitions:  $\varepsilon$  is linked with the cocycles  $z$ , and  $\varpi$  plays a role in the construction of our representations. For now, we write down explicitly seven Cartan subalgebras of  $\mathfrak{g}$  (one non-compact and six compact).

$$\mathfrak{h}_{s'_0} = \left\{ \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \mid x \in \mathbb{K} \right\}$$

$$\begin{aligned}
\mathfrak{h}_{s'_1} &= \left\{ \begin{bmatrix} 0 & x \\ x\varepsilon & 0 \end{bmatrix} \mid x \in \mathbb{K} \right\} \\
\mathfrak{h}_{s'_2} &= \left\{ \begin{bmatrix} 0 & x\varpi^{-1} \\ x\varepsilon\varpi & 0 \end{bmatrix} \mid x \in \mathbb{K} \right\} \\
\mathfrak{h}_{t'_0} &= \left\{ \begin{bmatrix} 0 & x \\ x\varpi & 0 \end{bmatrix} \mid x \in \mathbb{K} \right\} \\
\mathfrak{h}_{t'_1} &= \left\{ \begin{bmatrix} 0 & x\varepsilon \\ x\varpi\varepsilon^{-1} & 0 \end{bmatrix} \mid x \in \mathbb{K} \right\}. \\
\mathfrak{h}_{t'_2} &= \left\{ \begin{bmatrix} 0 & x \\ x\varpi\varepsilon & 0 \end{bmatrix} \mid x \in \mathbb{K} \right\} \\
\mathfrak{h}_{t'_3} &= \left\{ \begin{bmatrix} 0 & x\varepsilon \\ x\varpi & 0 \end{bmatrix} \mid x \in \mathbb{K} \right\}
\end{aligned}$$

We have labelled our Cartan subalgebras by cocycles  $z \in \{s'_0, s'_1, s'_2, t'_0, t'_1, t'_2, t'_3\}$ , one from each cohomology class in the case  $\text{sgn}(-1) = 1$ . In the case  $\text{sgn}(-1) = -1$ , the Cartan subalgebras labelled by  $t'_0$  and  $t'_1$  are conjugate; and the ones labelled by  $t'_2$  and  $t'_3$  are conjugate (which means that the corresponding cocycles represent the same cohomology class). We will always consider each one of these subalgebras separately, which means that we are doing one third more work than necessary half the time.

Our Cartan subalgebras are filtered by depth. To write the filtration lattices explicitly, we define, for each  $z \in \{s'_0, s'_1, s'_2, t'_0, t'_1, t'_2, t'_3\}$ , each integer  $n$  and unit  $u$  in  $\mathcal{O}_{\mathbb{K}}$ , an element  $Y_{z,n}(u) \in \mathfrak{h}_z$  by Table 5. Let  $\mathfrak{h}_{z,n} := \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^*\}$ . Then  $\mathfrak{h}_{z,n}$  is the set of elements of  $\mathfrak{h}_z$  with depth exactly  $n$  in  $\mathfrak{h}_z$ .

TABLE 5. Elements  $Y_{z,n}(u)$ .

$z$	$s_0$	$s_1$	$s_2$	
$Y_{z,n}(u)$	$\begin{bmatrix} \varpi^n u & 0 \\ 0 & -\varpi^n u \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^n u \\ \varepsilon \varpi^n u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^{n-1} u \\ \varepsilon \varpi^{n+1} u & 0 \end{bmatrix}$	
$z$	$t_0$	$t_1$	$t_2$	$t_3$
$Y_{z,n}(u)$	$\begin{bmatrix} 0 & \varpi^n u \\ \varpi^{n+1} u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon \varpi^n u \\ \varepsilon^{-1} \varpi^{n+1} u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^n u \\ \varepsilon \varpi^{n+1} u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon \varpi^n u \\ \varpi^{n+1} u & 0 \end{bmatrix}$

Further, let  ${}^G\mathfrak{h}_{z,n}$  (resp.  ${}^G\mathfrak{h}_{z,n}^+$ ,  ${}^G\mathfrak{h}_{z,n}^-$ ) denote the smallest  $G$ -invariant set containing  $\mathfrak{h}_{z,n}$ , where  $G$  acts on  $\mathfrak{g}$  by adjoint action. The disjoint union of all the sets  ${}^G\mathfrak{h}_{z,n}$  (as  $z$  ranges over  $\ker \alpha_{\mathbb{K}}$  and  $n$  ranges over all non-negative integers, with the exception of  $n = 0$  in  ${}^G\mathfrak{h}_{s_0,n}$ ,  ${}^G\mathfrak{h}_{s_1,n}$  and  ${}^G\mathfrak{h}_{s_1,n}$ ) coincides with the set of regular topologically nilpotent elements in  $\mathfrak{g}$ .

In fact, conjugation by  $G$  removes the ambiguity caused by the choice of specific cocycles (and therefore the choice of  $\varepsilon$ ): different representatives of the same cohomology class correspond to different, but conjugate, Cartan subalgebras. Hence, it would be more appropriate to label the sets  ${}^G\mathfrak{h}_{z,n}$  by the cohomology classes, and call them  ${}^G\mathfrak{h}_{\tau,n}$ , where  $\tau$  is the cohomology class represented by  $z$ . We would

like to stress that this produces a partition of the topologically nilpotent set that depends only on  $\text{sgn}(-1)$  (because the number of the distinct cohomology classes depends on this sign), and not on any choices. However, we do need the specific representatives  $Y_{z,n}(u)$  of each one of the sets  ${}^G\mathfrak{h}_{\tau,n}$  in order to carry out our calculations. Our choice of these representatives depends on  $\varepsilon$  and  $\varpi$ . The strategy is to carry out all the explicit calculations first, get to the level of logical formulas, and at that stage  $\varpi$  would disappear entirely. To remove the dependence of  $\varepsilon$ , in Section 2.3 we turn it into a parameter ranging over the residue field, and then average over it, also showing that in fact we are averaging a constant function.

We will soon see that our distributions are not all constant on  $\mathfrak{h}_{z,n}$ . In anticipation of the correct partition of regular, topologically nilpotent elements of  $\mathfrak{g}$  we define

$$\mathfrak{h}_{z,n,+} := \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^* \wedge \text{sgn}(u) = +1\},$$

and

$$\mathfrak{h}_{z,n,-} := \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^* \wedge \text{sgn}(u) = -1\},$$

and denote by  ${}^G\mathfrak{h}_{z,n,\pm}$  the corresponding  $G$ -invariant sets.

Note that with the way we defined our subalgebras, in the case  $\text{sgn}(-1) = -1$  when two pairs of them become conjugate, this conjugation switches the  $\pm$  sign:  ${}^G\mathfrak{h}_{t_0,n,+}$  corresponds to  ${}^G\mathfrak{h}_{t_1,n,-}$ , and  ${}^G\mathfrak{h}_{t_3,n,+}$  corresponds to  ${}^G\mathfrak{h}_{t_2,n,-}$  (we will see that this is reflected in our character table).

On the group side, we write  $\Gamma_{z,n}$  (resp.  $\Gamma_{z,n,+}$ ,  $\Gamma_{z,n,-}$ ) for the image of  $\mathfrak{h}_{z,n}$  (resp.  $\mathfrak{h}_{z,n,+}$ ,  $\mathfrak{h}_{z,n,-}$ ) under the exponential map, when defined; we denote the corresponding  $G$ -invariant sets by  ${}^G\Gamma_{z,n}$  (resp.  ${}^G\Gamma_{z,n,+}$ ,  ${}^G\Gamma_{z,n,-}$ ). The characteristic functions of the sets  ${}^G\Gamma_{z,n,\pm}$  will be denoted by  $f_{z,n,\pm}$  and the characteristic functions of the sets  ${}^G\mathfrak{h}_{z,n,\pm}$  will be denoted by  $\tilde{f}_{z,n,\pm}$  throughout the paper.

Given a depth zero representation  $\pi$  induced from a maximal compact subgroup  $G_i$ , in Section 2 we associate three virtual Chow motives (denoted  $M_{z,n,\pm}^{i,0}$ ,  $M_{z,n,\pm}^{i,1}$  and  $M_{z,n,\pm}^{i,\varepsilon}$ ) with each triple  $(z,n,\pm)$  as above, so that the value of the distribution character of  $\pi$  at  $f_{z,n,\pm}$  can be recovered from this triple of virtual Chow motives for almost all residual characteristics  $p$  (in fact, for all  $p \neq 2$ ). Moreover, we will see that the only way in which these motives depend on  $\pi$  is through the compact subgroup on which  $\pi$  has non-trivial compact restriction (*i.e.*, through the choice of the vertex in the building  $i = (0)$  or  $i = (1)$ , see Section 1.1).

In Section 3, we also associate with each  $(z,n,\pm)$  five virtual Chow motives, so that all orbital integrals that appear in the right-hand side of the Elliptic Character Formula can be recovered from these motives.

In Section 3.4, we put together all the results about groups over the finite fields that are crucial for our understanding of the  $p$ -adic “lifts”.

Since both sides of Elliptic Character Formula are invariant under conjugation by  $G$ , all we have to do is check the equality on each of the sets  $\Gamma_{z,n}$ . This is done in Section 4. Due to a very mechanical nature of this proof, we do not include all the details. In all proofs, we include all the details in one “unramified” case  $z = s_1$  and one “ramified” case  $z = t_2$ ; in the other cases we just indicate the differences and summarize the results in the tables.

## 2. MOTIVES CORRESPONDING TO OUR CHARACTERS

Throughout Section 2,  $i$  denotes either the standard vertex (0) or the vertex (1) in the Bruhat-Tits building for  $G$  (see equation 1.1); we also reserve the symbol  $j$  for the baricentre of the facet with boundary  $\{(0), (1)\}$ .

**2.1. Consequences of the Frobenius formula for the character.** This section follows the method of expressing the character as a sum over conjugacy classes in the reductive quotient that was used in [G04].

Let  $\pi$  be a supercuspidal depth zero representation of  $G$  and let  $\Theta_\pi$  be its distribution character in the sense of Harish-Chandra. Let  $f$  be a test function supported on some compact subset  $H$  of the set of regular topologically unipotent elements in  $G$ . We can assume that  $f = f_H$  is the characteristic function of such a set without loss of generality. Let  $\chi$  be the character of the representation of the finite group  $\bar{G}_i$  that gave rise to  $\pi$  (see Section 1.1). Since  $G_i$  is a maximal compact subgroup of  $G$ , the Frobenius formula gives the following expression for the character:

$$\begin{aligned}
 \Theta_\pi(f_H) &= \int_G \int_G f_H(ghg^{-1}) \chi_{i,0}(h) dh dg \\
 &= \int_G \int_H \chi_{i,0}(g^{-1}hg) dh dg \\
 (2.1) \quad &= \int_{G/G_i} \int_{G_i} \int_H \chi_{i,0}(k^{-1}g^{-1}h g k) dh dk dg \\
 &= m(G_i) \int_{G/G_i} \int_H \chi_{i,0}(g^{-1}hg) dh dg.
 \end{aligned}$$

Of course, the real work is to prove that these integrals converge. This is done for example in the proof of [BH96, Theorem A.14(ii)]. We will also see the convergence when we do the calculations by hand.

The next step is to rewrite the outside integral in equation (2.1) as a sum using the Cartan decomposition  $G = G_i A G_i$ , where  $A$  is the set of elements of the form  $a_\lambda := \text{diag}(\varpi^\lambda, \varpi^{-\lambda})$  with  $\lambda$  a non-negative integer. This is also done in [BH96, Theorem A.14(ii)], see equation (2.3) below. We will also take this formula one step further by collecting the terms corresponding to each value of the character of  $SL_2(\mathbb{F}_q)$  at unipotent elements.

Before we can carry out this plan, we need to introduce some more notation. Recall that there are three unipotent conjugacy classes in  $SL(2, \mathbb{F}_q)$ : the class  $U_0$  of the identity (one element), the class  $U_1$  of the element  $\begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix}$ , and the class  $U_\epsilon$  of the element  $\begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix}$ , where  $\epsilon$  is a non-square in  $\mathbb{F}_q$ . Suppose  $U$  is a unipotent conjugacy class in  $SL(2, \mathbb{F}_q)$ ,  $\lambda$  is a non-negative integer and  $i$  is a vertex in the Bruhat-Tits building for  $G$ . For any regular topologically unipotent element  $h$  of  $G$ , let  $N_{U,\lambda}^i(h)$  denote the number of right  $G_i$ -cosets  $gG_i$  inside the double coset  $G_i a_\lambda G_i$  that satisfy the following condition:

$$g^{-1}hg \in G_i \quad \wedge \quad \rho_{i,0}(g^{-1}hg) \in U.$$

**Proposition 2.1.** *Let  $\pi$  be a supercuspidal depth zero representation of  $G$  and let  $\Theta_\pi$  be its distribution character in the sense of Harish-Chandra. Let  $H$  be a compact set of regular topologically unipotent elements in  $G$  and let  $f_H$  be the characteristic function of  $H$ . Let  $\chi$  be the character of the representation of the finite group  $\bar{G}_i$*

that gave rise to  $\pi$  (see Section 1.1). Then

$$(2.2) \quad \Theta_\pi(f_H) = \sum_\lambda \sum_U \chi(U) \int_H N_{U,\lambda}^i(h) dh,$$

where  $U$  runs over unipotent conjugacy classes in  $SL(2, \mathbb{F}_q)$  and  $\lambda$  runs over non-negative integers. The sum over  $\lambda$  has only finitely many nonzero terms.

*Proof.* Consider a fixed double coset  $G_i a_\lambda G_i$ . We observe that if  $g_1, g_2 \in G_i$  and  $g_1 a_\lambda G_i = g_2 a_\lambda G_i$  then  $\chi_{i,0}(g_1^{-1} h g_1) = \chi_{i,0}(g_2^{-1} h g_2)$ . Hence, using Cartan decomposition, the double integral from the formula (2.1) can be rewritten as a double sum, and we obtain the formula (cf. [BH96, Theorem A.14(ii)]):

$$(2.3) \quad \Theta_\pi(f_H) = m(G_i) \sum_{a \in G_i \backslash G/G_i} \sum_{g \in G_i a G_i / G_i} \int_H \chi_{i,0}(g^{-1} h g) dh,$$

where the summation index in the outside sum runs over the set of representatives of the double cosets, that is, over  $A$ , and the summation index in the inner sum runs over the set of representatives of the left  $G_i$ -cosets inside the given double coset. We refer to [BH96, Theorem A.14(ii)] for the proof that there are only finitely many values of  $\lambda$  that give nonzero summands.

Now, consider the contribution of the double coset

$$G_i \begin{bmatrix} \varpi^\lambda & 0 \\ 0 & \varpi^{-\lambda} \end{bmatrix} G_i$$

to the sum in the right hand side of equation (2.3). The following two observations make the calculation of the character fairly simple. First, the function  $\chi_{i,0}$  vanishes outside the maximal compact subgroup  $G_i$ , while on  $G_i$ , the value  $\chi_{i,0}(g)$  depends only on  $\rho_{i,0}(g)$ . Second, the set  $H$  is contained in the set of topologically unipotent elements, so for every  $h$  in  $H$ , the element  $\rho_{i,0}(g^{-1} h g)$  is a unipotent element in  $\bar{G}_i$ , provided that  $g^{-1} h g$  lies in  $G_i$ . Using these observations it is possible to rewrite the formula for the character (equation (2.3)) in the following form:

$$(2.4) \quad \Theta_\pi(f_H) = \sum_\lambda \sum_U \chi(U) \int_H N_{U,\lambda}^i(h) dh.$$

□

In the next section, we will see that the numbers  $N_{U,\lambda}^i(h)$  give rise to geometric objects. Note that these numbers essentially depend only on the group; they depend on the representation only through the choice of the vertex  $i$  that indexes the maximal compact subgroup from which our representation is induced.

**2.2. Almost definable sets  $W_{U,\lambda}^i(h)$ .** In this section we begin the process of finding Chow motives related to the numbers  $N_{U,\lambda}^i(h)$  introduced in Section 2.1 by expressing these numbers as  $p$ -adic volumes of some sets  $W_{U,\lambda}^i(h)$ ; we will then use the comparison theorem of Denef and Loeser (see Section 1.7.3) to recover these  $p$ -adic volumes from the motivic volumes. But first we need to introduce yet more notation.

Suppose  $U$  is a unipotent conjugacy class in  $\bar{G}_i$ ,  $\lambda$  is a non-negative integer, and  $i$  is a vertex in the Bruhat-Tits building for  $G$ . For any regular topologically unipotent element  $h$  of  $G$ , define

$$(2.5) \quad W_{U,\lambda}^i(h) := \{y \in G_i \mid a_\lambda^{-1} y^{-1} h y a_\lambda \in G_i \wedge \rho_{i,0}((ya)^{-1} h (ya)) \in U\}.$$

**Lemma 2.2.** *With notation as above,*

$$(2.6) \quad N_{U,\lambda}^i(h) = (q+1)q^{2\lambda-1} \frac{\mathfrak{m}(W_{U,\lambda}^i(h))}{\mathfrak{m}(G_i)},$$

for all regular topologically unipotent  $h \in G$ .

*Proof.* Let  $a_\lambda$  denote the element  $\text{diag}(\varpi^\lambda, \varpi^{-\lambda})$ , as before. For each  $\lambda \in \mathbb{N} \cup \{0\}$ , we say that two elements  $y_1$  and  $y_2$  in  $G_i$  are  $\lambda$ -equivalent, if  $y_1 a_\lambda$  and  $y_2 a_\lambda$  are in the same left  $G_i$ -coset, that is, if  $a_\lambda^{-1} y_1^{-1} y_2 a_\lambda \in G_i$ . The  $\lambda$ -equivalence class of an element  $y$  is denoted by  $[y]_\lambda$ . Each set  $W_{U,\lambda}^i(h)$  is a finite disjoint union of  $\lambda$ -equivalence classes (see [G04]), and if it contains an element  $y$ , it contains the whole class  $[y]_\lambda$ . With this notation, the number  $N_{U,\lambda}^i(h)$  equals the number of  $\lambda$ -equivalence classes in the set  $W_{U,\lambda}^i(h)$ . Now, for  $\lambda > 0$ , the cardinality of  $G_i a_\lambda G_i / G_i$  is  $(q+1)q^{2\lambda-1}$ , as can be shown using the affine Bruhat decomposition for  $G$  (see [BT96], for example). Since all equivalence classes have equal volumes (see [G04, Lemma 4]), it follows that

$$(2.7) \quad N_{U,\lambda}^i(h) = (q+1)q^{2\lambda-1} \frac{\mathfrak{m}(W_{U,\lambda}^i(h))}{\mathfrak{m}(G_i)},$$

as claimed.  $\square$

The sets  $W_{U,\lambda}^i(h)$  are almost definable, but they depend on the parameter  $h$  that cannot be specified within the language. Still, it is possible to use motivic integration to calculate their volumes, by using the version of motivic integration that allows parameters.

**2.3. Motivic volumes of the sets  $W_{U,\lambda}^i(h)$ .** In this section, we write down the formulas defining the sets  $W_{U,\lambda}^i(h)$  of the previous section, and then calculate their motivic volumes. This is done case-by-case in  $z$ . We start with the most interesting case  $z = s'_1$ . We include all the details only for  $i = (0)$ . Before we start the calculation, two remarks are due.

First, recall that we had to fix a non-square unit  $\varepsilon$  and a uniformizer  $\varpi$ , as discussed in Section 1.8. In this section, we introduce a variable  $\delta$  that would be allowed to range over non-square units. As discussed in Section 1.7.1, a  $p$ -adic field together with the choice of the uniformizer is a structure for the language of Denef-Pas (*i.e.*, such a choice provides an interpretation of all the formulas). If we choose a non-square unit  $\varepsilon$  in the given field, carry out all the constructions that appeared so far, and then plug in the value of  $\varepsilon$  for  $\delta$  in our formulas, we will get the corresponding sets  $W_{U,\lambda}^i(h)$ . We will see, however, that their motivic volumes (and therefore, also  $p$ -adic volumes) are independent of the choice of  $\varepsilon$ .

Second, the comparison theorem relates the motivic volumes to  $p$ -adic volumes for all but finitely many primes  $p$ . There are two sources of “bad” primes in this statement: singularities of the varieties that come up, and quantifier elimination. Since we are doing all the calculations by hand, and our cases are very simple, we will see that the only prime that needs to be excluded is 2.

**2.3.1. The logical formulas for  $W_{U,\lambda}^{(0)}(h)$  for  $h \in \Gamma_{s'_1,n}$ .** Recall that all the elements of the set  $\Gamma_{s'_1,n}$  have the form

$$(2.8) \quad h = (1 + *) \begin{bmatrix} 1 + * & u\varpi^n \\ \varepsilon u\varpi^n & 1 + * \end{bmatrix},$$

where  $u \in \mathcal{O}_{\mathbb{K}}^*$ , and ‘\*’ stands for elements of order at least  $2n$ . We write  $y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{(0)}$  and use the entries  $a, b, c, d$  as free variables in our formulas. We begin by fixing  $U$  and  $h$  as above and finding explicit conditions on  $a, b, c, d$  and  $\lambda$  which ensure that  $y \in W_{U,\lambda}^{(0)}(h)$ . It can be verified directly that:

$$(2.9) \quad y^{-1}hy = (1 + *) \begin{bmatrix} 1 + \text{h.o.t} & u(d^2 - b^2\varepsilon)\varpi^n + \text{h.o.t} \\ u(a^2\varepsilon - c^2)\varpi^n + \text{h.o.t} & 1 + \text{h.o.t} \end{bmatrix},$$

where h.o.t means “higher order terms”, which refers to the terms with valuation greater than that of the leading term.

As we see from equation 2.5 we must now conjugate this element by  $a_\lambda$ . Conjugation by  $a_\lambda$  induces multiplication of the entry in the upper right-hand corner by  $\varpi^{-2\lambda}$ , and multiplication of the entry in the lower left-hand corner by  $\varpi^{2\lambda}$ . It follows that  $a_\lambda^{-1}y^{-1}hya_\lambda \in G_{(0)}$  if and only if

$$(2.10) \quad -2\lambda + \text{ord}(u(d^2 - b^2\varepsilon)\varpi^n) \geq 0 \quad \wedge \quad 2\lambda + \text{ord}(u(a^2\varepsilon - c^2)\varpi^n) \geq 0.$$

Consequently, we have the following list of cases:

**Case 1.**  $2\lambda < n$ . In this case, for any  $y \in G_{(0)}$  the element  $a_\lambda^{-1}y^{-1}hya_\lambda$  lies in  $G_{(0)}$  and projects to the identity under the reduction mod  $\varpi$ .

**Case 2.**  $2\lambda > n$ . We observe that  $\text{ord}(d^2 - b^2\varepsilon) = 0$ : indeed, either both  $a$  and  $d$  or  $b$  and  $c$  have to be units because  $ad - bc = 1$ , and  $\rho_{(0),0}(d^2 - b^2\varepsilon) \neq 0$  since  $\varepsilon$  is a non-square. Similarly,  $\text{ord}(a^2\varepsilon - c^2) = 0$ . This implies that if  $2\lambda > n$ , the set of  $ys$  satisfying the conditions is empty.

**Case 3.**  $2\lambda = n$ . (This case corresponds to the interesting situations.) In this case the image of  $a_\lambda^{-1}y^{-1}hya_\lambda$  under  $\rho_{(0),0}$  is an element of the form  $\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ , where  $\beta = \text{ac}(d^2 - b^2\varepsilon)\text{ac}(u) \in \mathbb{F}_q$ . Then for  $h \in \Gamma_{s'_1, n, +}$ ,  $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda)$  falls into  $U_1$  if  $\text{ac}(d^2 - b^2\varepsilon)$  is a square and into  $U_\varepsilon$  otherwise.

Notice that when  $\lambda$  is fixed and  $h$  is fixed, the condition that  $y \in W_{U,\lambda}^{(0)}(h)$  does not depend on  $u$ , as long as  $h$  is confined to one of the sets  $\Gamma_{s'_1, n, +}$  and  $\Gamma_{s'_1, n, -}$ .

We now reformulate what we have found concerning the character in terms of formulas in Pas’s language. We start by observing that the maximal compact subgroup  $G_{(0)}$  is defined by the formula

$$'ad - bc = 1 \quad \wedge \quad \text{ord}(a) \geq 0 \quad \wedge \quad \text{ord}(b) \geq 0 \quad \wedge \quad \text{ord}(c) \geq 0 \quad \wedge \quad \text{ord}(d) \geq 0.'$$

For the cases  $2\lambda < n$  or  $2\lambda > n$ , no other calculations are needed. Indeed, if  $2\lambda < n$ , then we are within Case 1, and therefore both sets  $W_{U_1, \varepsilon, \lambda}^{(0)}(h)$  are empty, and  $W_{U_0, \lambda}^{(0)}(h) = G_{(0)}$ . If  $2\lambda > n$ , then we are within Case 2, and the set of  $ys$  satisfying the conditions is also empty.

Let us now consider the case  $n$  even, and  $2\lambda = n$ . Let  $\psi(b, d, \delta)$  be the formula (in Pas’s language)

$$\psi(b, d, \delta) = '\exists x(d^2 - b^2\delta = x^2).'$$

Since we will often pass back and forth between the valued field and the finite field, let us introduce an abbreviation for the “reduction mod  $\varpi$ ” map: let

$$\bar{x} = \begin{cases} \text{ac}(x), & \text{ord}(x) = 0 \\ 0, & \text{ord}(x) > 0. \end{cases}$$

If we know that  $\delta$  is a non-square unit, then by Hensel’s Lemma, to check whether the triple  $(b, d, \delta)$  with  $(\bar{b}, \bar{d}) \neq (0, 0)$  satisfies the formula  $\psi$ , it is enough to know whether its reduction mod  $(\varpi)$  satisfies the same formula (where now the variables

are interpreted as residue-field variables). Hensel's Lemma is applicable because  $\bar{d}^2 - \bar{b}^2\bar{d}$  cannot be 0 when we assume that  $\bar{d}$  is a non-square and  $\bar{b}, \bar{d}$  are not simultaneously zero.

Let us consider the family of formulas depending on a parameter  $\eta$  that ranges over the set of non-squares in  $\mathbb{F}_q$  (note that this is a definable set):

$$(2.11) \quad \phi_\eta(a, b, c, d) = 'ad - bc = 1 \quad \wedge \exists \xi(\bar{b}^2 - \bar{d}^2\eta = \xi^2)'$$

For every value of  $\eta \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$ , if we let all the variables range over  $\mathcal{O}_{\mathbb{K}}$ , the formula  $\phi_\eta(a, b, c, d)$  defines the set  $W_{U_1, \lambda}^{(0)}(h)$  for any  $h \in \Gamma_{s'_1, n, +}$  if the unit  $\varepsilon$  that was fixed in order to define the sets  $\Gamma_{s'_1, n, \pm}$  has the property  $\bar{\varepsilon} = \eta$ . The same formula also defines the set  $W_{U_\varepsilon, \lambda}^{(0)}(h)$  with  $h \in \Gamma_{s'_1, n, -}$ . In the next subsection we find the motivic volumes of these sets.

**2.3.2. The motivic volumes of  $W_{U, \lambda}^{(0)}(h)$ ,  $h \in \Gamma_{s'_1, n, \pm}$ .** The calculation of these volumes is very simple. Recall that our Haar measure on  $G$  is normalized in such a way that the fibres of the projection  $G \rightarrow G(\mathbb{F}_q)$  have volume 1. The formula  $\phi_\eta$  imposes a condition on the variables  $b, d$  that apparently depends only on  $\bar{b}, \bar{d}$ . This means that either a whole fibre over a point  $(\bar{a}, \bar{b}, \bar{c}, \bar{d}) \in SL(2, \mathbb{F}_q)$  satisfies this condition, or the whole fibre does not satisfy it. Hence, to calculate the volume of the set  $W_{U_1, \lambda}^{(0)}(h)$ , all we need to do is count the number of points in  $[\frac{\bar{a}}{\bar{c}} \ \frac{\bar{b}}{\bar{d}}] \in SL(2, \mathbb{F}_q)$  that satisfy the condition  $\exists \xi : \bar{b}^2 - \bar{d}^2\eta = \xi^2$ , where  $\eta$  is a parameter that is a quadratic residue in  $\mathbb{F}_q$ . This calculation is carried out carefully in the Lemma below.

The next lemma is the only place in the present paper where interesting geometric objects come up from  $p$ -adic volumes. Given the argument above, it is not surprising that the  $p$ -adic volumes of the sets  $W_{U, \lambda}^{(0)}(h)$  with the ‘‘border-line’’ value of  $\lambda = n/2$  (for  $n$  even) are connected with the number of points of a conic over the finite field.

**Lemma 2.3.** *Suppose  $h \in \Gamma_{s'_1, n}$ . If  $n$  is even and  $\lambda = n/2$  then the motivic volumes of the sets  $W_{U_1, \lambda}^{(0)}(h)$  and  $W_{U_\varepsilon, \lambda}^{(0)}(h)$  both equal  $\frac{1}{2}\mathbb{L}(\mathbb{L} - 1)(\mathbb{L} + 1)$ , which is half of the motivic volume of the maximal compact subgroup  $G_{(0)}$ ; otherwise, the motivic volumes of the sets  $W_{U_1, \lambda}^{(0)}(h)$  and  $W_{U_\varepsilon, \lambda}^{(0)}(h)$  both equal 0.*

*Proof.* Since for any  $h \in \Gamma_{s'_1, n, \pm}$ ,

$$W_{U_1, \lambda}^{(0)}(h) \cup W_{U_\varepsilon, \lambda}^{(0)}(h) = G_{(0)},$$

it is enough to find the (motivic) volume of any one of these sets. Assume for simplicity that  $h \in \Gamma_{s'_1, n, +}$ . As shown in the previous subsection, we now have to find the (motivic) volumes of the sets defined by the formulas  $\phi_\eta(a, b, c, d)$ . It seems that the simplest way to do the ‘‘motivic’’ calculation is to consider the total volume of the disjoint union of all the subassignments defined by the formulas  $\phi_\eta$  over all the values of  $\eta$  in  $\mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$ , and then prove that in fact all these fibres have the same volume.

Consider the formula

$$(2.12) \quad \begin{aligned} &\Phi(a, b, c, d, \eta) \\ &= 'ad - bc = 1 \quad \wedge \exists \xi \neq 0(\bar{d}^2 - \bar{b}^2\eta = \xi^2) \quad \wedge \nexists \beta(\eta = \beta^2)'. \end{aligned}$$

In this formula, four variables  $a, b, c, d$  range over the valued field, the variable  $\eta$  ranges over the residue field, and all the quantifiers range over the residue field.

It defines a subassignment of  $h[4, 1, 0]$  (see Section 1.7.4 for this notation), which corresponds to the disjoint union of the subassignments of  $h[4, 0, 0]$  defined by the formulas  $\phi_\eta$  over all non-squares  $\eta$ .

*Step 1. Reduction to the residue field.* The formula  $\Phi$  can be broken up into two parts according to whether  $b$  is a unit:  $\Phi = (\Phi \wedge (\text{ord}(b) = 0)) \vee (\Phi \wedge (\text{ord}(b) > 0))$ .

We start by showing that the subassignment defined by  $\Phi \wedge (\text{ord}(b) = 0)$  is stable at level 0, *i.e.*, that it is essentially “inflated” from the finite field. In order to do this, we need to introduce an abstract variety  $V$  that plays the role of the “projection” of this formula to the residue field.

Let  $k$  be an arbitrary field of characteristic zero (the theory of arithmetic motivic integration tells us that we should think of  $k$  as a pseudofinite field). Consider the subvariety  $V$  of  $\mathbb{A}^4$  over  $k$  cut out by the equation  $x_1^2 - x_2^2 x_3 = x_4^2$ . It has no singularities outside the hyperplane  $x_2 = 0$  (note that this statement is true in any characteristic greater than 2). Recall the notation  $[\phi]$  for the motivic volume of a formula  $\phi$  in the language of rings. Let

$$(2.13) \quad \mathbb{M}_1 := [(x_1^2 - x_2^2 x_3 = x_4^2) \wedge (x_2 \neq 0) \wedge (x_3 \neq 0) \wedge (x_4 \neq 0) \wedge (\nexists \beta (x_3 = \beta^2))].$$

Consider the formula

$$\begin{aligned} \Phi_1(b, d, \eta, \xi) := \\ \cdot (\bar{d}^2 - \bar{b}^2 \eta = \xi^2) \wedge (\xi \neq 0) \wedge (\text{ord}(b) = 0) \wedge (\eta \neq 0) \wedge (\nexists \beta (\eta = \beta^2)) \end{aligned}$$

Set  $x_1 = \bar{d}$ ,  $x_2 = \bar{b}$ ,  $x_3 = \eta$ ,  $x_4 = \xi$ . This “reduction” takes the formula  $\Phi_1$  exactly to the ring formula that appears in the right-hand side of (2.13). Since it is mutually exclusive with the formula  $x_2 = 0$  that defines a set containing the singular locus of  $V$ , the subassignment of  $h[2, 2, 0]$  defined by the formula  $\Phi_1$  is stable at level 0, and its motivic volume equals  $\mathbb{M}_1$ .

Let  $\Phi_2(b, d, \eta)$  be the formula

$$\cdot \exists \xi (\bar{d}^2 - \bar{b}^2 \eta = \xi^2) \wedge (\xi \neq 0) \wedge (\text{ord}(b) = 0) \wedge (\eta \neq 0) \wedge (\nexists \beta (\eta = \beta^2)).$$

The formula  $\Phi_1$  is a double cover of  $\Phi_2$ , so  $\mu(\Phi_2) = \frac{1}{2}\mu(\Phi_1) = \frac{1}{2}\mathbb{M}_1$ .

Finally, consider the projection  $(d, b, c, a, \eta) \rightarrow (d, b, \eta)$ . The subassignment defined by  $\Phi \wedge (\text{ord}(b) = 0)$  projects to the subassignment defined by  $\Phi_2$ , and the volume of each fibre of this projection is  $\mathbb{L}$ : indeed, given that  $b$  is a unit, for every value of  $a$ , there is unique  $c$  such that  $ad - bc = 1$ . Hence, we have  $\mu(\Phi \wedge (\text{ord}(b) = 0)) = \frac{1}{2}\mathbb{L}\mathbb{M}_1$ .

*Step 2. Independence of the parameter  $\eta$ .* This step consists in the observation that for each  $\eta_1, \eta_2 \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$ , there is a definable bijection between the triples  $(x_1, x_2, x_4)$  and  $(x'_1, x'_2, x'_4)$  such that  $x_1^2 - x_2^2 \eta_1 = x_4^2$  and  $x'_1{}^2 - x'_2{}^2 \eta_2 = x'_4{}^2$ : the bijection is defined by the formula

$$\begin{aligned} \Psi(x_1, x_2, x_4, x'_1, x'_2, x'_4, \eta_1, \eta_2) \\ = \cdot \exists y (\eta_1 = \eta_2 y^2) \wedge (x'_1 = x_1) \wedge (x'_2 = x_2 y) \wedge (x'_4 = x_4) \wedge \nexists \beta (\eta_1 = \beta^2). \end{aligned}$$

The Corollary [CL05, 14.2.2] together with Remark [CL05, 14.2.3] implies that if the motivic volume is constant on the fibres, then the total volume is the volume of the fibre times the class of the base. It follows that for each  $\eta \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^{\times 2}$ , we have

$$\mu(\phi_\eta \wedge (\text{ord}(b) = 0)) = \frac{2}{\mathbb{L} - 1} \mu(\Phi \wedge (\text{ord}(b) = 0)).$$

*Step 3. A residue-field calculation: finding  $\mathbb{M}_1$ .* Note that it is in this step that we see the conic promised in the introduction.

We start by considering abstract varieties again. Recall the variety  $V$  defined in Step 1. Let us denote the coordinates on  $\mathbb{A}^3$  by  $(t, s, e)$ , and consider the subvariety  $V_2$  of  $\mathbb{A}^3$  defined by the equation  $t^2 - s^2 = e$ . Consider the birational map  $(x_1, x_2, x_3, x_4) \mapsto (x_2, \frac{x_1}{x_2}, x_3, \frac{x_4}{x_2})$  from the variety  $V$  to the variety  $V_2 \times \mathbb{A}^1$ . It is an isomorphism between the open sets  $x_2 x_3 \neq 0$  in  $V$  and  $e \neq 0$  in  $V_2 \times \mathbb{A}^1$ . Then the class  $\mathbb{M}_1$  equals  $[\nexists\beta \neq 0 : t^2 - s^2 = \beta^2](\mathbb{L} - 1)$ . It remains to calculate the class  $[\nexists\beta \neq 0 : t^2 - s^2 = \beta^2]$ .

The class  $\mathbb{L}^2$  of the  $(t, s)$ -plane breaks up into the sum of the three classes:

$$\mathbb{L}^2 = [\nexists\beta(t^2 - s^2 = \beta^2)] + [\exists\beta(t^2 - s^2 = \beta^2) \wedge \beta \neq 0] + [t^2 - s^2 = 0].$$

It is easy to see that  $[t^2 - s^2 = 0] = 2(\mathbb{L} - 1) + 1$ . We also have  $[\exists\beta(t^2 - s^2 = \beta^2) \wedge \beta \neq 0] = \frac{1}{2}(\mathbb{L} - 1)[x^2 - y^2 = 1] = \frac{1}{2}(\mathbb{L} - 1)(\mathbb{L} - 1)$ . Therefore,  $[\nexists\beta(t^2 - s^2 = \beta^2)] = \frac{1}{2}\mathbb{L}^2 - \mathbb{L} + \frac{1}{2}$ . Hence,  $\mathbb{M}_1 = \frac{1}{2}(\mathbb{L} - 1)^3$ .

Finally, we have:

$$(2.14) \quad \begin{aligned} \mu(\phi_\eta \wedge \text{ord}(b) = 0) &= \frac{2}{\mathbb{L} - 1} \mu(\Phi \wedge (\text{ord}(b) = 0)) \\ &= \frac{2}{\mathbb{L} - 1} \frac{1}{4} (\mathbb{L} - 1)^3 \mathbb{L} = \frac{1}{2} \mathbb{L} (\mathbb{L} - 1)^2. \end{aligned}$$

*Step 4. Completing the proof.* It is easy to calculate the motivic volume of the remaining part  $\phi_\eta \wedge (\text{ord}(b) > 0)$ . If  $\text{ord}(b) > 0$ , then the formula  $\psi(\bar{d}, \bar{b}, \eta)$  becomes  $[\exists\beta \neq 0(\bar{d}^2 = \beta^2)]$ . Clearly, its motivic volume is  $(\mathbb{L} - 1)$ . It remains to notice that if  $\text{ord}(b) > 0$ , then the variable  $c$  contributes the factor  $\mathbb{L}$ , both  $(a, d)$  have to be units, and once  $d$  is chosen,  $a$  is determined uniquely by the determinant condition. Altogether, we get  $\mu(\phi_\eta \wedge (\text{ord}(b) > 0)) = \mathbb{L}(\mathbb{L} - 1)$ . Finally, we get:

$$\begin{aligned} \mu(W_{U_\varepsilon, n/2}^{(0)}(h)) &= \mu(\phi_\eta) = \frac{1}{2} \mathbb{L} (\mathbb{L} - 1)^2 + \mathbb{L} (\mathbb{L} - 1) \\ &= \frac{1}{2} \mathbb{L} (\mathbb{L} - 1) (\mathbb{L} + 1), \end{aligned}$$

which completes the proof of the Lemma.  $\square$

**2.3.3. The motivic volumes of  $W_{U, \lambda}^{(0)}(h)$  for  $h \in \Gamma_{s'_2, n}$ .** In the case  $z = s'_2$  the answer is essentially the same as in the case  $z = s'_1$ , but the calculation is slightly more complicated. Here we sketch the calculation of the motivic volume of the sets  $W_{U, \lambda}^{(0)}(h)$  for  $h \in \Gamma_{s'_2, n, \pm}$ , indicating the differences with the case  $z = s'_1$ .

Exactly as before, we consider the element  $a_\lambda^{-1} y^{-1} \gamma_{s'_2, n} y a_\lambda$ , and use the entries of the matrix  $y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{(0)}$  as free variables in our logical formulas.

The conditions (2.10) are now replaced with

$$(2.15) \quad \begin{aligned} -2\lambda + \text{ord}(d^2 u \varpi^{n-1} - b^2 u \varepsilon \varpi^{n+1}) &\geq 0 \\ 2\lambda + \text{ord}(a^2 u \varepsilon \varpi^n + 1 - c^2 u \varpi^{n-1}) &\geq 0. \end{aligned}$$

The second condition is satisfied automatically if  $\lambda \geq 0$ , so we only need to focus on the first one. Similarly to the case  $z = s'_1$ , this condition implies that when  $\lambda > \frac{n+1}{2}$ , the element  $a_\lambda^{-1} y^{-1} h y a_\lambda$  is outside  $G_{(0)}$ ; if  $\lambda < \frac{n-1}{2}$ , this element is in  $G_{(0)}$  and projects to  $U_0$ .

Suppose for now that  $n$  is odd. Unlike the case  $z = s'_1$ , now there are two interesting cases:  $\lambda = \frac{n-1}{2}$  and  $\lambda = \frac{n+1}{2}$ . If  $\lambda = \frac{n-1}{2}$ , and  $\text{ord}(d) = 0$ , then  $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_1$  if  $\text{sgn}(u) = 1$  and  $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_\varepsilon$  if  $\text{sgn}(u) = -1$ . If  $\text{ord}(d) > 0$ , then  $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_0$ . If  $\lambda = \frac{n+1}{2}$ , then  $a_\lambda^{-1}y^{-1}hya_\lambda$  is in  $G_{(0)}$  only if  $\text{ord}(d) > 0$ . In this case, the projection of the element  $a_\lambda^{-1}y^{-1}hya_\lambda$  depends on the sign of  $\text{ac}(d^2 - b^2\varepsilon)$  and on  $\text{sgn}(u)$ , which is similar to the case  $z = s'_1$ . The only difference is that here there is an additional condition  $\text{ord}(b) = 0$  (if  $\text{ord}(d) > 0$  then the determinant condition forces  $\text{ord}(b) = 0$ ). By Lemma 2.3 (see equation (2.14)), the motivic volume of the subset of  $G$  defined by the formula ' $\exists \xi \neq 0(\text{ac}(d^2 - b^2\varepsilon) = \xi^2) \wedge (\text{ord}(b) = 0) \wedge (\text{ord}(d) > 0)$ ' equals  $\frac{1}{2}(\mathbb{L} - 1)^2$ . The motivic volume of its complement, *i.e.*, the set defined by the formula ' $\nexists \xi \neq 0(\text{ac}(d^2 - b^2\varepsilon) = \xi^2) \wedge (\text{ord}(b) = 0) \wedge (\text{ord}(d) > 0)$ ' equals  $\mathbb{L}(\mathbb{L}^2 - 1) - \frac{1}{2}(\mathbb{L} - 1)^2 = \frac{1}{2}\mathbb{L}^2 - \frac{1}{2}$ . The last two sentences are of course an abbreviation. In truth, we have to replace  $\varepsilon$  with a variable  $\delta$  and do everything exactly the same way as it was done in the previous case. Since this is already part of Lemma 2.3, we do not repeat this argument here.

For simplicity, suppose that  $\text{sgn}(u) = 1$ . Putting all these calculations together, we get:

$$(2.16) \quad \begin{aligned} \mu(W_{\frac{n+1}{2}, U_\varepsilon}) &= \frac{1}{2}\mathbb{L}^2 - \frac{1}{2}; \\ \mu(W_{\frac{n-1}{2}, U_\varepsilon}) &= 0; \\ \mu(W_{\frac{n-1}{2}, U_1}) &= \mathbb{L}^2(\mathbb{L} - 1); \\ \mu(W_{\frac{n+1}{2}, U_1}) &= \frac{1}{2}(\mathbb{L} - 1)^2. \end{aligned}$$

The case  $n$  even, and the calculation of  $\mu(W_{\lambda, U_0})^{(0)}(h)$  are similar to the case  $z = s'_1$ .

#### 2.3.4. The ramified cases.

**Lemma 2.4.** *Let  $i$  be a vertex in the Bruhat-Tits building for  $G$ . Suppose  $h \in \Gamma_{z,n,\pm}$ , with  $z \in \{t'_0, t'_1, t'_2, t'_3\}$ . Then the sets  $W_{U,\lambda}^i(h)$  are definable for all non-negative integers  $\lambda$ , their motivic volumes are independent of  $h$  and the choice of  $\varepsilon$ , and can be explicitly computed.*

*Proof.* We prove this lemma only for  $i = (0)$ . The proof in the case  $i = (1)$  is very similar; the results of these calculations become part of the expressions for  $M_{z,n,\pm}^{(1),0}$ ,  $M_{z,n,\pm}^{(1),1}$  and  $M_{z,n,\pm}^{(1),\varepsilon}$  appearing in Tables 8 and 9. So everywhere in this proof  $i = (0)$ , and we drop the superscript  $i$  from the notation  $W_{U,\lambda}^i(h)$ .

The argument is very similar to the unramified case; the only difference is that the actual calculation of the motivic volumes of the corresponding sets  $W_{U,\lambda}(h)$  is simpler. Here we carry out the proof for the case  $z = t'_2$ . The other three ramified cases are almost identical to it. First, as in the previous subsection, we write the elements  $h$  of the set  $\Gamma_{t'_2,n}$  explicitly as:

$$(2.17) \quad h = (1 + *) \begin{bmatrix} 1 + * & u\varpi^n \\ \varepsilon u\varpi^{n+1} & 1 + * \end{bmatrix}, \text{ where } u \in \mathcal{O}_{\mathbb{K}}^*,$$

and  $*$  denotes the terms of order at least  $2n$ , as before.

As in the proof of the previous lemma, we let  $y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  be a variable running over  $G_{(0)}$  (so that the symbols for its entries  $a, b, c, d$  will become the formal variables in the Pas's language formulas defining the sets  $W_{U,\lambda}^{(0)}(h)$ ).

As before, we compute  $y^{-1}hy$ :

$$(2.18) \quad y^{-1}hy = (1 + *) \begin{bmatrix} 1 + \text{h.o.t} & d^2u\varpi^n - b^2\varepsilon u\varpi^{n+1} + \text{h.o.t} \\ a^2u\varepsilon\varpi^{n+1} - c^2u\varpi^n + \text{h.o.t} & 1 + \text{h.o.t} \end{bmatrix},$$

This leads to the following conditions on  $y, \lambda$  for the element  $a_\lambda^{-1}y^{-1}hya_\lambda$  to be in  $G_{(0)}$ :

$$(2.19) \quad \begin{aligned} -2\lambda + \text{ord}(ud^2\varpi^n - b^2\varepsilon u\varpi^{n+1}) &\geq 0 \text{ and} \\ 2\lambda + \text{ord}(ua^2\varepsilon\varpi^{n+1} - c^2u\varpi^n) &\geq 0. \end{aligned}$$

As before, it is convenient to consider the cases  $n$  even and  $n$  odd separately.

Suppose  $n$  is even. Looking at the left-hand side of the inequalities (2.19), we see that if  $\lambda < n/2$ , then the set  $W_{U_0,\lambda}(h)$  coincides with the whole of  $G_{(0)}$ , and the sets  $W_{U_1,\lambda}(h)$ ,  $W_{U_\varepsilon,\lambda}(h)$  are empty.

If  $\lambda > n/2$ , then, since  $n$  is even and  $\lambda$  is an integer,  $2\lambda$  is at least  $n + 2$ , which forces the element  $a_\lambda^{-1}y^{-1}hya_\lambda$  outside  $G_{(0)}$ , and all three sets are empty.

Finally, in the case  $\lambda = n/2$ , the outcome depends on the entry  $d$ : if  $\text{ord}(d) = 0$ , then the term  $d^2u\varpi^n$  in the expression  $d^2u\varpi^n - b^2\varepsilon u\varpi^{n+1}$  dominates, and therefore  $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_1$  if  $\text{sgn}(u) = 1$ , and  $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_\varepsilon$  if  $\text{sgn}(u) = -1$ . If  $\text{ord}(d) > 0$ , then  $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_0$ .

Hence, for  $h \in \Gamma_{t'_2,n,\pm}$ , the set  $W_{U_0,n/2}(h)$  is defined by the formula ' $\text{ord}(d) > 0$ ' in conjunction with the formulas defining  $G_{(0)}$ . Note that the volume of this set is the same as that of  $G_{(01)}$ , *i.e.*, equals  $\mathbb{L}(\mathbb{L} - 1)$ . The sets  $W_{U_1,n/2}(h)$ ,  $W_{U_\varepsilon,n/2}(h)$  are defined by the formula ' $\text{ord}(d) = 0$ ' in conjunction with the formulas defining  $G_{(0)}$  for  $h \in \Gamma_{t'_2,n,+}$  and  $h \in \Gamma_{t'_2,n,-}$ , respectively, and are respectively empty for  $h \in \Gamma_{t'_2,n,-}$  and  $h \in \Gamma_{t'_2,n,+}$ .

The case  $n$  odd is very similar. If  $\lambda = \frac{n+1}{2}$  (the most interesting case), and  $\text{ord}(d) = 0$ , the second one of the conditions (2.19) is not satisfied, so all three sets  $W_{U,\frac{n+1}{2}}(h)$  are empty. If  $\text{ord}(d) > 0$ , then automatically  $\text{ord}(b) = 0$ , the leading term in the expression  $d^2u\varpi^n - b^2\varepsilon u\varpi^{n+1}$  is  $-b^2u\varepsilon\varpi^{n+1}$ , which has sign opposite to  $\text{sgn}(-1)\text{sgn}(u)$ .

We get: if  $\lambda \leq (n-1)/2$  then

$$W_{U_0,\lambda} = G_{(0)}, \quad \text{and} \quad W_{U_1,\lambda}(h) = W_{U_\varepsilon,\lambda}(h) = \emptyset, \quad \text{for any } h \in \Gamma_{t'_2,n};$$

if  $\lambda > (n+1)/2$  then all three sets are empty; if  $\lambda = (n+1)/2$ , we have

$$(2.20) \quad \begin{aligned} W_{U_1,\frac{n+1}{2}}(h) &= G_{(0)} \cap \{\text{ord}(d) > 0\}, & W_{U_\varepsilon,\frac{n+1}{2}}(h) &= \emptyset, & h &\in \Gamma_{t'_2,n,\text{sgn}(-1)}; \\ W_{U_\varepsilon,\frac{n+1}{2}}(h) &= G_{(0)} \cap \{\text{ord}(d) > 0\}, & W_{U_1,\frac{n+1}{2}}(h) &= \emptyset, & h &\in \Gamma_{t'_2,n,-\text{sgn}(-1)}. \end{aligned}$$

It follows that  $\mu(W_{U_1,\frac{n+1}{2}}(h))$  equals 0 or equals  $\mu(G_{(01)}) = \mathbb{L}(\mathbb{L} - 1)$  depending on whether  $h \in \Gamma_{t'_2,n,\text{sgn}(-1)}$  or  $h \in \Gamma_{t'_2,n,-\text{sgn}(-1)}$ ; the same is true for  $\mu(W_{U_\varepsilon,\frac{n+1}{2}}(h))$ .

As we see from this proof, a different choice of  $\varepsilon$  could not have affected the motivic volumes of these sets; also, clearly no "bad" primes were acquired.  $\square$

2.3.5. *The case  $i = (1)$ .* In all the proofs in the present section we have been assuming that  $i = (0)$ . The only major difference of the case  $i = (1)$  is that the element  $y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  now belongs to  $G_{(1)}$ , not  $G_{(0)}$ . The requirement is that the element  $a_\lambda^{-1}y^{-1}hya_\lambda$  belongs to  $G_{(1)}$ , and the reduction map applied to this element is  $\rho_{(1),0}$  instead of  $\rho_{(0),0}$ . Therefore all the cases look slightly different, but no new varieties appear in the calculations of the motivic volumes of the new sets.

#### 2.4. Motives corresponding to the distribution characters.

**Proposition 2.5.** *The Harish-Chandra character of each depth zero supercuspidal representation is constant on each set  ${}^G\Gamma_{z,n,\pm}$ . Moreover, there exist virtual motives  $M_{z,n,\nu}^{i,0}$ ,  $M_{z,n,\nu}^{i,1}$  and  $M_{z,n,\nu}^{i,\varepsilon}$  (where  $z$  is any cocycle defined in Section 1.8,  $n$  is a positive integer for  $z \in \{s'_0, s'_1, s'_2\}$  and a non-negative integer for  $z \in \{t'_0, t'_1, t'_2, t'_3\}$ , the sign  $\nu$  is  $\pm$  and  $i$  is a vertex  $(0)$  or  $(1)$ ) such that*

$$(2.21) \quad \frac{1}{m({}^G\Gamma_{z,n,\nu})} \Theta_\pi(f_{z,n,\nu}) \\ = \chi|_{U_0} \text{TrFrob} M_{z,n,\nu}^{i,0} + \chi|_{U_1} \text{TrFrob} M_{z,n,\nu}^{i,1} + \chi|_{U_\varepsilon} \text{TrFrob} M_{z,n,\nu}^{i,\varepsilon}.$$

The virtual motives  $M_{z,n,\nu}^{i,0}$ ,  $M_{z,n,\nu}^{i,1}$  and  $M_{z,n,\nu}^{i,\varepsilon}$  are explicitly given in Tables 6 and 7 in the case  $i = (0)$ , and in Tables 8 and 9 in the case  $i = (1)$ .

*Proof.* We will show the details of the proof of this proposition in the case  $i = (0)$  only. The results of the similar calculations in the case  $i = (1)$  are summarized in the tables.

Let us apply Proposition 2.1 to the test functions  $f_{z,n,\nu}$  (so that the set  $H$  in that Proposition is  ${}^G\Gamma_{z,n,\nu}$ ). Note that in  $G$ , for all  $z$  but  $s'_0$ ,  $\mathfrak{h}_z$  is elliptic, *i.e.*, it is a compact Cartan subalgebra. We consider the elliptic cases first. As we will explicitly see below, the sum over  $\lambda$  that appears in the expression (2.2) has only finitely many terms in these cases, so we can permute the two sums, and obtain

$$(2.22) \quad \Theta_\pi(f_{z,n,\nu}) = \sum_U \chi(U) \sum_\lambda \int_{{}^G\Gamma_{z,n,\nu}} N_{U,\lambda}^i(h) dh,$$

where the index  $U$  runs over  $U_0$ ,  $U_1$ , and  $U_\varepsilon$ . That is, the character is already expressed as a linear combination of the values  $\chi|_{U_0}$ ,  $\chi|_{U_1}$ , and  $\chi|_{U_\varepsilon}$ . All we need to do, is “evaluate” the coefficients  $\sum_\lambda \int_{{}^G\Gamma_{z,n,\nu}} N_{U,\lambda}^i(h)$ . We recall Lemma 2.2 which relates the numbers  $N_{U,\lambda}^i(h)$  to the volumes of the sets  $W_{U,\lambda}^i(h)$ . Then we evaluate their *motivic* volumes for  $h$  in each of the sets  ${}^G\Gamma_{z,n,\nu}$  and sum them over  $\lambda$  with coefficients that come from Lemma 2.2.

We start with  $z = s'_1$ . Let  $h$  be an element of  $\Gamma_{s'_1,n,\pm}$ , and let  $\lambda$  be a non-negative integer. By the comparison theorem, the equality (2.7) of Lemma 2.2 can be written in a “motivic” form: for all primes  $p \neq 2$  (recall that  $q$  is a power of  $p$ ), we have  $N_{U,\lambda}^i(h) = \text{TrFrob}_q M_{U,\lambda}^i$ , where

$$M_{U,\lambda}^i = [(q+1)q^{2\lambda-1}] \frac{\mu(W_{U,\lambda}^i(h))}{\mu(G_i)} = (\mathbb{L}+1)\mathbb{L}^{2\lambda-1} \frac{\mu(W_{U,\lambda}^i(h))}{\mu(G_i)}.$$

For now, let  $i = (0)$ . Note that, *a priori*, the right hand side depends on the element  $h$ . However, we have shown in Lemma 2.3 that for all  $h \in \Gamma_{s'_1,n,\pm}$ , and for every

$U$ , the motivic volume of  $W_{U,\lambda}^i(h)$  does not depend on  $h$  and equals:

$$\begin{aligned} & \frac{1}{2}\mathbb{L}(\mathbb{L}^2 - 1), & \text{if } \lambda = n/2, n \text{ even, } U = U_1 \text{ or } U_\varepsilon, \\ & \mathbb{L}(\mathbb{L}^2 - 1), & \text{if } \lambda < n/2, U = U_0 \\ & 0, & \text{otherwise.} \end{aligned}$$

Let us define the virtual Chow motive  $M_{s'_1, n, \pm}^{(0), 0}$  that corresponds to the conjugacy class  $U_0$  by the formulas

$$(2.23) \quad \begin{aligned} M_{s'_1, n, \pm}^{(0), 0} &= 1 + (\mathbb{L} + 1) \sum_{\lambda=1}^{n/2-1} \mathbb{L}^{2\lambda-1} = \frac{\mathbb{L}^{n-1} - 1}{\mathbb{L} - 1} & \text{if } n \text{ is even,} \\ M_{s'_1, n, \pm}^{(0), 0} &= 1 + (\mathbb{L} + 1) \sum_{\lambda=1}^{(n-1)/2} \mathbb{L}^{2\lambda-1} = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} & \text{if } n \text{ is odd.} \end{aligned}$$

Also let

$$(2.24) \quad M_{s'_1, n, \pm}^{(0), 1} = M_{s'_1, n, \pm}^{(0), \varepsilon} = \begin{cases} \frac{1}{2}(\mathbb{L} + 1)\mathbb{L}^{n-1}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Then, combining the equations above, we obtain, for  $p \neq 2$ ,

$$(2.25) \quad \begin{aligned} & \frac{\Theta_\pi(f_{s'_1, n, \nu})}{m(G\Gamma_{s'_1, n, \nu})} = \Theta_\pi(h) \\ & = \chi|_{U_0} \text{TrFrob } M_{s'_1, n, \nu}^{(0), 0} + \chi|_{U_1} \text{TrFrob } M_{s'_1, n, \nu}^{(0), 1} + \chi|_{U_\varepsilon} \text{TrFrob } M_{s'_1, n, \nu}^{(0), \varepsilon} \end{aligned}$$

where we write  $\Theta_\pi$  both for the distribution character and for the locally integrable function on the regular set that represents it. Note that there is no difference in the formulas for  $\Theta_\pi(f_{s'_1, n, +})$  and  $\Theta_\pi(f_{s'_1, n, -})$ , because  $M_{s'_1, n, \pm}^{(0), 1}$  in any case coincides with  $M_{s'_1, n, \pm}^{(0), \varepsilon}$ . The proposition in the case  $z = s'_1$  is proved.

In the case  $z = s'_2$ , the calculation is very similar to the case  $z = s'_1$ , except that when  $n$  is odd, the expressions for  $M_{s'_2, n, \nu}^{(0), 1}$  and  $M_{s'_2, n, \nu}^{(0), \varepsilon}$  contain one or two terms depending on  $\nu$ , yet the final answer is the same in the both cases. Using equation (2.16) we get:

$$\begin{aligned} M_{s'_2, n, +}^{(0), 1} = M_{s'_2, n, -}^{(0), \varepsilon} &= \frac{1}{2} \frac{(\mathbb{L} - 1)^2}{\mathbb{L}(\mathbb{L}^2 - 1)} \mathbb{L}^{2 \frac{n+1}{2} - 1} (\mathbb{L} + 1) + \frac{\mathbb{L}}{\mathbb{L} + 1} \mathbb{L}^{2 \frac{n-1}{2} - 1} (\mathbb{L} + 1) \\ &= \frac{1}{2} \mathbb{L}^{n-1} (\mathbb{L} + 1); \\ M_{s'_2, n, -}^{(0), 1} = M_{s'_2, n, +}^{(0), \varepsilon} &= \frac{1}{2} \frac{(\mathbb{L}^2 - 1)}{\mathbb{L}(\mathbb{L}^2 - 1)} \mathbb{L}^{2 \frac{n+1}{2} - 1} (\mathbb{L} + 1) \\ &= \frac{1}{2} \mathbb{L}^{n-1} (\mathbb{L} + 1). \end{aligned}$$

The values of  $M_{s'_2, n, \nu}^{(0), 0}$  are computed similarly to the case  $z = s'_1$ .

Let us now prove the Proposition for the ramified elements. All ramified cases are very similar to each other. We show the details for the case  $z = t'_2$ . The

argument is exactly the same as in the case  $z = s'_1$ , but we have to use Lemma 2.4 instead of Lemma 2.3. In the case  $n$  even, we get:

(2.26)

$$\begin{aligned} M_{t'_2, n, \pm}^{(0), 0} &= 1 + \sum_{\lambda=1}^{n/2-1} \mathbb{L}^{2\lambda-1} + \frac{\mu(G_{(01)})}{\mu(G_{(0)})} \mathbb{L}^{2\frac{n}{2}-1} (\mathbb{L} + 1) \\ &= \frac{\mathbb{L}^{n-1} - 1}{\mathbb{L} - 1} + \frac{1}{\mathbb{L} + 1} \mathbb{L}^{2\frac{n}{2}-1} (\mathbb{L} + 1) = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1}; \\ M_{t'_2, n, +}^{(0), 1} &= M_{t'_2, n, -}^{(0), \varepsilon} = \frac{\mu(G_{(0)}) - \mu(G_{(01)})}{\mu(G_{(0)})} \mathbb{L}^{2\frac{n}{2}-1} (\mathbb{L} + 1) = \frac{\mathbb{L}}{\mathbb{L} + 1} \mathbb{L}^{2\frac{n}{2}-1} (\mathbb{L} + 1) = \mathbb{L}^n; \\ M_{t'_2, n, -}^{(0), 1} &= M_{t'_2, n, +}^{(0), \varepsilon} = 0. \end{aligned}$$

If  $n$  is odd, a similar calculation yields:

$$\begin{aligned} M_{t'_2, n, \pm}^{(0), 0} &= 1 + (\mathbb{L} + 1) \sum_{\lambda=1}^{\frac{n-1}{2}} \mathbb{L}^{2\lambda-1} (\mathbb{L} + 1) = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1}. \\ M_{t'_2, n, \text{sgn}(-1)}^{(0), 1} &= M_{t'_2, n, -\text{sgn}(-1)}^{(0), \varepsilon} = \frac{\mathbb{L}}{\mathbb{L} + 1} \mathbb{L}^{2\frac{n+1}{2}-1} (\mathbb{L} + 1) = \mathbb{L}^n. \\ M_{t'_2, n, -\text{sgn}(-1)}^{(0), 1} &= M_{t'_2, n, \text{sgn}(-1)}^{(0), \varepsilon} = 0. \end{aligned} \tag{2.27}$$

The Proposition for  $z = t'_2$  follows.

Finally, let us address the case  $s = s'_0$ . The elements of the set  $\Gamma_{s'_0, n}$  have the form  $h = \begin{bmatrix} 1+u\varpi^n & 0 \\ 0 & (1+u\varpi^n)^{-1} \end{bmatrix}$ , where  $u$  is a unit. Following the pattern of the previous section, we take  $y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{(0)}$ , and write down the conditions ensuring that the element  $a_\lambda^{-1} y^{-1} h y a_\lambda$  belongs to  $G_{(0)}$  and projects to a given conjugacy class under the map  $\rho_{(0), 0}$ . As before, we see that this depends on the entry in the right-hand corner of the matrix  $a_\lambda^{-1} y^{-1} h y a_\lambda$ , which equals  $\varpi^{-2\lambda} b d ((1 + u\varpi^n) - (1 + u\varpi^n)^{-1})$ . We have (when  $p \neq 2$ ):

$$\begin{aligned} \text{ord}(\varpi^{-2\lambda} b d ((1 + u\varpi^n) - (1 + u\varpi^n)^{-1})) &= -2\lambda + n + \text{ord}(b d), \\ \text{ac}(\varpi^{-2\lambda} b d ((1 + u\varpi^n) - (1 + u\varpi^n)^{-1})) &= 2\text{ac}(b d) \text{ac}(u). \end{aligned} \tag{2.28}$$

Here the situation is quite different from the elliptic cases, because the valuation of  $b d$  can be arbitrarily large, and therefore there are *infinitely many* values of  $\lambda$  such that  $a_\lambda^{-1} y^{-1} h y a_\lambda$  belongs to  $G_{(0)}$ . We know, of course, that the sum in the equation (2.2) has to be finite anyway. Here we will see explicitly that it happens because both for  $\chi = Q_T$  and  $\chi = Q_G$  (and therefore, for any linear combination of these two functions also), for large values of  $\lambda$  the sum of three terms  $\sum_U \chi|_U N_{U, \lambda}^{(0)}(h)$  vanishes. Indeed, if  $\chi = Q_G$ , it is easy to see that this sum is always zero, because  $Q_G$  vanishes on the class  $U_0$  and takes opposite values on  $U_1$  and  $U_\varepsilon$ . Suppose  $\chi = Q_T$ . Then for each positive integer  $k$ , we will need the volumes of the subsets of  $G_{(0)}$  defined by the formulas  $\{\text{ord}(b) \geq k\}$  and  $\{\text{ord}(d) \geq k\}$ . Note that these sets are disjoint and have equal volumes. From the point of view of motivic integration, it is easy to see that these sets are stable at level  $k$  in the language of [DL01], and their motivic volumes equal  $(\mathbb{L} - 1) \mathbb{L}^{-(k-2)}$ . The only varieties appearing in this calculation are affine spaces, so we acquire no bad primes. Now it is easy to see that when  $2\lambda > n$ , the value  $\chi|_{U_0}$  appears with the coefficient  $(\mathbb{L} - 1) \mathbb{L}^{-(k-2)}$ , and the

values  $\chi|_{U_1}$ ,  $\chi|_{U_\varepsilon}$  each appear with the coefficient  $\frac{1}{2}(\mathbb{L}-1)^2\mathbb{L}^{-(k-1)}$ , which leads to the cancellation in the case  $\chi = Q_T$ . Finally, we are again in a situation similar to all the previous cases, where we only need to sum over all  $\lambda$  not exceeding  $n/2$ . We omit the details of getting the answers that appear in the first rows of Tables 6 and 8. This ends the proof of the Proposition 2.5.  $\square$

TABLE 6. Virtual motives for the characters of  $\pi(0)_\theta$ ,  $\pi(0)_+$  and  $\pi(0)_-$  at  $Y_{z,n}(u)$  for  $z \in \{s'_0, s'_1, s'_2\}$ .

$z$	$M_{z,n,\nu}^{(0),0}$	$M_{z,n,\nu}^{(0),1}$	$M_{z,n,\nu}^{(0),\varepsilon}$
$s'_0$	$\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$	$\mathbb{L}^n$
$s'_1$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}, \quad n \text{ odd}$ $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}, \quad n \text{ even}$	$0, \quad n \text{ odd}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ even}$	$0, \quad n \text{ odd}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ even}$
$s'_2$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}, \quad n \text{ odd}$ $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}, \quad n \text{ even}$	$0, \quad n \text{ odd}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ even}$	$0, \quad n \text{ odd}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ even}$

TABLE 7. Virtual motives for the characters of  $\pi(0)_\theta$ ,  $\pi(0)_+$  and  $\pi(0)_-$  at  $Y_{z,n}(u)$  for  $z \in \{t'_0, t'_1, t'_2, t'_3\}$ . (Recall that  $\zeta^2 = \text{sgn}(-1)$ .)

$z$	$M_{z,n,\nu}^{(0),0}$	$M_{z,n,\nu}^{(0),1}$	$M_{z,n,\nu}^{(0),\varepsilon}$
$t'_0$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n} = \nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n} = \nu$ $\mathbb{L}^n, \quad \text{otherwise}$
$t'_1$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n} = -\nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n} = -\nu$ $\mathbb{L}^n, \quad \text{otherwise}$
$t'_2$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n} = (-1)^n \nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n} = (-1)^n \nu$ $\mathbb{L}^n, \quad \text{otherwise}$
$t'_3$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n} = (-1)^{n+1} \nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n} = (-1)^{n+1} \nu$ $\mathbb{L}^n, \quad \text{otherwise}$

### 3. MOTIVES FOR THE FOURIER TRANSFORMS OF OUR ORBITAL INTEGRALS

In this section we use the notation of [CH04], except that we denote points in the Bruhat-Tits building by  $i$  (or  $j$ ) rather than  $x$  (or  $y$ ).

TABLE 8. Virtual motives for the characters of  $\pi(1)_\theta$ ,  $\pi(1)_+$  and  $\pi(1)_-$  at  $Y_{z,n}(u)$  for  $z \in \{s'_0, s'_1, s'_2\}$ .

$z$	$M_{z,n,\nu}^{(1),0}$	$M_{z,n,\nu}^{(1),1}$	$M_{z,n,\nu}^{(1),\varepsilon}$
$s'_0$	$\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$	$\mathbb{L}^n$
$s'_1$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}, \quad n \text{ even}$ $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}, \quad n \text{ odd}$	$0, \quad n \text{ even}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ odd}$	$0, \quad n \text{ even}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ odd}$
$s'_2$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}, \quad n \text{ even}$ $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}, \quad n \text{ odd}$	$0, \quad n \text{ even}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ odd}$	$0, \quad n \text{ even}$ $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, \quad n \text{ odd}$

 TABLE 9. Virtual motives for the characters of  $\pi(1)_\theta$ ,  $\pi(1)_+$  and  $\pi(1)_-$  at  $Y_{z,n}(u)$  for  $z \in \{t'_0, t'_1, t'_2, t'_3\}$ .

$z$	$M_{z,n,\nu}^{(1),0}$	$M_{z,n,\nu}^{(1),1}$	$M_{z,n,\nu}^{(1),\varepsilon}$
$t'_0$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n+2} = \nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n+2} = \nu$ $\mathbb{L}^n, \quad \text{otherwise}$
$t'_1$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n} = -\nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n} = -\nu$ $\mathbb{L}^n, \quad \text{otherwise}$
$t'_2$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n+2} = (-1)^{n+1}\nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n+2} = (-1)^{n+1}\nu$ $\mathbb{L}^n, \quad \text{otherwise}$
$t'_3$	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n, \quad \zeta^{2n+2} = (-1)^n\nu$ $0, \quad \text{otherwise}$	$0, \quad \zeta^{2n+2} = (-1)^n\nu$ $\mathbb{L}^n, \quad \text{otherwise}$

**3.1. The Fourier transform of good orbital integrals.** For any rational number  $s$ , let  $\mathfrak{g}_s$  denote the union of the spaces  $\mathfrak{g}_{i,s}$  as  $i$  ranges over all points in the extended Bruhat-Tits building  $I(G, \mathbb{K})$  for  $G$  (see [MP94] for the definition of  $\mathfrak{g}_{i,s}$ ). Let  $\mathcal{H}(\mathfrak{g})$  denote the Hecke algebra of locally constant, compactly supported functions  $f : \mathfrak{g} \rightarrow \bar{\mathbb{Q}}_\ell$ . For any pair of rational numbers  $s \leq r$ , let  $\mathcal{H}(\mathfrak{g})_r^s$  denote the  $\bar{\mathbb{Q}}_\ell$ -vector space of elements of  $\mathcal{H}(\mathfrak{g})$  such that  $f$  is supported by  $\mathfrak{g}_s$  and  $\hat{f}$  is supported by  $\mathfrak{g}_{-r}$ . (Recall that the Fourier transform is taken with respect to an additive character of  $\mathbb{K}$  with conductor  $\mathcal{O}_{\mathbb{K}}$ .) Then the Fourier transform defines

an isomorphism of  $\bar{\mathbb{Q}}_\ell$ -vector spaces

$$\begin{aligned} \mathcal{H}(\mathfrak{g})_r^s &\rightarrow \mathcal{H}(\mathfrak{g})_{-s}^{-r} \\ f &\mapsto \hat{f}. \end{aligned}$$

We will write  $\mathcal{H}(\mathfrak{g})^s$  for the union of the spaces  $\mathcal{H}(\mathfrak{g})_r^s$  with  $s \leq r$ , and  $\mathcal{H}(\mathfrak{g})_r$  for the union of the spaces  $\mathcal{H}(\mathfrak{g})_r^s$  with  $s \leq r$ .

For any  $\varphi : \bar{\mathfrak{g}}_{i,r} \rightarrow \bar{\mathbb{Q}}_\ell$ , we write  $\varphi_{i,r}$  for the element of  $\mathcal{H}(\mathfrak{g})_r^r$  such that  $\varphi_{i,r}(Y) = (\varphi \circ \rho_{i,r})(Y)$  if  $Y \in \mathfrak{g}_{i,r}$  and  $\varphi_{i,r}(Y) = 0$  otherwise. As explained in [CH04, §1], on the level of reductive quotients we have another Fourier transform taking functions on  $\bar{\mathfrak{g}}_{i,r}$  to functions on  $\bar{\mathfrak{g}}_{i,-r}$ . With respect to these definitions we have

$$(3.1) \quad \widehat{\varphi}_{i,r} = \text{vol}(\mathfrak{g}_{i,r}) \hat{\varphi}_{i,-r},$$

where  $\text{vol}$  refers to the measure on  $\mathfrak{g}$ . The reader is referred to [CH04, §1] for elaboration and proofs.

**Proposition 3.1.** *Suppose  $X$  is a regular, elliptic, good element of  $g(\mathbb{K})$ . Let  $i$  be the point corresponding to the centraliser of  $X$  in  $G$  and let  $r$  be the depth of  $X$ . Let  $\bar{X}$  denote the image of  $X$  under  $\rho_{i,r} : \mathfrak{g}_{i,r} \rightarrow \bar{\mathfrak{g}}_{i,r}$  and let  $\varphi : \bar{\mathfrak{g}}_{i,r} \rightarrow \bar{\mathbb{Q}}$  denote the characteristic function of the  $\bar{G}_i$ -orbit of  $\bar{X} \in \bar{\mathfrak{g}}_{i,r}$  divided by the cardinality of that orbit. If  $f \in \mathcal{H}(\mathfrak{g})^{-r}$  then*

$$(3.2) \quad \hat{\Phi}_G(X, f) = \int_G \int_{\mathfrak{g}} f(\text{Ad}(g)Y) \hat{\varphi}_{i,-r}(Y) dY dg.$$

*Proof.* Suppose  $f \in \mathcal{H}(\mathfrak{g})^{-r}$ . Then  $f \in \mathcal{H}(\mathfrak{g})_{-r}^{-s}$  for some  $-r \leq -s$ . Thus,  $\hat{f} \in \mathcal{H}(\mathfrak{g})_r^s$ , so  $\hat{f} \in \mathcal{H}(\mathfrak{g})_r$ . Now, by [CH04, Prop 1.22] and elementary properties of the Fourier transform,

$$(3.3) \quad \begin{aligned} \hat{\Phi}_G(X, f) &= \Phi_G(X, \hat{f}) \\ &= \text{vol}(\mathfrak{g}_{i,r})^{-1} \int_G \int_{\mathfrak{g}} \hat{f}(\text{Ad}(g)Y) \varphi_{i,r}(Y) dY dg \\ &= \text{vol}(\mathfrak{g}_{i,r})^{-1} \int_G \int_{\mathfrak{g}} f(\text{Ad}(g)Y) \widehat{\varphi}_{i,r}(Y) dY dg. \end{aligned}$$

By [CH04, Prop 1.13],

$$\widehat{\varphi}_{i,r}(Y) = \text{vol}(\mathfrak{g}_{i,r}) \hat{\varphi}_{i,-r}(Y),$$

so

$$(3.4) \quad \hat{\Phi}_G(X, f) = \int_G \int_{\mathfrak{g}} f(\text{Ad}(g)Y) \hat{\varphi}_{i,-r}(Y) dY dg,$$

as claimed.  $\square$

*Remark 3.2.* We will sometimes write  $\varphi_{\bar{X}}$  (resp.  $\hat{\varphi}_{\bar{X}}$ ) for the function  $\varphi$  (resp.  $\hat{\varphi}$ ) appearing in Proposition 3.1 above; in that case,  $\varphi_{i,r}$  becomes  $(\varphi_{\bar{X}})_{i,r}$  and  $\hat{\varphi}_{i,-r}$  becomes  $(\hat{\varphi}_{\bar{X}})_{i,-r}$ .

**3.2. Application to our orbital integrals.** In order to apply [CH04, Prop 1.22] to the Lie algebra  $\mathfrak{g}$  and the orbital integrals appearing in Theorem 1.6, we must find the function  $(\hat{\varphi}_{\bar{X}_z})_{i_z, -r_z}$  for each  $X_z$  appearing in Theorem 1.6, where  $i_z$  is the point in the Bruhat-Tits building for  $X_z$  and  $r_z$  is the depth of  $X_z$  in  $\mathfrak{g}$ .

3.2.1. *Case:  $z = s_1$ .* Recall that

$$X_{s_1}(v) := \begin{bmatrix} 0 & u \\ -\varepsilon u & 0 \end{bmatrix}.$$

The point  $i_{s_1}$  is the standard vertex of the Bruhat-Tits building for  $G$  and the depth  $r_{s_1}$  of  $X_{s_1}(v)$  is 0; in other words,  $i_{s_1} = (0)$  and  $r_{s_1} = 0$ . Thus,

$$\mathfrak{g}_{i_{s_1}, r_{s_1}} = \mathfrak{g}_{(0), n} = \left\{ \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \mid x, y, z \in \mathcal{O}_{\mathbb{K}} \right\}.$$

The reduction map  $\rho_{i_{s_1}, r_{s_1}}$  is given by

$$\begin{bmatrix} z & x \\ y & -z \end{bmatrix} \mapsto \begin{bmatrix} \bar{z} & \bar{x} \\ \bar{y} & -\bar{z} \end{bmatrix},$$

where  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  denote the image of  $x$ ,  $y$  and  $z$  respectively under  $\mathcal{O}_{\mathbb{K}} \rightarrow \mathbb{F}_q$ . Let  $\bar{X}_{s_1}(v)$  denote the image of  $X_{s_1}(v)$  under  $\rho_{i_{s_1}, r_{s_1}}$ . The  $G_{i_{s_1}}(\mathbb{F}_q)$ -orbit of  $\bar{X}_{s_1}(v)$  in  $\bar{\mathfrak{g}}_{i_{s_1}, r_{s_1}}$  is

$$\left\{ \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \in sl(2, \mathbb{F}_q) \mid xy + z^2 = -\varepsilon \bar{v}^2 \right\},$$

which has cardinality  $q(q-1)$ . Thus,  $\varphi_{\bar{Y}_{s_1}(v)} : \bar{\mathfrak{g}}_{i_{s_1}, r_{s_1}} \rightarrow \bar{\mathbb{Q}}_\ell$  is given by

$$(3.5) \quad \varphi_{\bar{X}_{s_1}(v)} \left( \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \right) = \begin{cases} \frac{1}{q(q-1)} & xy + z^2 = -\varepsilon \bar{v}^2 \\ 0 & \text{otherwise.} \end{cases}$$

In order to find the (relative) Fourier transform of this function in the sense of [CH04] we observe that the Killing form  $\langle X, Y \rangle := \text{trace}(XY)$  gives a pairing between lattices

$$\begin{aligned} \mathfrak{g}_{i_{s_1}, r_{s_1}} \times \mathfrak{g}_{i_{s_1}, -r_{s_1}} &\rightarrow \mathcal{O}_{\mathbb{K}} \\ \left( \begin{bmatrix} z & x \\ y & -z \end{bmatrix}, \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \right) &\mapsto xb + ya + 2zc, \end{aligned}$$

which in turn gives a bilinear form  $\bar{\mathfrak{g}}_{i_{s_1}, r_{s_1}} \times \bar{\mathfrak{g}}_{i_{s_1}, -r_{s_1}} \rightarrow \mathbb{F}_q$ . The relative Fourier transform is taken with respect to this form. The image of the set of topologically nilpotent elements in  $\mathfrak{g}_{i_{s_1}, -r_{s_1}}$  under  $\rho_{i_{s_1}, -r_{s_1}}$  is the cone

$$\left\{ \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \in sl(2, \mathbb{F}_q) \mid ab + c^2 = 0 \right\}.$$

In Section 3.4 we will see that, if  $ab + c^2 = 0$  then

$$(3.6) \quad Q_T \left( \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \right) = (1-q) \hat{\varphi}_{\bar{X}_{s_1}(v)} \left( \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \right),$$

where  $Q_T$  is given in equation (1.6). This completes our description of the relevant properties of  $\hat{\varphi}_{\bar{X}_{s_1}(v)}$ .

3.2.2. *Case:*  $z = s_2$ . Since this case is very similar to the case above, we only summarize the results here. Recall (from Section 1.4) that

$$\bar{X}_{s_2}(v) := \begin{bmatrix} 0 & u\varpi \\ -\varepsilon u\varpi^{-1} & 0 \end{bmatrix}.$$

The point  $i_{s_2}$  is the vertex (1) of the Bruhat-Tits building for  $G$  (see Section 1.3) and the depth  $r_{s_2}$  of  $\bar{X}_{s_2}(v)$  is 0. Thus,

$$\mathfrak{g}_{i_{s_2}, r_{s_2}} = \mathfrak{g}_{(1), 0} = \left\{ \begin{bmatrix} z & x\varpi \\ y\varpi^{-1} & -z \end{bmatrix} \mid x, y, z \in \mathcal{O}_{\mathbb{K}} \right\}.$$

The reduction map  $\rho_{i_{s_2}, r_{s_2}}$  is given by

$$\begin{bmatrix} z & x\varpi \\ y\varpi^{-1} & -z \end{bmatrix} \mapsto \begin{bmatrix} \bar{z} & \bar{x} \\ \bar{y} & -\bar{z} \end{bmatrix}.$$

The function  $\varphi_{\bar{X}_{s_2}(v)} : \bar{\mathfrak{g}}_{i_{s_2}, r_{s_2}} \rightarrow \bar{\mathbb{Q}}_\ell$  is exactly as in the preceding case, so

$$(3.7) \quad Q_T \left( \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \right) = (1 - q) \hat{\varphi}_{\bar{X}_{s_2}(v)} \left( \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \right),$$

as above. This completes our description of the relevant properties of  $\hat{\varphi}_{\bar{X}_{s_2}(v)}$ .

3.2.3. *Case:*  $z \in \{t_0, t_1, t_2, t_3\}$ . Then the point  $i_z$  is (01) and the depth  $r_z$  is  $\frac{1}{2}$ . Thus,

$$\mathfrak{g}_{i_z, r_z} = \mathfrak{g}_{(01), \frac{1}{2}} = \left\{ \begin{bmatrix} z\varpi & x \\ y\varpi & -z\varpi \end{bmatrix} \mid x, y, z \in \mathcal{O}_{\mathbb{K}} \right\}.$$

Thus,  $\bar{\mathfrak{g}}_{i_z, r_z} = \mathbb{A}^2(\mathbb{F}_q)$  and the reduction map  $\rho_{i_z, r_z} : \mathfrak{g}_{i_z, r_z} \rightarrow \mathbb{A}^2(\mathbb{F}_q)$  is given by

$$\begin{bmatrix} z\varpi & x \\ y\varpi & -z\varpi \end{bmatrix} \mapsto (\bar{x}, \bar{y}).$$

The reduction map on  $G_{i_z} \rightarrow GL(1, \mathbb{F}_q)$  is given by

$$\begin{bmatrix} a & b \\ \varpi c & d \end{bmatrix} \mapsto \bar{a}.$$

Thus, the action of  $\bar{G}_{i_z}$  on  $\bar{\mathfrak{g}}_{i_z, r_z}$  corresponds to the action of  $GL(1, \mathbb{F}_q)$  on  $\mathbb{A}^2(\mathbb{F}_q)$  given by  $t \cdot (x, y) := (t^2x, t^{-2}y)$ . It follows immediately from the definitions above that

$$\begin{aligned} \varphi_{\bar{X}_{t_0}(v)}(x, y) &= \begin{cases} \frac{2}{q-1} & xy = -\bar{v}^2 \wedge \text{sgn}(x) = \text{sgn}(v) \\ 0 & \text{otherwise} \end{cases} \\ \varphi_{\bar{X}_{t_1}(v)}(x, y) &= \begin{cases} \frac{2}{q-1} & xy = -\bar{v}^2 \wedge \text{sgn}(x) = \text{sgn}(\varepsilon v) \\ 0 & \text{otherwise} \end{cases} \\ \varphi_{\bar{X}_{t_2}(v)}(x, y) &= \begin{cases} \frac{2}{q-1} & xy = -\bar{\varepsilon}\bar{v}^2 \wedge \text{sgn}(x) = \text{sgn}(v) \\ 0 & \text{otherwise} \end{cases} \\ \varphi_{\bar{X}_{t_3}(v)}(x, y) &= \begin{cases} \frac{2}{q-1} & xy = -\bar{\varepsilon}\bar{v}^2 \wedge \text{sgn}(x) = \text{sgn}(\varepsilon v) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

**3.3. Two functions on  $\mathbb{A}^2(\mathbb{F}_q)$ .** In this section we introduce two functions that play a crucial role in our proof of Theorem 1.6. Using notation from the preceding Subsection, define

$$(3.8) \quad \begin{aligned} \varphi(0) &:= \varphi_{\bar{X}_{t_0}} - \varphi_{\bar{X}_{t_1}} + \varphi_{\bar{X}_{t_2}} - \varphi_{\bar{X}_{t_3}} \\ \varphi(1) &:= \varphi_{\bar{X}_{t_0}} - \varphi_{\bar{X}_{t_1}} - \varphi_{\bar{X}_{t_2}} + \varphi_{\bar{X}_{t_3}}. \end{aligned}$$

To find the Fourier transform of these functions (in the sense of [CH04]), observe that the Killing form  $\langle X, Y \rangle := \text{trace}(XY)$  gives a pairing between lattices

$$\begin{aligned} \mathfrak{g}_{i_z, r_z} \times \mathfrak{g}_{i_z, -r_z} &\rightarrow \mathcal{O}_{\mathbb{K}} \\ \left( \begin{bmatrix} z\varpi & x \\ y\varpi & -z\varpi \end{bmatrix}, \begin{bmatrix} c & a\varpi^{-1} \\ b & -c \end{bmatrix} \right) &\mapsto xb + ya + 2\varpi zc, \end{aligned}$$

which in turn gives the bilinear form

$$\begin{aligned} \bar{\mathfrak{g}}_{i_z, r_z} \times \bar{\mathfrak{g}}_{i_z, -r_z} &\rightarrow \mathbb{F}_q \\ ((x, y), (a, b)) &\mapsto xb + ya. \end{aligned}$$

The relative Fourier transform is taken with respect to this form. The image of the set of topologically nilpotent elements in  $\mathfrak{g}_{i_z, -r_z}$  under  $\rho_{i_z, -r_z}$  is the normal crossing

$$\{(a, b) \in \mathbb{A}^2(\mathbb{F}_q) \mid ab = 0\}.$$

In Section 3.4 we will see that

$$(3.9) \quad \begin{aligned} \hat{\varphi}_{(0)}(a, 0) &= 0 \\ \hat{\varphi}_{(0)}(0, b) &= \frac{4}{q-1} \sqrt{q} \zeta \text{sgn}(b) \\ \hat{\varphi}_{(1)}(a, 0) &= \frac{4}{q-1} \sqrt{q} \zeta^3 \text{sgn}(a) \\ \hat{\varphi}_{(1)}(0, b) &= 0. \end{aligned}$$

**3.4. Some finite field calculations.** In this section we defend equations (3.6), (3.7) and (3.9).

Recall the definition of the function  $\gamma_{\pm} : \mathbb{A}^2(\mathbb{F}_q) \rightarrow \bar{\mathbb{Q}}_{\ell}$  from equation (1.2). Recall also that we equip  $\mathbb{A}^2(\mathbb{F}_q)$  with the bilinear form  $\langle (x, y), (a, b) \rangle = xb + ya$ , as explained in Section 3.2. Finally, recall the definition of  $\varphi_{\bar{X}_z(v)}$  for  $z \in \{t_0, t_1, t_2, t_3\}$ . Then

$$\begin{aligned} \hat{\varphi}_{\bar{X}_{t_0}(v)}(a, b) &= \sum_{(x, y) \in \mathbb{A}^2(\mathbb{F}_q)} \bar{\psi} \langle (x, y), (a, b) \rangle \varphi_{\bar{X}_{t_0}(v)}(x, y) \\ &= \frac{2}{q-1} \sum_{\substack{xy = -\bar{v}^2 \\ \text{sgn}(x) = \text{sgn}(v)}} \bar{\psi}(xb) \bar{\psi}(ya) \\ &= \frac{2}{q-1} \sum_{\text{sgn}(x) = \text{sgn}(v)} \bar{\psi}(xb) \bar{\psi}(-\bar{v}^2 x^{-1} a). \end{aligned}$$

Thus,

$$\begin{aligned}
\hat{\varphi}_{\bar{X}_{t_0}(v)}(a, 0) &= \frac{2}{q-1} \sum_{\text{sgn}(x)=\text{sgn}(v)} \bar{\psi}(0)\bar{\psi}(-\bar{v}^2x^{-1}a) \\
&= \frac{2}{q-1} \sum_{\text{sgn}(t)=\text{sgn}(-v)} \bar{\psi}(ta) \\
&= \frac{2}{q-1} \gamma_{\text{sgn}(-v)}(a),
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varphi}_{\bar{X}_{t_0}(v)}(0, b) &= \frac{2}{q-1} \sum_{\text{sgn}(x)=\text{sgn}(v)} \bar{\psi}(xb)\bar{\psi}(0) \\
&= \frac{2}{q-1} \gamma_{\text{sgn}(v)}(b).
\end{aligned}$$

Similar arguments show that

$$\begin{aligned}
\hat{\varphi}_{\bar{X}_{t_1}(v)}(a, 0) &= \frac{2}{q-1} \gamma_{\text{sgn}(-\varepsilon v)}(a) \\
\hat{\varphi}_{\bar{X}_{t_1}(v)}(0, b) &= \frac{2}{q-1} \gamma_{\text{sgn}(\varepsilon v)}(b)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varphi}_{\bar{X}_{t_2}(v)}(a, 0) &= \frac{2}{q-1} \gamma_{\text{sgn}(-\varepsilon v)}(a) \\
\hat{\varphi}_{\bar{X}_{t_2}(v)}(0, b) &= \frac{2}{q-1} \gamma_{\text{sgn}(v)}(b)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varphi}_{\bar{X}_{t_3}(v)}(a, 0) &= \frac{2}{q-1} \gamma_{\text{sgn}(-v)}(a) \\
\hat{\varphi}_{\bar{X}_{t_3}(v)}(0, b) &= \frac{2}{q-1} \gamma_{\text{sgn}(\varepsilon v)}(b).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left( \hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} + \hat{\varphi}_{\bar{X}_{t_2}(v)} - \hat{\varphi}_{\bar{X}_{t_3}(v)} \right)(a, 0) \\
&= \frac{2}{q-1} \left( \gamma_{\text{sgn}(-v)} - \gamma_{\text{sgn}(-\varepsilon v)} + \gamma_{\text{sgn}(-\varepsilon v)} - \gamma_{\text{sgn}(-v)} \right)(a) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
& \left( \hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} + \hat{\varphi}_{\bar{X}_{t_2}(v)} - \hat{\varphi}_{\bar{X}_{t_3}(v)} \right) (0, b) \\
&= \frac{2}{q-1} \left( \gamma_{\text{sgn}(v)} - \gamma_{\text{sgn}(\varepsilon v)} + \gamma_{\text{sgn}(v)} - \gamma_{\text{sgn}(\varepsilon v)} \right) (b) \\
&= \frac{4}{q-1} \left( \gamma_{\text{sgn}(v)} - \gamma_{\text{sgn}(\varepsilon v)} \right) (b) \\
&= \frac{4}{q-1} \text{sgn}(v) \widehat{\text{sgn}}(b) \\
&= \frac{4}{q-1} \text{sgn}(v) \sqrt{q} \zeta \text{sgn}(b).
\end{aligned}$$

Letting  $v = 1$  we recover the first two parts of equation (3.9). Likewise,

$$\begin{aligned}
& \left( \hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} - \hat{\varphi}_{\bar{X}_{t_2}(v)} + \hat{\varphi}_{\bar{X}_{t_3}(v)} \right) (a, 0) \\
&= \frac{2}{q-1} \left( \gamma_{\text{sgn}(-v)} - \gamma_{\text{sgn}(-\varepsilon v)} - \gamma_{\text{sgn}(-\varepsilon v)} + \gamma_{\text{sgn}(-v)} \right) (a) \\
&= \frac{4}{q-1} \left( \gamma_{\text{sgn}(-v)} - \gamma_{\text{sgn}(-\varepsilon v)} \right) (a) \\
&= \frac{4}{q-1} \text{sgn}(-v) \widehat{\text{sgn}}(a) \\
&= \frac{4}{q-1} \text{sgn}(v) \sqrt{q} \zeta^3 \text{sgn}(a),
\end{aligned}$$

and

$$\begin{aligned}
& \left( \hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} - \hat{\varphi}_{\bar{X}_{t_2}(v)} + \hat{\varphi}_{\bar{X}_{t_3}(v)} \right) (0, b) \\
&= \frac{2}{q-1} \left( \gamma_{\text{sgn}(v)} - \gamma_{\text{sgn}(\varepsilon v)} - \gamma_{\text{sgn}(v)} + \gamma_{\text{sgn}(\varepsilon v)} \right) (b) \\
&= 0.
\end{aligned}$$

Letting  $v = 1$  we recover the last two parts of equation (3.9).

Equation (3.6) (and therefore equation (3.7) also) is exactly Springer's Hypothesis in our context. In fact, it is quite easy to verify this equality by direct calculation of the relevant Fourier transforms, but since this paper is already long, we omit the details.

**3.5. Motives for the Fourier transforms of orbital integrals.** As we see from Section 3.2, for our purposes, there are exactly three important points in the Bruhat-Tits building: (0), (01) and (1). For the remainder of this section, the symbol  $i$  is reserved for the vertices (0) and (1) while  $j = (01)$ .

Let  $f$  be a test function supported on the topologically nilpotent elements in  $\mathfrak{g}(\mathbb{K})$ . All we need to know in order to find  $\hat{\Phi}(X_z, f)$  is the values of the corresponding function  $\hat{\varphi}_{X_z}$  on the corresponding reductive quotient, and the information about the projections of the elements of the form  $h^{-1}Yh$  for the elements  $Y$  in the support of  $f$ .

Recall the notation  $\mathfrak{h}_{z,n} = \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^*\} \subset \mathfrak{g}$ , and  $\tilde{f}_{z,n,\pm} = \exp^* f_{z,n,\pm}$  (the characteristic function of the set  ${}^G\mathfrak{h}_{z,n,\pm}$ , where  $G$  acts by adjoint action).

**Proposition 3.3.** *Suppose  $z \in \{s'_0, s'_1, s'_2, t'_0, t'_1, t'_2, t'_3\}$ . Let  $\tilde{f}_{z,n,\pm}$  be the characteristic function of the set  ${}^G\mathfrak{h}_{z,n,\pm}$  and let  $\mathfrak{u}_0, \mathfrak{u}_1$  and  $\mathfrak{u}_\varepsilon$  denote the nilpotent orbits in  $sl(2, \mathbb{F}_q)$  corresponding to the unipotent conjugacy classes  $U_0, U_1$  and  $U_\varepsilon$  respectively. Let  $\kappa$  be an arbitrary linear combination of the functions  $Q_T$  and  $Q_G$  viewed as functions on  $sl(2, \mathbb{F}_q)$  (see Section 3.4 for their definition). Then*

$$(3.10) \quad \int_G \int_{\mathfrak{g}} \tilde{f}_{z,n,\pm}(gYg^{-1})\kappa_{i,0}(Y)dYdg \\ = \kappa|_{\mathfrak{u}_0} \text{TrFrob}M_{z,n,\pm}^{i,0} + \kappa|_{\mathfrak{u}_1} \text{TrFrob}M_{z,n,\pm}^{i,1} + \kappa|_{\mathfrak{u}_\varepsilon} \text{TrFrob}M_{z,n,\pm}^{i,\varepsilon},$$

where  $M_{z,n,\pm}^{i,0}, M_{z,n,\pm}^{i,1}$  and  $M_{z,n,\pm}^{i,\varepsilon}$  are defined in Tables 6 and 7 in the case  $i = (0)$  and Tables 8 and 9 in the case  $i = (1)$ .

*Proof.* This Proposition essentially follows from Proposition 2.5. We have

$$\int_G \int_{\mathfrak{g}} \tilde{f}_{z,n,\pm}(gYg^{-1})\kappa_{i,0}(Y)dYdg \\ = \int_G \int_{\mathfrak{g}} \tilde{f}_{z,n,\pm}(Y)\kappa_{i,0}(g^{-1}Yg)dYdg \\ = \int_G \int_{G\Gamma_{z,n,\pm}} (\kappa_{i,0} \circ \exp^*)(g^{-1}\gamma g)d\gamma dg.$$

Note that in the above integral, the function  $\kappa_{i,0} \circ \exp^*$  is evaluated only at topologically unipotent elements. The exponential map is measure-preserving on this set, and that's why the integral can be rewritten as a double integral over the group  $G$ . This is exactly the expression that appears in the equation (2.1), and therefore by Proposition 2.5 it has the required form.  $\square$

**Corollary 3.4.** *Suppose  $z \in \{s_1, s_2\}$  and  $z' \in \{s'_0, s'_1, s'_2, t'_0, t'_1, t'_2, t'_3\}$ . For each  $n \in \mathbb{N}$ , the Fourier transform  $\hat{\Phi}_G(X_z, \cdot)$  of the orbital integral at  $X_z$  is constant on the set  ${}^G\mathfrak{h}_{z',n,\pm}$ , and*

$$\hat{\Phi}(X_z, \tilde{f}_{z',n,\pm}) \\ = \hat{\varphi}_{\bar{X}_z}|_{\mathfrak{u}_0} \text{TrFrob}M_{z',n,\pm}^{i,0} + \hat{\varphi}_{\bar{X}_z}|_{\mathfrak{u}_1} \text{TrFrob}M_{z',n,\pm}^{i,1} + \hat{\varphi}_{\bar{X}_z}|_{\mathfrak{u}_\varepsilon} \text{TrFrob}M_{z',n,\pm}^{i,\varepsilon},$$

where  $i = (0)$  if  $z = s_1$  and  $i = (1)$  if  $z = s_2$ . See Remark 3.2 for the definition of  $\varphi_{\bar{X}_z}$ .

*Proof.* From the Section 3.2, we see that our elements  $X_z$  correspond to vertices  $i$  in the building, and their depth is  $r = 0$ . First, recall that by Proposition 3.1, we have

$$\Phi(X_z, \tilde{f}_{z',n,\pm}) = \int_G \int_{\mathfrak{g}} \tilde{f}_{z',n,\pm}(gYg^{-1})(\hat{\varphi}_{\bar{X}_z})_{i,0}(Y).$$

Now we can plug in  $\kappa = \hat{\varphi}_{\bar{X}_z}$  in the Proposition 3.3. By Springer hypothesis (see Section 3.4), the function  $\hat{\varphi}_{\bar{X}_z}$  restricted to the set of nilpotent elements is a constant multiple of the Green's polynomial  $Q_T$  (thought of as a function on the Lie algebra), so the assumptions of the Proposition are satisfied.  $\square$

Let us now consider the elements  $X_z$  of the ramified elliptic tori, that is,  $z = t_0, t_1, t_2$  or  $t_3$ . Then (see Section 3.2),  $j = (01)$  and  $r = \frac{1}{2}$ , and  $\mathfrak{g}_{j,-r}(\mathbb{F}_q) = \mathbb{A}^2(\mathbb{F}_q)$ ; in the rest of this section we write  $j$  for  $(01)$  and  $r$  for  $\frac{1}{2}$ . The image of the set of topologically nilpotent elements in  $\mathfrak{g}_{j,-1/2}$  under  $\rho_{j,-r}$  is contained in  $\{(x, y) \in$

$\mathbb{A}^2(\mathbb{F}_q) \mid xy = 0$ . This set is the union of the following five orbits of the action of  $GL(1, \mathbb{F}_q)$  on  $\mathbb{A}^2(\mathbb{F}_q)$ :

$$(3.11) \quad \begin{aligned} V^0 &:= \{(0, 0)\} \\ V^{1,+} &:= \{(0, x) \mid \text{sgn}(x) = 1\} & V^{1,-} &:= \{(0, x) \mid \text{sgn}(x) = -1\} \\ V^{2,+} &:= \{(x, 0) \mid \text{sgn}(x) = 1\} & V^{2,-} &:= \{(x, 0) \mid \text{sgn}(x) = -1\} \end{aligned}$$

**Proposition 3.5.** *Let  $\kappa$  be a  $GL(1)$ -invariant function defined on  $\mathbb{A}^2(\mathbb{F}_q)$  with respect to the action defined in Section 3.4. Then, for each  $z \in \{s'_1, s'_2, t'_0, t'_1, t'_2, t'_3\}$ , each  $\nu = \pm$ , and each non-negative integer  $n$  in the case  $z \in \{t'_0, t'_1, t'_2, t'_3\}$ , and each positive integer  $n$  in the case  $z \in \{s'_1, s'_2\}$ , there exist virtual Chow motives  $\mathcal{N}_{z,n,\nu}^0$ ,  $\mathcal{N}_{z,n,\nu}^{1,\pm}$  and  $\mathcal{N}_{z,n,\nu}^{2,\pm}$ , such that*

$$\begin{aligned} & \int_{G(\mathbb{K})} \int_{\mathfrak{g}(\mathbb{K})} \tilde{f}_{z,n,\nu}(Ad(g)Y) \kappa_{j,-r}(Y) dY dg \\ &= \kappa|_{V_0} \text{TrFrob} \mathcal{N}_{z,n,\nu}^0 + \sum_{\alpha=\pm} \kappa|_{V^{1,\alpha}} \text{TrFrob} \mathcal{N}_{z,n,\nu}^{1,\alpha} + \sum_{\alpha=\pm} \kappa|_{V^{2,\alpha}} \text{TrFrob} \mathcal{N}_{z,n,\nu}^{2,\alpha}, \end{aligned}$$

Moreover,  $\mathcal{N}_{z,n,\nu}^0$  and  $\mathcal{N}_{z,n,\nu}^{1,\pm}$  and  $\mathcal{N}_{z,n,\nu}^{2,\pm}$  are rational functions of  $\mathbb{L}$ . The virtual Chow motives  $\mathcal{N}_{z,n,\nu}^{1,\pm}$  and  $\mathcal{N}_{z,n,\nu}^{2,\pm}$  are given in Tables 10 and 11.

*Proof.* We rewrite the left-hand side as the sum over the nilpotent orbits in the reductive quotient, as we did with the character in Section 2.1. In order to do this, we make the following definition. Suppose  $V$  is one of the  $GL(1, \mathbb{F}_q)$ -orbits in  $\mathbb{A}^2(\mathbb{F}_q)$  appearing in equation (3.11). Let  $n$  be an integer and recall that  $j$  denotes the point (01) in the Bruhat-Tits building for  $G$ . For any regular topologically nilpotent element  $Y$  of  $\mathfrak{g}$ , let  $N_{V,\lambda}^j(Y)$  denote the number of  $G_i$ -cosets  $gG_j$  inside the double coset space  $G_i \backslash G/G_j$  that satisfy the following condition:

$$g^{-1}Yg \in \mathfrak{g}_{j,-r} \quad \wedge \quad \rho_{j,-r}(g^{-1}Yg) \in V.$$

Instead of using the Cartan decomposition  $G = G_i A G_i$  as in Section 2.1 we exploit the  $G_j$ -invariance of this function, and do the manipulations similar to Section 2.1. The reason we have to do this is that the function  $\kappa_{j,-r}$  is not  $G_i$ -invariant; it is, however,  $G_j$ -invariant.

Let  $\tilde{A}$  be the set of diagonal matrices of the form  $\text{diag}(\varpi^\lambda, \varpi^{-\lambda})$  where  $\lambda$  is an arbitrary integer. Then  $G$  has the decomposition  $G = G_i \tilde{A} G_j$  (note the difference with Cartan decomposition where  $\lambda$  is non-negative). Using the  $G_j$ -invariance of the function  $\kappa_{j,-r}$ , we obtain, for an arbitrary test function  $f$ :

$$(3.12) \quad \begin{aligned} & \int_G \int_{\mathfrak{g}} f(Y) \kappa_{j,-r}(g^{-1}Yg) dY dg \\ &= \int_{G/G_j} \int_{G_j} \int_{\mathfrak{g}} f(Y) \kappa_{j,-r}(y^{-1}h^{-1}Yhy) dY dy dh \\ &= m(G_j) \sum_{a \in G_i \backslash G/G_j} \sum_{h \in G_i a G_j / G_j} \int_{\mathfrak{g}} f(Y) \kappa_{j,-r}(h^{-1}Yh) dY. \end{aligned}$$

Note that the summation index in the outside sum in fact runs over  $\mathbb{Z}$ . As with the case of the character, the sum in fact contains only finitely many nonzero terms, since at the moment we are considering only the elliptic elements  $Y$ .

The rest of the argument follows the pattern of the proof of Proposition 2.5, taking equation (3.12) as the starting point. It also proceeds case-by-case. Here we carry out the proof for the test functions  $\tilde{f}_{z,n,\pm}$  with  $z = s'_1$  and  $z = t'_2$ . The other cases are very similar; the results of these calculations are recorded in Tables 10 and 11.

As in Section 2.1, we can continue the chain of equalities (3.12) by writing the integral inside the sum as a sum over  $GL_1(\mathbb{F}_q)$ -orbits:

$$\begin{aligned} & \mathfrak{m}(G_j) \sum_{a \in G_i \backslash G/G_j} \sum_{h \in G_i a G_j / G_j} \int_{\mathfrak{g}} \tilde{f}_{z,n,\pm}(Y) \kappa_{j,-r}(h^{-1} Y h) dY \\ &= \sum_{a \in G_i \backslash G/G_j} \sum_V \kappa|_V \int_{G \mathfrak{h}_{z,n,\pm}} N_{V,\lambda}(Y) dY \\ &= \sum_V \kappa|_V \sum_{\lambda=-\infty}^{\infty} \int_{G \mathfrak{h}_{z,n,\pm}} N_{V,\lambda}(Y) dY. \end{aligned}$$

Note that since there are, in fact, only finitely many non-zero terms, the permutation of the two sums is valid.

Now it remains to “calculate” the numbers  $N_{V,\lambda}(Y)$ , *i.e.*, to express them in terms of motivic volumes of some definable sets. This is done by brute force, in a manner similar to the calculation of the character. Our calculation will make it transparent that these numbers are constant on each of the sets  $G \mathfrak{h}_{z,n,\pm}$ .

We will need the formula for the number of  $G_j$ -cosets inside each double coset  $G_i a_\lambda G_j$  ( $\lambda \in \mathbb{Z}$ ): the cardinality  $\#G_i a_\lambda G_j / G_j$  equals  $q^{2\lambda-1}$  if  $\lambda > 0$ , and  $q^{2|\lambda|}$  if  $\lambda \leq 0$ , as can be shown using the affine Bruhat decomposition for  $G$  (see [BT96], for example). Recall that with the notation of Section 1.7.2, we can write  $[G_i a_\lambda G_j / G_j] = \mathbb{L}^{2\lambda-1}$  when  $\lambda$  is a positive integer,  $[G_i a_\lambda G_j / G_j] = \mathbb{L}^{-2\lambda}$  when  $\lambda \leq 0$ .

Let  $ha_\lambda$  be a representative of a coset  $G_i a_\lambda G_j / G_j$ ; write  $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_i$ .

A statement completely analogous to Lemma 2.2 relates the numbers  $N_{V,\lambda}$  to the motivic volumes of the sets  $\{h \mid \rho_{j,-r}(a_\lambda^{-1} h^{-1} Y_{z,n}(u) h a_\lambda) = (x, 0)\}$ , and  $\{h \mid \rho_{j,-r}(a_\lambda^{-1} h^{-1} Y_{z,n}(u) h a_\lambda) = (0, x)\}$ , with  $x$  square or non-square, respectively (of course, we also need to show that these four sets are definable). Then to obtain the virtual motives  $\mathcal{N}_{z,n,\nu}^{1,\pm}$ , and  $\mathcal{N}_{z,n,\nu}^{2,\pm}$ , we need to sum these motivic volumes over all values of  $\lambda$ .

The following is the list of possibilities for the element  $\rho_{j,-r}(a_\lambda^{-1} h^{-1} Y_{z,n}(u) h a_\lambda)$  of  $\mathbb{A}^2(\mathbb{F}_q)$  in the two cases  $z = t'_2$  and  $z = s'_1$ :

**Case:**  $z = t'_2$ . The conditions on  $a, b, c, d$  and  $\lambda$  for  $a_\lambda^{-1} h^{-1} Y_{z,n}(u) h a_\lambda$  to be in  $G_{j,-r}$  are:

$$(3.13) \quad \begin{aligned} & -2\lambda + \text{ord}(d^2 u \varpi^n - b^2 \varepsilon u \varpi^{n+1}) \geq -1, \text{ and} \\ & 2\lambda + \text{ord}(-c^2 u \varpi^n + a^2 \varepsilon u \varpi^{n+1}) \geq 0. \end{aligned}$$

Applying the reduction map  $\rho_{j,-r}$  to  $a_\lambda^{-1} h^{-1} Y_{z,n}(u) h a_\lambda$ , we see that there are a few possibilities:

1.  $n$  is even: Recall the notation  $\nu = \text{sgn}(u)$ .
  - (a) If  $-n/2 < \lambda \leq n/2$ , then  $\rho_{j,-r}(a_\lambda^{-1} h^{-1} Y_{z,n}(u) h a_\lambda) = (0, 0)$ .

(b) Suppose  $\lambda = n/2 + 1$ . If  $\text{ord}(d) > 0$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (x, 0)$$

with  $\text{sgn}(x) = \text{sgn}(-\varepsilon u) = -\text{sgn}(-u) = -\zeta^2 \text{sgn}(u)$ ; on the other hand, if  $\text{ord}(d) = 0$ , we have  $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda \notin G_{j,-r}$ .

(c) Suppose  $\lambda = -n/2$ . If  $\text{ord}(c) > 0$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, 0);$$

on the other hand, if  $\text{ord}(c) = 0$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, y),$$

with  $\text{sgn}(y) = \text{sgn}(-u) = \zeta^2 \nu$ .

The case (b) allows us to calculate the virtual motives responsible for the two orbits of the form  $(x, 0)$ : the virtual Chow motive  $\mathcal{N}_{t'_2, n, \nu}^{1,+}$  corresponding to the orbit of  $(1, 0)$  equals 0 if  $\zeta^2 \nu = 1$ , and equals  $\frac{\mu(G_j)}{\mu(G_i)} [G_i a_{\frac{n}{2}+1} G_j / G_j] = \frac{1}{\mathbb{L}+1} \mathbb{L}^{n+1}$  if  $\zeta^2 \nu = -1$ . For the orbit of  $(\varepsilon, 0)$ , the answer is the reverse:  $\mathcal{N}_{t'_2, n, \nu}^{1,-}$  equals 0 if  $\zeta^2 \nu = -1$ , and equals  $\frac{1}{\mathbb{L}+1} \mathbb{L}^{n+1}$  if  $\zeta^2 \nu = 1$ .

From the case (c), we get the motives corresponding to the other two orbits:  $\mathcal{N}_{t'_2, n, \nu}^{2,+} = \frac{\mu(G_i) - \mu(G_j)}{\mu(G_i)} [G_i a_{-\frac{n}{2}} G_j / G_j] = \frac{\mathbb{L}}{\mathbb{L}+1} \mathbb{L}^n$  if  $\zeta^2 \nu = 1$ , and 0 if  $\zeta^2 \nu = -1$ , and for  $\mathcal{N}_{t'_2, n, \nu}^{2,-}$  these answers are reversed.

2.  $n$  is odd:

(a) If  $-(n+1)/2 < \lambda < (n+1)/2$ , then  $\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, 0)$ .

(b) If  $\lambda = (n+1)/2$ , there are two possibilities: if  $\text{ord}(d) = 0$  then we have

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (x, 0)$$

with  $\text{sgn}(x) = \text{sgn}(u)$ ; on the other hand, if  $\text{ord}(d) > 0$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, 0).$$

(c) Suppose  $\lambda = -(n+1)/2$ . If  $\text{ord}(c) = 0$ , we have  $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda \notin G_{j,-r}$ ; if  $\text{ord}(c) > 0$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, y),$$

with  $\text{sgn}(y) = -\text{sgn}(u)$ .

As before, from the calculations in the subcase (b), we see that the virtual Chow motive  $\mathcal{N}_{t'_2, n, \nu}^{1,+}$  that corresponds to the orbit of  $(1, 0)$  equals 0 if  $\nu = -1$ , and equals  $\frac{\mu(G_i) - \mu(G_j)}{\mu(G_i)} [G_i a_{\frac{n+1}{2}} G_j / G_j] = \frac{\mathbb{L}}{\mathbb{L}+1} \mathbb{L}^{n+1-1} = \frac{1}{\mathbb{L}+1} \mathbb{L}^n$  if  $\nu = 1$ . For the orbit of  $(\varepsilon, 0)$ , the answer is the reverse. The subcase (c) yields  $\mathcal{N}_{t'_2, n, \nu}^{2,+} = \frac{\mu(G_j)}{\mu(G_i)} [G_i a_{-\frac{n+1}{2}} G_j / G_j] = \frac{1}{\mathbb{L}+1} \mathbb{L}^{n+1}$  if  $\nu = -1$ , and 0 if  $\nu = 1$ .

**Case:**  $z = s'_1$ . In this case, the conditions (3.13) are replaced with:

$$(3.14) \quad \begin{aligned} -2\lambda + \text{ord}(d^2 \varpi^n - b^2 \varepsilon u \varpi^n) &\geq -1, \text{ and} \\ 2\lambda + \text{ord}(-c^2 u \varpi^n + a^2 \varepsilon u \varpi^n) &\geq 0. \end{aligned}$$

1. Suppose  $n$  is even. Then we obtain:

(a) If  $-n/2 < \lambda \leq n/2$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, 0).$$

(b) If  $\lambda = -n/2$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, y)$$

with  $\text{sgn}(y) = \text{sgn}(u)\text{sgn}(\text{ac}(-c^2 + a^2\varepsilon))$ .

We see that  $\mathcal{N}_{s'_1, n, \nu}^{1, \pm}$  are both zero when  $n$  is even, for any value of  $\nu$ . In order to find  $\mathcal{N}_{s'_1, n, \nu}^{2, \pm}$ , we need to calculate the ratio of the volume of the subset of  $G_i$  defined by the formula ' $\# \eta \neq 0, \text{ac}(-c^2 + a^2\varepsilon) = \eta^2$ ' to the total volume of  $G_{(0)}$ . Note that  $-c^2 + a^2\varepsilon = -(c^2 - a^2\varepsilon)$ . (This formula should be understood as an abbreviation. We should first consider the formula with an extra free variable  $\delta$ : ' $\# \eta \neq 0, \text{ac}(-c^2 + a^2\delta) = \eta^2$ ', do the motivic calculation, then plug in our value of  $\varepsilon$ . This is exactly what was done in Lemma 2.3, so we do not need to repeat this argument.) By Lemma 2.3, the expression  $c^2 - a^2\varepsilon$  is a square of a nonzero element for exactly half of the elements of  $G_i$ . Then this ratio is  $1/2$ . Therefore, in the both cases  $\nu = 1$  and  $\nu = -1$ , the answer is the same:

$$\mathcal{N}_{s'_1, n, \nu}^{2, \pm} = \frac{1}{2}[G_i a_{n/2} G_j / G_j] = \frac{1}{2} \mathbb{L}^{2|-n/2|} = \frac{1}{2} \mathbb{L}^n.$$

2. Suppose  $n$  is odd. In this case, we have

(a) If  $-(n+1)/2 < \lambda < (n+1)/2$  then

$$\rho_{j,-1/2}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0, 0).$$

(b) If  $\lambda = -(n+1)/2$  then the element  $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda$  is not in  $G_i$  for any  $h$ .

(c) If  $\lambda = (n+1)/2$  then

$$\rho_{j,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (x, 0),$$

with  $\text{sgn}(x) = \text{sgn}(u)\text{sgn}(\text{ac}(d^2 - \varepsilon b^2))$ .

Similarly to the previous case, we get:  $\mathcal{N}_{s'_1, n, \nu}^{2, \pm} = 0$ ;  $\mathcal{N}_{s_1, n, \nu}^{1, \pm} = \frac{1}{2} \mathbb{L}^n$  (again using lemma 2.3 to show that the volume of the subset of  $G_i$  defined by the formula ' $\# \eta \neq 0, \text{ac}(d^2 - b^2\varepsilon) = \eta^2$ ' equals half of the volume of  $G_i$ ).

The calculation in the case  $z = s_2$  follows the pattern of the case  $z = s_1$ , except there are the same additional complications as we saw in the case of the character. Not surprisingly, the answer is still the same as in the case of  $s_1$  except the roles of the cases  $n$  even and  $n$  odd are switched. The results of similar calculations for the remaining ramified cases are summarized in Tables 10 and 11.

The virtual Chow motives  $\mathcal{N}_{z, n, \nu}^0$  can be computed following arguments as above. However, since they do not appear in any of our further calculations (all the functions  $\kappa$  we are interested in vanish at the origin), we omit this calculation.  $\square$

**Corollary 3.6.** *Let  $\mathfrak{h}_z$  be ramified elliptic (so  $z \in \{t_0, t_1, t_2, t_3\}$ ). For each  $n \in \mathbb{N}$ , the Fourier transform  $\hat{\Phi}_G(X_z)$  of the orbital integral at  $X_z$  is constant on the set  ${}^G\mathfrak{h}_{z', n, \nu}$  for  $\nu = \pm$ ,  $z' \in \{s'_1, s'_2, t'_0, t'_1, t'_2, t'_3\}$  and a positive (resp., non-negative, if  $z' \in \{t'_0, t'_1, t'_2, t'_3\}$ ) integer  $n$ , and we have*

$$\begin{aligned} & \hat{\Phi}(X_z, \tilde{f}_{z', n, \nu}) \\ &= \hat{\phi}_{X_z}|_{V_0} \text{TrFrob} \mathcal{N}_{z', n, \nu}^0 + \sum_{\alpha=\pm} \hat{\phi}_{X_z}|_{V^{1, \nu}} \text{TrFrob} \mathcal{N}_{z', n, \nu}^{1, \alpha} + \sum_{\alpha=\pm} \hat{\phi}_{X_z}|_{V^{2, \nu}} \text{TrFrob} \mathcal{N}_{z', n, \nu}^{2, \alpha}, \end{aligned}$$

*Proof.* From the Section 3.2, we see that our elements  $X_z$  correspond to the point  $j = (01)$  in the building, and their depth is  $r = 1/2$ . First, recall that by Proposition 3.1, we have

$$\Phi(X_z, \tilde{f}_{z',n,\pm}) = \int_G \int_{\mathfrak{g}} \tilde{f}_{z',n,\pm}(gYg^{-1})(\hat{\varphi}_{\bar{X}_z})_{j,-r}(Y).$$

Note that the support of the functions  $\tilde{f}_{z',n,\pm}$  for all  $z$  and all  $n > 0$  is contained in  $\mathfrak{h}_{j,-r}$ , so the Proposition 3.1 is applicable. Now it remains to plug in the function  $\kappa = \hat{\varphi}_{\bar{X}_z}$ , which is now a  $GL(1, \mathbb{F}_q)$ -invariant function on  $\mathbb{A}^2(\mathbb{F}_q)$  in the Proposition 3.5 to complete the proof.  $\square$

TABLE 10. The virtual motives for the Fourier transform of orbital integrals at elements  $Y_{z,n}(u)$ , for  $z \in \{s'_1, s'_2\}$ . Here  $\nu = \text{sgn}(u)$ .

$z$	$\mathcal{N}_{z,n,\nu}^{1,\pm}$	$\mathcal{N}_{z,n,\nu}^{2,\pm}$
$s'_1$	$\mathcal{N}_{z,n,\nu}^{1,+} = \mathcal{N}_{z,n,\nu}^{1,-} = \frac{1}{2}\mathbb{L}^n$ $n$ odd	$\mathcal{N}_{z,n,\nu}^{2,+} = \mathcal{N}_{z,n,\nu}^{2,-} = 0$ $n$ odd
	$\mathcal{N}_{z,n,\nu}^{1,+} = \mathcal{N}_{z,n,\nu}^{1,-} = 0$ $n$ even	$\mathcal{N}_{z,n,\nu}^{2,+} = \mathcal{N}_{z,n,\nu}^{2,-} = \frac{1}{2}\mathbb{L}^n$ $n$ even
$s'_2$	$\mathcal{N}_{z,n,\nu}^{1,+} = \mathcal{N}_{z,n,\nu}^{1,-} = 0$ $n$ odd	$\mathcal{N}_{z,n,\nu}^{2,+} = \mathcal{N}_{z,n,\nu}^{2,-} = \frac{1}{2}\mathbb{L}^n$ $n$ odd
	$\mathcal{N}_{z,n,\nu}^{1,+} = \mathcal{N}_{z,n,\nu}^{1,-} = \frac{1}{2}\mathbb{L}^n$ $n$ even	$\mathcal{N}_{z,n,\nu}^{2,+} = \mathcal{N}_{z,n,\nu}^{2,-} = 0$ $n$ even

#### 4. MOTIVIC PROOF OF THE CHARACTER FORMULA

Now we are ready to prove Theorem 1.6, and to calculate the coefficients that appear in Table 4 in the process. Recall that Elliptic Character Formula is an equality of two distributions on the topologically nilpotent regular set in the Lie algebra. These distributions are represented by locally integrable functions, which are constant on the sets  ${}^G\mathfrak{h}_{z,n,\pm}$ , see Proposition 2.5, Corollary 3.4 and Corollary 3.6. We prove the Elliptic Character Formula by checking the equality on each of these sets.

Let  $\pi$  be a representation as in Section 1.1. We start with the case when  $\pi$  is obtained from a Deligne-Lusztig representation, where the proof is straightforward, and requires no consideration of separate cases. If  $\pi$  is of the type  $\pi(i)_\theta$  with  $i = (0)$  or  $i = (1)$ , then by Proposition 2.5, we have, for each  $z' \in \{s'_0, s'_1, s'_2, t'_0, t'_1, t'_2, t'_3\}$

$$\frac{1}{m({}^G\Gamma_{z',n,\nu})} \Theta_{\pi(i)_\theta}(f_{z',n,\nu}) = - \sum_U Q_T|_U \text{TrFrob} M_{z',n,\nu}^{i,U}.$$

On the other hand, for  $z = s_1$  or  $s_2$ , by Corollary 3.4,

$$\hat{\Phi}_G(X_z, \tilde{f}_{z',n,\pm}) = \sum_U \varphi_{\bar{X}_z} \text{TrFrob} M_{z',n,\pm}^{i,U},$$

TABLE 11. The virtual motives for the Fourier transform of orbital integrals at elements  $Y_{z,n}(u)$ , for  $z \in \{t'_0, t'_1, t'_2, t'_3\}$ . (Recall that  $\zeta^2 = \text{sgn}(-1)$  and  $\nu = \text{sgn}(u)$ .)

$z$	$\mathcal{N}_{z,n,\nu}^{1,\pm}$	$\mathcal{N}_{z,n,\nu}^{2,\pm}$
$t'_0$	$\mathcal{N}_{z,n,\nu}^{1,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{1,-} = 0$ , if $\zeta^{2n+2} = \nu$ $\mathcal{N}_{z,n,\nu}^{1,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{1,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}_{z,n,\nu}^{2,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{2,-} = 0$ if $\zeta^{2n+2} = \nu$ $\mathcal{N}_{z,n,\nu}^{2,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{2,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise
$t'_1$	$\mathcal{N}_{z,n,\nu}^{1,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{1,-} = 0$ , if $\zeta^{2n+2} = -\nu$ $\mathcal{N}_{z,n,\nu}^{1,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{1,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}_{z,n,\nu}^{2,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{2,-} = 0$ , if $\zeta^{2n+2} = -\nu$ $\mathcal{N}_{z,n,\nu}^{2,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{2,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise
$t'_2$	$\mathcal{N}_{z,n,\nu}^{1,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{1,-} = 0$ , if $\zeta^{2n+2} = (-1)^{n+1}\nu$ $\mathcal{N}_{z,n,\nu}^{1,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{1,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}_{z,n,\nu}^{2,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{2,-} = 0$ , if $\zeta^{2n+2} = (-1)^n\nu$ $\mathcal{N}_{z,n,\nu}^{2,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{2,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise
$t'_3$	$\mathcal{N}_{z,n,\nu}^{1,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{1,-} = 0$ , if $\zeta^{2n+2} = (-1)^n\nu$ $\mathcal{N}_{z,n,\nu}^{1,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{1,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}_{z,n,\nu}^{2,+} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}_{z,n,\nu}^{2,-} = 0$ , if $\zeta^{2n+2} = (-1)^{n+1}\nu$ $\mathcal{N}_{z,n,\nu}^{2,+} = 0$ and $\mathcal{N}_{z,n,\nu}^{2,-} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise

where  $i = (0)$  if  $z = s_1$ , and  $i = (1)$  if  $z = s_2$ . As shown in Sections 3.2.1 and 3.2.2,

$$Q_T = (1 - q)\hat{\varphi}_{\bar{X}_{s_1}} = (1 - q)\hat{\varphi}_{\bar{X}_{s_2}}.$$

It follows immediately that

$$(4.1) \quad \frac{1}{m({}^G\mathfrak{h}_{z',n,\nu})} \Theta_\pi(\tilde{f}_{z',n,\nu} \circ \exp) = \begin{cases} (1-q)\hat{\Phi}_G(X_{s_1}, \tilde{f}_{z',n,\nu}) & \text{if } \pi = \pi(0)_\theta \\ (1-q)\hat{\Phi}_G(X_{s_2}, \tilde{f}_{z',n,\nu}) & \text{if } \pi = \pi(1)_\theta \end{cases}$$

This proves the theorem in the case when  $\pi$  comes from a Deligne-Lusztig representation.

Let us now turn to the non-Deligne-Lusztig case. Let  $\pi = \pi(0)_+$  or  $\pi = \pi(1)_+$  (the other two cases can be obtained by changing the sign in front of  $Q_G$  everywhere below). Then

$$(4.2) \quad \begin{aligned} & \frac{1}{m({}^G\mathfrak{h}_{z',n,\nu})} \Theta_\pi(\tilde{f}_{z',n,\nu} \circ \exp) \\ &= -\frac{1}{2} \sum_U Q_T(U) \text{TrFrob} M_{z',n,\nu}^{i,U} - \frac{1}{2} \sum_U Q_G(U) \text{TrFrob} M_{z',n,\nu}^{i,U}. \end{aligned}$$

As we have seen in the equation (4.1), the first term (i.e. the part of the character that comes from inflating  $Q_T$ ) is a multiple of  $\hat{\Phi}_G(X_{s_1}, \tilde{f}_{z',n,\nu})$  in the case  $\pi = \pi(0)_+$ , and of  $\hat{\Phi}_G(X_{s_2}, \tilde{f}_{z',n,\nu})$  in the case  $\pi = \pi(1)_+$ . The coefficient is  $\frac{q-1}{2}$  since on the left  $Q_T$  appears with the coefficient  $-\frac{1}{2}$ .

It remains to express the second term,  $-\frac{1}{2} \sum_U Q_G(U) \text{TrFrob} M_{z',n,\nu}^{i,U}$ , as a linear combination of the Fourier transforms of orbital integrals. In order to do that, recall the functions  $\varphi_{(0)}$  and  $\varphi_{(1)}$  defined Section 3.3. By Corollary 3.6, if  $z'$  is elliptic, we have

$$(4.3) \quad \begin{aligned} & \sum_{\substack{l=1,2 \\ \alpha=\pm}} \hat{\varphi}_{(0)}|_{V^{l,\alpha}} \text{TrFrob} \mathcal{N}_{z',n,\nu}^{l,\alpha} \\ &= \hat{\Phi}_G(X_{t_0}, f_{z',n,\nu}) - \hat{\Phi}_G(X_{t_1}, f_{z',n,\nu}) + \hat{\Phi}_G(X_{t_2}, f_{z',n,\nu}) - \hat{\Phi}_G(X_{t_3}, f_{z',n,\nu}) \end{aligned}$$

and

$$(4.4) \quad \begin{aligned} & \sum_{\substack{l=1,2 \\ \alpha=\pm}} \hat{\varphi}_{(1)}|_{V^{l,\alpha}} \text{TrFrob} \mathcal{N}_{z',n,\nu}^{l,\alpha} \\ &= \hat{\Phi}_G(X_{t_0}, f_{z',n,\nu}) - \hat{\Phi}_G(X_{t_1}, f_{z',n,\nu}) - \hat{\Phi}_G(X_{t_2}, f_{z',n,\nu}) + \hat{\Phi}_G(X_{t_3}, f_{z',n,\nu}). \end{aligned}$$

Also, if  $z' = s_0$ , then, since the both functions  $\hat{\varphi}_{(0)}$  and  $\hat{\varphi}_{(1)}$  vanish at the origin and take the opposite values on the orbits  $V^{1,+}, V^{1,-}$ , and  $V^{2,+}, V^{2,-}$ , it is easy to see in a way that mimics the proof of Proposition 2.5 in the case  $z = s'_0$ , that the right-hand side of equation (4.3) and (4.4) vanishes on  $\mathfrak{h}_{s'_0,n}$  for  $n > 0$ .

We claim that for any  $z$  and any positive (resp., non-negative)  $n$ ,

$$(4.5) \quad -\frac{1}{2} \sum_U Q_G(U) \text{TrFrob} M_{z',n,\nu}^{i,U} = c \sum_{\substack{l=1,2 \\ \alpha=\pm}} \hat{\varphi}_i|_{V^{l,\alpha}} \text{TrFrob} \mathcal{N}_{z,n,\nu}^{l,\alpha},$$

with some constant  $c$  that we will calculate below (we will see that  $c = -\frac{q^2-1}{8q}$ ). Note that the left-hand side depends on the choice of the vertex  $i$ . On the right, it is the constant  $c$  and the function  $\hat{\varphi}_i$  that depend on  $i$ . This equation could be called the motivic version of our character formula. It is the core of the proof – here we are comparing the inflation of two functions that live on different reductive quotients. Note that once we prove this claim, Theorem 1.6 will follow immediately. On the other hand, the proof of the claim is automatic: all we need to do is plug in

the values of the functions  $\varphi_i$  from Section 3.3, the values of  $Q_G$  from Section 1.3, and the motivic coefficients from Tables 6 and 7 if  $i = (0)$  and Tables 8 and 9 if  $i = (1)$  on the left, and from Tables 10 and 11 on the right.

The equality (4.5) has to be checked on each of the sets  $\mathfrak{h}_{z,n,\nu}$  (recall that  $\nu = \pm$ ).

Observe that the function  $\hat{\varphi}_{(0)}$  vanishes at  $V^{1,\pm}$ , and the function  $\hat{\varphi}_{(1)}$  vanishes at  $V^{2,\pm}$ , so that the right-hand side in any case has only two nonzero terms. On the left, since  $Q_G$  vanishes at the identity, there are also only two nonzero terms.

For  $z = s'_0$  the equality we want to prove is trivial, since the both sides vanish, as discussed above. For  $z = s'_1$  and  $z = s'_2$ , the equality also turns out to be trivial. On the right, the two nonzero opposite values of the function  $\hat{\varphi}_{(0)}$  or  $\hat{\varphi}_{(1)}$  appear with the same coefficient, so the right-hand side equals zero. On the left, the function  $Q_G$  takes opposite values on  $U_1$  and  $U_\varepsilon$ , and they also appear on the left-hand side with equal coefficients, so the left-hand side of equation (4.5) is also zero. This, however, gives no information of the constant  $c$ .

For  $z \in \{t'_0, t'_1, t'_2, t'_3\}$ , we see from Tables 10 and 11 that only one of the nonzero values of  $\hat{\varphi}_i$  appears with a nonzero coefficient, and we see from Tables 6, 7, 8 and 9 that also only one nonzero term appears on the left. The left-hand side of equation (4.5) equals  $-\frac{1}{2}\sqrt{q}\zeta^3 q^n \text{SIGN}_1$ , where  $\text{SIGN}_1 = 1$  if the coefficient  $M_{z,n,\nu}^{i,U_1}$  is nonzero, and  $\text{SIGN}_1 = -1$  if  $M_{z,n,\nu}^{i,U_\varepsilon}$  is nonzero.

The right-hand side equals:

$$\begin{cases} c \frac{4}{q-1} \sqrt{q} \zeta \frac{q^{n+1}}{q+1} \text{SIGN}_2 & \text{if } i = (0), \\ c \frac{4}{q-1} \sqrt{q} \zeta^3 \frac{q^{n+1}}{q+1} \text{SIGN}_2 & \text{if } i = (1), \end{cases}$$

where  $\text{SIGN}_2$  is the sign that depends on which one of the virtual Chow motives  $\mathcal{N}^{1,2,\pm}$  is nonzero. Let  $c = -\frac{q^2-1}{8q}$ , which is the constant that appears in Table 4. Then, to finish the proof of the theorem it remains to show that on every set  $\mathfrak{h}_{z,n,\nu}$  with  $z \in \{t'_0, \dots, t'_3\}$ , we have the identity  $\text{SIGN}_1 = \zeta^2 \text{SIGN}_2$  in the case  $i = (0)$ , and  $\text{SIGN}_1 = \text{SIGN}_2$  in the case  $i = (1)$ . Here is the comparison of the two SIGNs in the case  $i = (0)$ :

$$\begin{array}{ll} z = t_0 : & \text{SIGN}_1 = \nu \zeta^{2n} & \text{SIGN}_2 = \zeta^{2(n+1)}. \\ z = t_1 : & \text{SIGN}_1 = -\nu \zeta^{2n} & \text{SIGN}_2 = -\nu \zeta^{2(n+1)}. \\ z = t_2 : & \text{SIGN}_1 = \nu \zeta^{2n} (-1)^n & \text{SIGN}_2 = \nu \zeta^{2(n+1)} (-1)^n. \\ z = t_3 : & \text{SIGN}_1 = \nu \zeta^{2n} (-1)^{n+1} & \text{SIGN}_2 = \nu \zeta^{2(n+1)} (-1)^{n+1}. \end{array}$$

We see that in all cases  $\text{SIGN}_1 = \zeta^2 \text{SIGN}_2$ , which completes the proof in the case  $i = (0)$ .

The case  $i = (1)$  is identical, except we need to use Tables 8 and 9 instead of Tables 6 and 7 to calculate  $\text{SIGN}_1$ , and the first column of Tables 10 and 11 instead of the second column, to calculate  $\text{SIGN}_2$ .

## 5. FINAL COMMENTS

Let us return to the choices we made in order to state Theorem 1.6. We began with an odd prime  $p$ , a  $p$ -adic field  $\mathbb{K}$ , a prime  $\ell$  different from  $p$  (e.g.,  $\ell = 2$ ), and an algebraic closure  $\mathbb{Q}_\ell$  of  $\mathbb{Q}_\ell$  in the pre-ambule to Section 1.

Reluctantly, in Section 1.1, we fixed a uniformizer  $\varpi$  for  $\mathbb{K}$ , in order to be precise about our depth zero representations.

Next, in Section 1.2 we fixed an additive character  $\bar{\psi} : \mathbb{F}_q \rightarrow \bar{\mathbb{Q}}_\ell$  and a square root  $\sqrt{q}$  of  $q$  in  $\bar{\mathbb{Q}}_\ell$ . These choices (via Gauss sums) determined a fourth-root of unity  $\zeta \in \bar{\mathbb{Q}}_\ell$  such that  $\zeta^2 = \text{sgn}(-1)$  (see Remark 1.2). This, in turn, determined how we labelled the two representations  $\sigma_+$  and  $\sigma_-$  in the Lusztig series for  $(T, \theta_0)$  and therefore our definition of  $Q_G$  (see Section 1.3), and therefore our definition of  $\pi(0)_+$ ,  $\pi(0)_-$ ,  $\pi(1)_+$  and  $\pi(1)_-$  (see Remark 1.3).

Independently, we fixed cocycles  $\{s_1, s_2, t_0, t_1, t_2, t_3\}$  in  $Z^1(\mathbb{K}, N)$  such that their cohomology classes lay in the kernel of the map  $H^1(\mathbb{K}, N) \rightarrow H^1(\mathbb{K}, G)$  induced by inclusion  $N \rightarrow G$ . This choice determined a non-square unit  $\varepsilon$  in  $\mathcal{O}_{\mathbb{K}}$  (see Remark 1.5). We remind the reader that the cohomology class for  $t_0$  equal the cohomology class for  $t_1$  and the cohomology class for  $t_2$  equal the cohomology class for  $t_3$  if  $\text{sgn}(-1) = -1$ .

Finally, for each  $z \in \{s_1, s_2, t_0, t_1, t_2, t_3\}$  we chose an element  $X_z \in \mathfrak{g}$  with depth zero in its Cartan subalgebra; they are listed in Table 3. This last step amounted to the choice of a unit  $v$  in  $\mathcal{O}_{\mathbb{K}}$  (see Section 1.8).

With all these choices we made, the values of the coefficients  $c_z(\pi)$  in our Elliptic Character Formula are presented in Table 4. (Also, the Fourier transform of the orbital integral  $\Phi_G(X_z, -)$  is taken with respect to an additive character  $\psi : \mathbb{K} \rightarrow \bar{\mathbb{Q}}_\ell$  with conductor  $\mathcal{O}_{\mathbb{K}}$  such that the induced additive character of  $\mathbb{F}_q$  is  $\bar{\psi}$ . Also, we must use compatible Killing forms, and the correct measures everywhere, as we did.) From Table 4 we see that the coefficients  $c_z(\pi)$  are all rational. Further, they are rational functions of  $q$ , given correctly for all primes except  $p = 2$ .

It is reasonable to ask if these properties continue to hold had we made different choices above. In particular, we wish to say a few words about how the Elliptic Character Formula changes if the definition of the  $X_z$ s were modified. Since we have gathered complete information about the Fourier transforms of all regular elliptic orbital integrals  $\hat{\Phi}_G(X, -)$  (when  $X$  has depth zero in its Cartan subalgebra) evaluated at topologically nilpotent elements  $Y$ , we can explore the dependence of the coefficients in the Elliptic Character Formula of the orbits we chose. (Of course, each  $X_z$  can be replaced by any element in the same adjoint orbit as  $X_z$ , since the distributions  $\hat{\Phi}_G(X_z, -)$  and  $\Theta_\pi$  are invariant under the group action.)

To begin, consider the local constancy of  $X \mapsto \hat{\Phi}_G(X, f)$  for a fixed Schwartz function  $f$  supported by topologically nilpotent elements. Using equations (3.6) and (2.1) we see that

$$(5.1) \quad \int_G \int_{\mathfrak{g}} f(\text{Ad}(g)Y) (Q_T)_{(0),0}(Y) dY dg = (1 - q) \hat{\Phi}_G(X_{s_1}(v), f).$$

Moreover, since  $Q_T$  (see Section 1.3) does not depend on  $v \in \mathcal{O}_{\mathbb{K}}^*$  it follows that  $\hat{\Phi}_G(X_{s_1}(v), f)$  is independent of  $v$ ; thus, for any  $f$  as above,

$$(5.2) \quad \int_{\mathcal{O}_{\mathbb{K}}^*} \hat{\Phi}_G(X_z(v), f) dv = (q - 1) \hat{\Phi}_G(X_{s_1}, f),$$

where the Haar measure on  $\mathcal{O}_{\mathbb{K}}^*$  is chosen so that  $\mathcal{O}_{\mathbb{K}}^*$  has measure  $q - 1$ .

From our point of view, it would have been more natural to re-write equation (5.1) (and therefore the first line in Table 4) in the form

$$(5.3) \quad \Theta_{\pi(0)_\theta}(f) = \int_{\mathcal{O}_{\mathbb{K}}^*} \hat{\Phi}_G(X_{s_1}(v), f) dv,$$

from which we see that our choice for  $v$  was completely unimportant here. (See Sections 1.1 and 1.3 for the definition of  $\pi(0)_\theta$ .) Similarly, we can re-write the last line of Table 4 in the form

$$(5.4) \quad \Theta_{\pi(1)_\theta}(f) = \int_{\mathcal{O}_{\mathbb{K}}^*} \hat{\Phi}_G(X_{s_2}(v), f) dv.$$

These expressions depend on our choice for the cocycles  $s_1$  and  $s_2$  only. Therefore, once the cocycle  $s_1$  is chosen, the element  $X_{s_1}$  may be replaced by *any* element in the Cartan uniquely determined by the cocycle  $s_1$  as long as that element has depth zero in that Cartan; likewise, for the cocycle  $s_2$ .

Regarding the totally ramified Cartan subgroups, things are a bit more subtle. From Table 4 we see that if  $f$  is a Schwartz function supported by topologically nilpotent elements, then

$$\begin{aligned} & \int_G \int_{\mathfrak{g}} f(\text{Ad}(g)Y) (Q_G)_{(01), \frac{1}{2}}(Y) dY dg \\ &= \frac{q^2 - 1}{4q} \left( \hat{\Phi}_G(X_{t_0}, f) - \hat{\Phi}_G(X_{t_1}, f) + \hat{\Phi}_G(X_{t_2}, f) - \hat{\Phi}_G(X_{t_3}, f) \right). \end{aligned}$$

Invoking the local constancy of  $X \mapsto \hat{\Phi}_G(X, f)$  we may re-write the equation above in the form

$$\begin{aligned} & \frac{2q}{q+1} \int_G \int_{\mathfrak{g}} f(\text{Ad}(g)Y) (Q_G)_{(0), 0}(Y) dY dg \\ &= \int_{\{\text{sgn}(v)=+1\}} \hat{\Phi}_G(X_{t_0}(v), f) dv - \int_{\{\text{sgn}(v)=+1\}} \hat{\Phi}_G(X_{t_1}(v), f) dv \\ &+ \int_{\{\text{sgn}(v)=+1\}} \hat{\Phi}_G(X_{t_2}(v), f) dv - \int_{\{\text{sgn}(v)=+1\}} \hat{\Phi}_G(X_{t_3}(v), f) dv. \end{aligned}$$

Similar observations apply to representations induced from  $G_{(1)}$ . In this way we see that only  $\text{sgn}(v)$  was important in our definition of the ramified orbits appearing in Theorem 1.6. More precisely, once the cocycle  $t_0$  is chosen, the element  $X_{t_0}$  may be replaced by  $X_{t_0}(v)$  as long as  $\text{sgn}(v) = 1$ ; likewise for the cocycles  $t_1$ ,  $t_2$  and  $t_3$ . However, if  $\text{sgn}(v) = -1$ , then the function  $Y \rightarrow \hat{\Phi}_G(X_z(v), Y)$  is *not* equal to the function  $Y \rightarrow \hat{\Phi}_G(X_z(v), Y)$  (these are functions on the set of regular topologically nilpotent elements of  $\mathfrak{g}$ ).

Having dealt with  $v$  we return to our choice of non-square unit  $\varepsilon$ . From the local constancy of the Fourier transform of the orbital integrals we have the following immediate consequence. For any Schwartz function  $f$  supported by topologically nilpotent elements of  $\mathfrak{g}$ ,

$$(5.5) \quad \int_{\{\text{sgn}(\delta)=-1\}} \hat{\Phi}_G\left(\begin{bmatrix} 0 & 1 \\ \delta\varepsilon & 0 \end{bmatrix}, f\right) d\delta = \frac{q-1}{2} \hat{\Phi}_G(X_{t_2}, f).$$

Similar observations hold for all our orbits. Consequently, and especially from the point of view of motivic integration, we could have replaced each of our orbital integral with an integral over  $v$  and  $\delta$  ( $\varepsilon$  is a particular value of  $\delta$ ). In fact, that is exactly what we did when we used motivic integration with parameters in Section 2.3.2 and allowed the parameter to vary over the set of all non-squares in the

residue field. The result is easily related back to orbital integrals appearing in representation theory because of relations of the following form.

$$(5.6) \quad \int_{\{\text{sgn}(\delta)=-1\}} \int_{\{\text{sgn}(v)=+1\}} \hat{\Phi}_G([\delta \begin{smallmatrix} 0 & \\ \varpi v & v \end{smallmatrix}], f) dv d\delta = \frac{(q-1)^2}{4} \hat{\Phi}_G(X_{t_2}, f).$$

Similar observations hold for all our orbits. Consequently, replacing orbital integrals with these ‘smeared’ orbital integrals would have the effect of changing the coefficients in the elliptic character expansion in very simple ways.

What *is* crucial in the elliptic character expansion is the pattern of the signs appearing in front of the orbits indexed by the cocycles  $t_0, t_1, t_2$  and  $t_3$ . However, without careful choices for  $X_z$  it is *false* that the coefficients of an elliptic character expansion are rational functions in  $q$ . To illustrate the point, suppose we replace our  $X_z$ s with the elements appearing in Table 12. Then the coefficients in the elliptic character expansion involve  $\text{sgn}_q(-1)$ , as we see from Table 13, where we write  $\zeta^2$  for  $\text{sgn}(-1)$ , to save space. Observe that  $\text{sgn}(-1)$  cannot be expressed as a rational function in  $q$ . The point we are making here is that, in this case, although  $c_{z'}(\pi)$  is a rational number, it is *not* a rational function in  $q$ . However, since we have been switching freely between polynomials in  $q$  and virtual Chow motives, we can notice that the coefficients  $c_z(\pi)$ , as well as  $c_{z'}(\pi)$  can be interpreted as elements of the ring  $\text{Mot}$ . Indeed, if we replace the polynomials in  $q$  with the corresponding elements of the ring  $\text{Mot}$  as we have been doing so far, we can see that the denominators of  $c_z(\pi)$  and  $c_{z'}(\pi)$  are invertible in the ring  $\text{Mot}$ . Nothing more is needed for the coefficients  $c_z(\pi)$ . For  $c_{z'}(\pi)$ , we still need to find the motivic expression for  $\zeta^2$ . This is easily done by considering the 0-dimensional variety defined by the equation  $x^2 = -1$  (this variety appears for the same purpose in [Hb05]). Let  $\mathbb{S}$  denote its class in the ring  $\text{Mot}$ . Then  $\zeta^2 = -\text{TrFrob}(1 - \mathbb{S})$ . As Table 13 shows, this expression appears only when we study depth zero supercuspidal representations which are not induced from Deligne-Lusztig representations.

TABLE 12. Elements  $X_{z'}$  *not* appearing in Theorem 1.6.

$z'$	$s'_1$	$s'_2$	$t'_0$	$t'_1$	$t'_2$	$t'_3$
$X_{z'}$	$\begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^{-1} \\ \varepsilon \varpi & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon \\ \varpi \varepsilon^{-1} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ \varpi \varepsilon & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon \\ \varpi & 0 \end{bmatrix}$

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TABLE 13. Coefficients  $c_{z'}(\pi)$  *not* appearing in Theorem 1.6.

$c_{z'}(\pi)$	$s'_1$	$t'_0$	$t'_1$	$t'_2$	$t'_3$	$s'_2$
$\pi(0)_\theta$	$q-1$	0	0	0	0	0
$\pi(0)_+$	$\frac{q-1}{2}$	$-\frac{q^2-1}{8q}\zeta^2$	$+\frac{q^2-1}{8q}\zeta^2$	$-\frac{q^2-1}{8q}\zeta^2$	$+\frac{q^2-1}{8q}\zeta^2$	0
$\pi(0)_-$	$\frac{q-1}{2}$	$+\frac{q^2-1}{8q}\zeta^2$	$-\frac{q^2-1}{8q}\zeta^2$	$+\frac{q^2-1}{8q}\zeta^2$	$-\frac{q^2-1}{8q}\zeta^2$	0
$\pi(1)_+$	0	$-\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$\frac{q-1}{2}$
$\pi(1)_-$	0	$+\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$-\frac{q^2-1}{8q}$	$+\frac{q^2-1}{8q}$	$\frac{q-1}{2}$
$\pi(1)_\theta$	0	0	0	0	0	$q-1$

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