ON THE TRANSVERSAL HELLY NUMBERS OF DISJOINT AND OVERLAPPING DISKS

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In honour of Helge Tverberg's seventieth birthday

ABSTRACT. A family of disks is said to have the property $T(k)$ if any k members of the family have a common line transversal. We call a family of unit diameter disks t-disjoint if the distances between the centers are greater than t. We consider for each natural number $k \geq 3$, the infimum t_k of the distances t for which any finite family of t-disjoint unit diameter disks with the property $T(k)$ has a line transversal. We determine exact values of t_3 and t_4, and give general lower and upper bounds on the sequence t_k, showing that $t_k = O(1/k)$ as $k \to \infty$.

1. INTRODUCTION

Transversal properties of families of disjoint unit disks have been studied by a number of authors (see references [1]–[15]) with special attention to Helly type problems. For a more recent survey on geometric transversals, we refer to B. Wenger [16].

Consider a finite family $\mathcal{F}$ of closed solid circles of diameter 1, called *disks*, in the plane. We say that the family is *t-disjoint* if the distance between every pair of centers is larger than $t > 0$. For $t = 1$, the disks are disjoint in the original sense. If $t > 1$ or $t < 1$, we say that the disks of the family are *superdisjoint* or *overlapping*, respectively. A family of disks is said to have property T if the family has a (*line*) *transversal*: a straight line intersecting all members of the family. The family is called a *$T(k)$-family* if any k members of the family have a common transversal. If for families of a certain type, property $T(k)$ implies property T but property $T(k-1)$ does not, we say that families of that type have transversal Helly number k.

Danzer [4] proved that for a disjoint family of disks $T(5) \Rightarrow T$ but $T(4) \not\Rightarrow T$, (see also [1]), thus disjoint families of disks have transversal Helly number 5. Presently, similar problems will be considered for superdisjoint and overlapping families of disks. Our concern is to investigate the relation between t-disjointness and property $T(k)$. It has been found that a lower level of disjointness can be compensated by a higher transversal Helly number; that is, a smaller t by a larger k. We mention that Danzer's result was extended for disjoint translates of an arbitrary compact convex set in the plane by H. Tverberg [15]. This leads us to the following quite general problem.

1991 *Mathematics Subject Classification.* 52A35, 52A37, 52A10.

Key words and phrases. line transversal, width, disk.

The transversal problem for t-disjoint families in normed spaces. *Let B be a convex body in $\mathbb{R}^d$. We say that a family of translates of B, $\{B + \mathbf{x}_i \mid i \in I\}$ is a t-disjoint family if the length of the difference $\mathbf{x}_i - \mathbf{x}_j$ measured in the norm generated by $B + (-B)$ is larger than $t > 0$ for every pair of indices $i, j \in I$, $i \neq j$. Let $0 < l < d$ be positive integers given. Then determine the smallest real number $t_k(B, l)$ (if it exists at all) with the following property: For any $t_k(B, l)$-disjoint family of translates of B the condition that any k members have an l-dimensional affine subspace intersecting them implies the existence of an l-dimensional affine subspace that intersects all members of the family in $\mathbb{R}^d$.*

In this paper we focus on the case of 2-dimensional Euclidean disks of diameter 1. For this it is useful to consider the set of centers rather than the family of disks and interpret the $T(k)$ property as the requirement that any k centers are covered by a strip of width 1.

The *width* $w(K)$ of a compact planar set K is the minimum of the distances between pairs of its parallel support lines. A compact set and its convex hull have the same width. If K is a finite set then clearly, only directions parallel with sides of the convex hull $conv(K)$ have to be considered. If PQ is a side of a convex polygon $conv(K)$, and R is a vertex on the opposite support line parallel to PQ, then the triangle PQR is called a *bridge triangle* with *base* PQ and *opposite vertex* R. A bridge triangle is said to be *stable* if the orthogonal projection of the opposite vertex R onto the line $\langle P, Q\rangle$ belongs to the closed segment PQ, otherwise, we shall call it *unstable*. Every bridge triangle (with given base side) spans a unique strip bounded by the support lines parallel to the base side. If a strip, covering a polygon has no stable bridge triangle spanning it, then it does not yield the minimum width of the polygon. Indeed, if a strip covering a finite set K is bounded by the parallel lines e and f and it has no stable bridge triangle, then $e \cap K$ and $f \cap K$ can be separated weakly by a straight line EF, $(E \in e,\ F \in f)$, orthogonal to e and f. Then one can choose a covering strip of K with boundary lines going through E and F respectively, which is different from the strip between e and f. This strip will be narrower than the original one.

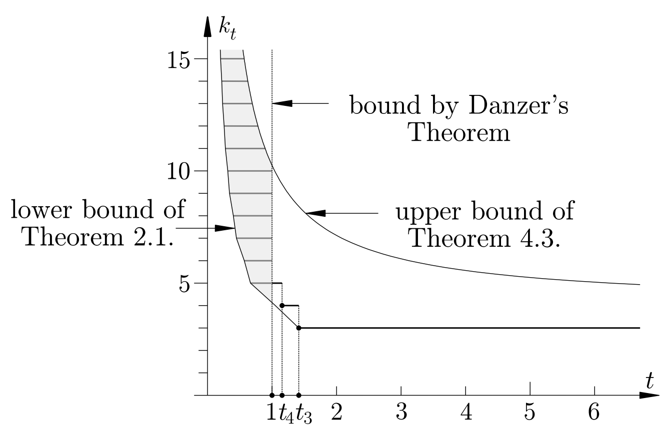

FIGURE 1. The unknown part of the graph of the function k_t is covered by the horizontal lines in the shaded domain.

In the present paper, we show that for every positive number $t > 0$, t-disjoint families of disks have a finite transversal Helly number k_t. We give general lower and upper bounds for this monotone function of t, and in some cases, we determine sharp values.

The function k_t is completely described by the sequence of its points of discontinuity $t_k = \inf\{t \,|k_t \le k\}$, $k \ge 3$.

In Section 2, we give a general lower bound for t_k, which is asymptotically equal to π/k and we show that the bound for t_3 is sharp. This will yield $t_3 = \sqrt{2}$. We remark that the value of t_3 was determined also by B. Grünbaum in [7]. The methods of that paper are based on the topological Helly theorem and are different from ours. In [7], Grünbaum studied t-disjoint families for $t > 1$ and called these families t-thin. In Section 3, we prove that $t_4 = 2/\sqrt{3}$. In Section 4, we give a general upper bound for t_k, which is asymptotically equal to c/k with $c \approx 6.2508$. This means that the ratio between our upper and lower bounds is less than 2 if k is large.

By these theorems, all *superdisjoint* cases are settled. Danzer's proof that $T(5) \Rightarrow T$ for 1-disjoint families suggests that from $k \ge 5$, we are in the *overlapping* range. This would show that $t_k = 1$ never holds; that is, there is *no* k for which common *overlapping* provides the "dividing line".

The results of the paper are depicted in Figure 1.

2. General lower bound for t_k and the exact value of t_3

Theorem 2.1. *If $k \ge 3$ is odd then*

$$t_k \ge s_{k+1} = \frac{2\sin\frac{\pi}{k+1}}{1+\cos\frac{2\pi}{k+1}},$$

and if $k \ge 4$ is even then

$$t_k \ge s_{k+1} = \frac{2\sin\frac{\pi}{k+1}}{\cos\frac{2\pi}{k+1}+\cos\frac{\pi}{k+1}}.$$

Proof. Let $R(k+1)$ be a regular $(k+1)$-gon of side length s_{k+1}, where s_{k+1} is the largest value such that a strip of width 1 covers k vertices of $R(k+1)$ (see Fig. 2). Clearly, the family of disks centered at the $k+1$ vertices of $R(k+1)$ has property

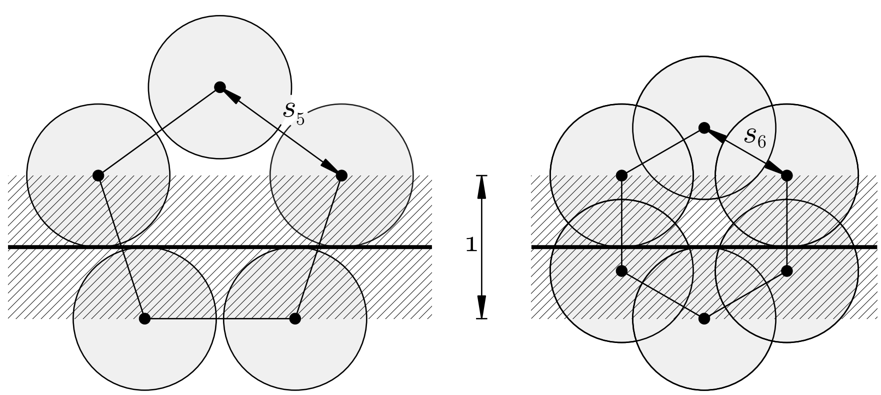

Figure 2

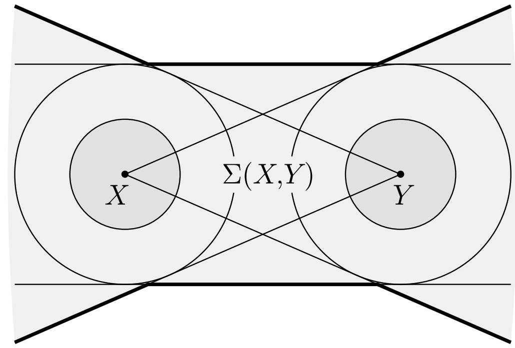

FIGURE 3. Center sheaf, the locus of the center of a third disk with a common transversal

$T(k)$ and it does not have a line transversal. Since this family is $(s_{k+1} - \varepsilon)$-disjoint for any positive $\varepsilon > 0$, $t_k \geq s_{k+1}$. Explicit computation yields the indicated values of s_{k+1}. □

Remark 2.2. A similar construction in [9] shows that the disjoint property of the disks in Danzer's theorem cannot be omitted.

The lower bound given in the theorem is asymptotically equal to $s_{k+1} \sim \pi/k$. For $k = 3$, it yields $t_3 \geq \sqrt{2}$. We show that this bound is sharp. For the proof, we need some preparatory results.

Definition. The *center sheaf* $\Sigma(X, Y)$ of two distinct points X and Y is the locus of the center of a disk D such that D and the disks with centers X and Y have a common line transversal. A center sheaf is bounded by parts of (at most) six lines: two of them non-separating common tangents to the two disks, of radius 1, concentric with the given disks, and four further lines that pass through X or Y and are tangent to the other disk of radius 1 (see Fig. 3).

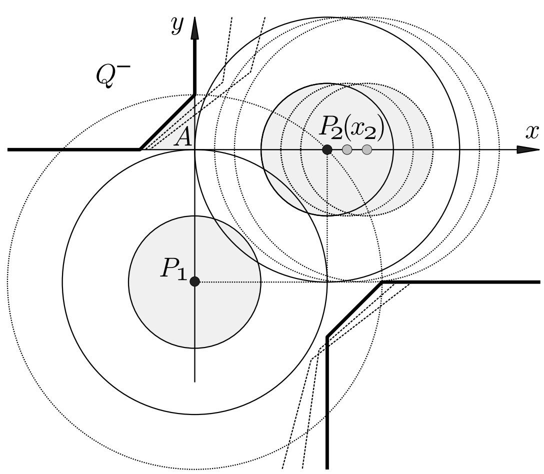

FIGURE 4. Domain $A(x_2)$ is maximal for $x_2 = 1$

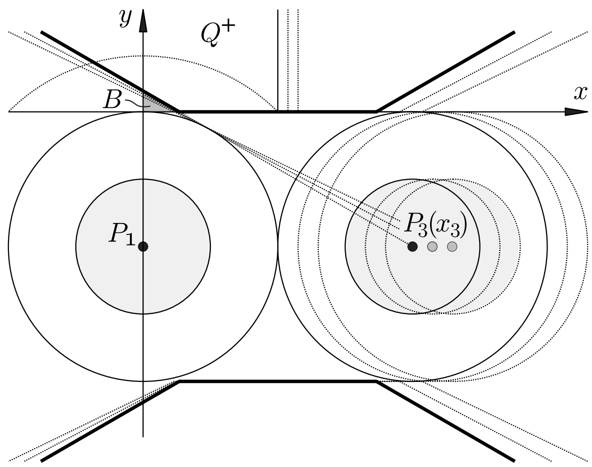

FIGURE 5. Domain $B(x_3)$ is maximal for $x_3 = 2$

Lemma 2.1. *Let $P_1(0,-1)$, $P_2(x_2,0)$, $x_2 \geq 1$, and denote by $A(x_2)$, the intersection of the quadrant $Q^- = \{(x,y) \mid x \leq 0 \leq y\}$ and $\Sigma(P_1,P_2)$. Then $intA(1)$ is covered by the disk of radius $\sqrt{2}$ about P_1 and $x_2 > 1$ implies $A(x_2) \subset A(1)$.*

Proof. The claim easily follows from the fact that, of the boundary lines of $\Sigma(P_1,P_2)$, only the common upper tangent of the disks of radius 1 about P_1 and P_2 has points in common with the open quadrant Q^-, (see Fig. 4). □

Lemma 2.2. *Let $P_1(0,-1)$, $P_3(x_3,-1)$, $x_3 \geq 2$, and denote by $B(x_3)$, the intersection of the half-strip $Q^+ = \{(x,y) \mid 0 \leq x \leq x_3/2,\ y \geq 0\}$ and $\Sigma(P_1,P_3)$. Then $intB(2)$ is covered by the closed disk of radius $\sqrt{2}$ about P_1, and $x_3 > 2$ implies $B(x_3) \subset B(2)$.*

Proof. Consider the initial position $P_3(2,-1)$. Of the boundary lines of $\Sigma(P_1,P_3)$, only the upper tangent of the disk of radius 1 about P_1 and through P_3 cuts into the half-strip Q^+ (see Fig. 5). □

Theorem 2.3. *Let a $\sqrt{2}$-disjoint family of unit diameter disks have the property $T(3)$. Then it has the property T.*

Proof. Consider for each triple of centers, the narrowest covering strip and denote by w the maximum of the width of these strips. By the $T(3)$ property, $w \leq 1$. If $w < 1$, then replace all disks by a disk of radius $w/2$ and rescale to get a family of disks of diameter 1. It is clear that the new family of disks has the $T(3)$ property, and if it has a transversal then so has the original family. Thus, we may assume without loss of generality that $w = 1$.

Let X_1, X_2 and Y be three centers whose narrowest covering strip S has maximal width 1. Suppose that X_1 and X_2 are on one boundary line of S and Y is on the other. Since S is the narrowest strip covering the triangle X_1X_2Y, the orthogonal projection of Y on the line $\langle X_1, X_2\rangle$ is between X_1 and X_2. Each of the other disks share a transversal with any two of the disks around X_1, X_2 and Y, and thus, their centers lie in the intersection of $\Sigma(X_1,X_2)$, $\Sigma(X_1,Y)$ and $\Sigma(X_2,Y)$. On

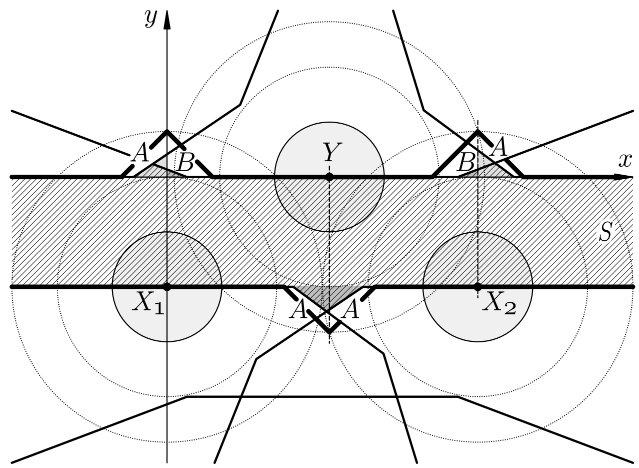

FIGURE 6. The widest of the narrowest strips covering three centers

the other hand, since the family is $\sqrt{2}$-disjoint, all other centers lie outside of the disks of radius $\sqrt{2}$ about X_1, X_2 and Y. Figure 6 depicts the typical shape of $\Sigma(X_1, X_2) \cap \Sigma(X_1, Y) \cap \Sigma(X_2, Y)$, (the hatched domain). Applying Lemma 2.1 at the regions labelled by A and Lemma 2.2 at regions labelled by B in the figure, all points of this intersection, which are not in the strip S, are covered by the disks of radius $\sqrt{2}$ around X_1, X_2 and Y. Therefore, none of these points can be the center of a disk of the family, and thus, the axis of S is a transversal of the family. □

3. THE EXACT VALUE OF t_4

Theorem 3.1. *Let a $2/\sqrt{3}$-disjoint family of unit diameter disks have the property $T(4)$. Then it has the property T. On the other hand, if $t < 2/\sqrt{3}$, then there exists a t-disjoint family of unit diameter disks with the property $T(4)$, which does not have the property T.*

The proof is split into two parts.

Proposition 3.1. $t_4 \geq 2/\sqrt{3}$.

Proof. The inequality follows from the construction shown in Figure 7. Fixing an arbitrary positive ε, the five centers are chosen so that $P_1P_2P_4$ and $P_2P_3P_5$ are regular triangles of side $2/\sqrt{3}$, and the distance between P_4 and P_5 is strictly between $2/\sqrt{3}$ and $2/\sqrt{3} - \varepsilon$. The disks with these five centers form a $(2/\sqrt{3} - \varepsilon)$-disjoint $T(4)$ family without a transversal. (In the figure, L_i is a line intersecting all the disks but the one with center P_i.) □

Lemma 3.1. *Suppose that all sides of a triangle are of length at least $2/\sqrt{3}$ and that no two lengths are equal. If one of the heights is 1 and the two angles at the base belonging to this height are acute, then the two other heights are greater than 1.*

Proof. Let A, B, C denote the vertices of the triangle and BC be the side with acute angles at its ends and corresponding height 1. The lower bound of $2/\sqrt{3}$ for the length of the sides implies that both angles at B and at C are less than $\pi/3$.

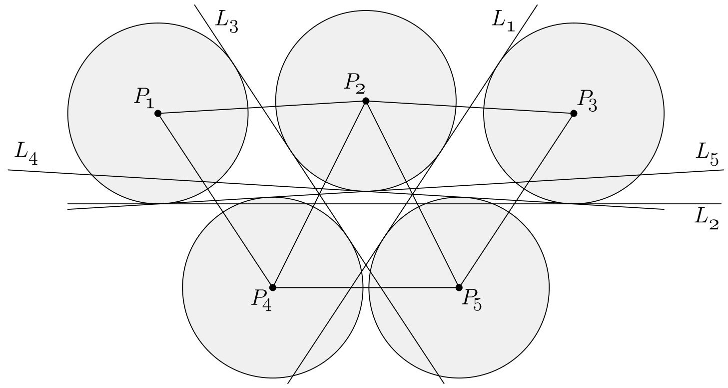

FIGURE 7. Counterexample for $t_4 = 2/\sqrt{3} - \varepsilon$

Then the angle opposite to BC is greater than $\pi/3$, and BC is the largest side. Consequently, the height belonging to BC is the smallest of the three heights. □

Proposition 3.2. $t_4 \le 2/\sqrt{3}$.

Proof. We show that a $2/\sqrt{3}$-disjoint $T(4)$-family of disks has a transversal.

Following the usual reduction technique (see Tverberg [15]), we may assume that the centers of the family are in general position. In particular, we may assume that no two of the altitudes of the triangles spanned by the centers are equal. Consider, for each subfamily of four disks, the narrowest strip covering the four centers. We say that a subfamily is *critical*, if the narrowest covering strip of the four centers has maximal width, denoted by w, of all such strips. The $T(4)$ property is equivalent to the inequality $w \le 1$, and by rescaling the system of centers (as in the proof of Theorem 2.3), we may assume without loss of generality that $w = 1$. The narrowest strip covering a critical four-tuple of centers has a stable bridge triangle ABC with base BC and opposite vertex A. Since the height of ABC, belonging to the base BC, is equal 1, and the centers are in general position, both the narrowest strip S covering the critical four-tuple of centers and its bridge triangle are uniquely determined.

Applying Lemma 3.1 with ABC, we obtain that S is the only strip of width at most 1 which covers ABC. By the $T(4)$ property, any four centers A, B, C, D are covered by a strip of width at most 1. Thus, any other center D is contained in the strip S, and the midline of S intersects all the disks. □

4. GENERAL UPPER BOUND FOR t_k

Theorem 4.1. *Let* $t > 0$, $c = 2\sqrt{2} + 4\sqrt{\sqrt{3} - 1} \approx 6.2508$, *and* $n > c/t + 3$ *be a natural number. Then for any family of* t*-disjoint disks,* $T(n)$ *implies* $T(n + 1)$.

Proof. Observe first that $n > 3$. Thus, if $t > \sqrt{2}$, then $T(n) \Longrightarrow T(3) \Longrightarrow T \Longrightarrow T(n+1)$ by Theorem 2.3 for families of at least $(n+1)$ t-disjoint disks. Consequently, it is sufficient to consider the case $t \le \sqrt{2}$. Then $n \ge 8$.

We make use of the following theorem due to J. Jerónimo [12], the proof of which is based on a theorem of L. Santaló on the transversals of parallelepipeds with parallel edges [14].

Theorem 4.2. *The width of the set of centers of a $T(6)$ family of unit diameter disks is at most $\sqrt{2}$.*

We suppose that there is a t-disjoint family $\mathcal{F}$ of $n+1$ disks having the $T(n)$, but not the $T(n+1)$ property, and seek a contradiction. Let H be the set of centers of the disks of $\mathcal{F}$. By our assumptions, H has width w greater than 1, and any proper subset of H has width at most 1. Theorem 4.2 implies also $w \leq \sqrt{2}$. Since the width of a set depends only on its convex hull, points of H are the vertices of the convex hull P of H.

Let S_h be the narrowest strip covering H. Without loss of generality, we may assume that S_h is horizontal, its upper boundary is the x-axis, the origin belongs to H and two other points of H are $P_1(x_1, -w)$ and $P_2(x_2, -w)$, such that $x_1 \leq 0 \leq x_2$.

Order the points of H by the lexicographic ordering $((x,y) \prec (x',y') \iff x < x'$ or $(x = x'$ and $y < y'))$, and let $\tilde{H} \subset H$ be the subset of H obtained by removing the two largest and two smallest points. Denote by $\tilde{S}_v$, the narrowest vertical strip covering $\tilde{H}$, and let $\tilde{l}$ be its width. $\tilde{H}$ and its convex hull $\tilde{P}$ are contained in the rectangle $S_h \cap \tilde{S}_v$ of perimeter $2(w+\tilde{l})$, and the perimeter of $\tilde{P}$ is at least $(n-3)t > c$. Thus, w and $\tilde{l}$ satisfy $2(\sqrt{2}+\tilde{l}) \geq 2(w+\tilde{l}) > c$.

In particular, $\tilde{l} > 2\sqrt{\sqrt{3}-1}$.

Let A, B, C denote the three largest points of $(H, \prec)$. We may assume without loss of generality, that each of them is lying at distance at least $\tilde{l}/2 > \sqrt{\sqrt{3}-1}$ away from the y-axis. These points form a triangle, two sides of which coincide with sides of H. Let B be the common end of these sides, and $H^* = H \setminus \{B\}$. We claim that H^* is a proper subset of H whose width is greater than 1.

Removing B from H causes only minor change to the set of the covering strips (or bridge triangles) that determine the width of H^*. The set of bridge triangles is reduced by the two triangles with bases AB or BC, and by some other bridge triangles with B as opposite vertex. However, new directions through A and C have to be added to the set. Our goal is to show that none of the new covering strips has width at most 1; a contradiction.

It is sufficient to deal with new covering strips possessing a stable bridge triangle, since only these strips may provide the minimal width. Let α denote the angle determined by the x-axis and such a covering strip S^* of H^*.

Assume first that $\alpha \geq \alpha_0$, where $\alpha_0 = \arctan\left(\sqrt{\sqrt{3}+1}\right)$. Suppose also that the slope of the boundary lines of the strip is positive. Let $P \prec Q \prec R \prec S$ be the vertices of the rectangle $S_h \cap \tilde{S}_v$ (see Fig. 8). Denote by a and b the boundary lines of S^*, where a is left of b. Let a_0 be the line through P parallel to a. Since H^* has a point left of the line PQ, a is to the left of a_0, and the intersection point $T = QS \cap a$ is to the left of the point $T_0 = QS \cap a_0$. Draw the lines c and c_0 through T and T_0 respectively, perpendicular to a. It is clear that c is below c_0, and therefore, the intersection point $U = SR \cap c$ is below the point $U_0 = SR \cap c_0$. At least one of those vertices of a stable bridge triangle which lie on b is below c because of stability. On the other hand, b crosses the triangle ABC cutting off B, so A and C are the only vertices that can lie on b. These vertices are to the right of the line SR, so b to the right of the line b_0, parallel to b and passing through U_0.

The width of the strip S^* is not less than the distance $|U_0T_0|$ between a_0 and b_0, and

$$|U_0T_0| = \frac{|T_0S|}{\sin\alpha} > \frac{2\sqrt{\sqrt{3}-1} - |QT_0|}{\sin\alpha} \geq \frac{2\sqrt{\sqrt{3}-1} - \sqrt{2}\cot(\alpha)}{\sin\alpha}. \tag{1}$$

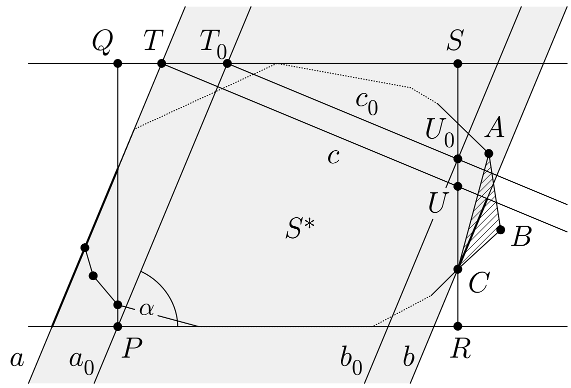

FIGURE 8. The case $\alpha \geq \alpha_0$

Differentiating the function on the right hand side, we note that it is a mononotone increasing function of α. Thus,

$$|U_0T_0| > \frac{2\sqrt{\sqrt{3}-1} - \sqrt{2}\cot(\alpha)}{\sin\alpha} \geq \frac{2\sqrt{\sqrt{3}-1} - \sqrt{2}\cot(\alpha_0)}{\sin\alpha_0} = 1.$$

This settles the case when $\alpha \geq \alpha_0$ and the sides of the strip S^* have positive slope.

The subcase when the slope of the sides of S^* is negative is treated similarly, with slight modifications. Then we define a_0 as the line through Q parallel to a, T and T_0 are defined as $T = PR \cap a$ and $T_0 = PR \cap a_0$. Then c will be above c_0, U will be above U_0. Finally, to get the lower bound (1) on $|U_0T_0|$, we should express $|U_0T_0|$ as $|T_0R|/\sin\alpha$.

Let $\alpha < \alpha_0$ and consider a stable bridge triangle spanning the strip S^* with height h belonging to the base side. The line of height connects (x_1, y_1) and (x_2, y_2) with abscissae $x_1 \leq 0$ and $x_2 \geq \tilde{l}/2$, and determines an angle β with the x-axis. We note that $\beta > \pi/2 - \alpha_0$, and thus,

$$h = \frac{|x_2 - x_1|}{\cos(\beta)} > \frac{\tilde{l}}{2\sin(\alpha_0)} = 1.$$

□

Theorem 4.3. *Let $k \geq 5$ and $(k-3)t > c \approx 6.2508$. Then $T(k)$ implies T for t-disjoint families of unit diameter disks, and $t_k \leq c/(k-3)$.*

Proof. As the conditions of Theorem 4.1 are satisfied by t and $n = k+1, k+2, \ldots$, it follows that $T(n)$ implies $T(n+1)$ for $n \geq k$. □

5. Acknowledgement

K. Bezdek and T. Bisztricky gratefully acknowledge that this research was partially supported by the Natural Sciences and Engineering Research Council of Canada. The work of K. Bezdek, B. Csikós and A. Heppes was supported by the Hungarian Science Foundation OTKA, Grant No. T 037752, T 038397, T043556 and T047102. B. Csikós and A. Heppes thank also the University of Calgary (October 15-31, 2004) for partial support of the research.

The authors are grateful to J. Jerónimo for making his paper [12] available and for calling their attention to the paper [7].

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