

ON THE MONOTONICITY OF THE VOLUME OF HYPERBOLIC CONVEX POLYHEDRA

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ABSTRACT. We give a proof of the monotonicity of the volume of nonobtuse-angled compact convex polyhedra in terms of their dihedral angles. More exactly we prove the following. Let P and Q be nonobtuse-angled compact convex polyhedra of the same simple combinatorial type in hyperbolic 3-space. If each (inner) dihedral angle of Q is at least as large as the corresponding (inner) dihedral angle of P , then the volume of P is at least as large as the volume of Q . Moreover, we extend this result to nonobtuse-angled hyperbolic simplices of any dimension.

1. INTRODUCTION

A compact convex polyhedron in hyperbolic 3-space \mathbb{H}^3 is the convex hull of finitely many points with nonempty interior or equivalently is the bounded intersection of finitely many closed halfspaces with nonempty interior. The different dimensional faces of a compact convex polyhedron are called vertices, edges and faces. Any two faces meeting along an edge are called adjacent faces. Finally, a compact convex polyhedron in \mathbb{H}^3 is called nonobtuse-angled if the inner dihedral angles at all edges do not exceed $\frac{\pi}{2}$. In a sequence of highly influential papers Andreev gave a description of the geometry of nonobtuse-angled compact convex polyhedra in \mathbb{H}^3 . (For more details on the Koebe-Andreev-Thurston theorem see for example [6].) In [3] he proved that the planes of any two nonadjacent faces of a nonobtuse-angled compact convex polyhedron do not intersect in \mathbb{H}^3 . It is easy to see that the combinatorial type of any nonobtuse-angled compact convex polyhedron is simple that is at any vertex exactly 3 faces (resp. 3 edges) meet. More importantly it is shown in [1] that if P and Q are nonobtuse-angled compact convex polyhedra of the same simple combinatorial type in \mathbb{H}^3 such that the corresponding inner dihedral angles of P and Q are equal, then P and Q are congruent. All these results have been extended to higher dimensions by Andreev in [1] and [3]. However, the most striking result of Andreev is a 3-dimensional one. Namely, for a given simple combinatorial type of a polyhedron, not a tetrahedron, Andreev [1] determined necessary and sufficient conditions on the inner dihedral angles under which there exists a compact convex polyhedron with the given dihedral angles not greater than $\frac{\pi}{2}$ and with the given simple combinatorial type. ([2] extends this result to finite-volume convex polyhedra in \mathbb{H}^3 .)

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Based on this, in this note we prove the monotonicity of the volume of 3-dimensional nonobtuse-angled compact convex polyhedra in terms of their inner dihedral angles. More exactly we prove the following:

Theorem 1. *Let P and Q be nonobtuse-angled compact convex polyhedra of the same simple combinatorial type in \mathbb{H}^3 . If each inner dihedral angle of Q is at least as large as the corresponding inner dihedral angle of P , then the volume of P is at least as large as the volume of Q .*

The author believes that the conclusion of Theorem 1 fails to hold if one drops the assumption that P and Q are nonobtuse-angled. However, it is highly possible that the following conjecture holds as a natural extension of Theorem 1 to higher dimensions. A proof of that, however, would require fundamentally new ideas as well.

Conjecture 1. *Let P and Q be nonobtuse-angled compact convex polytopes of the same simple combinatorial type in \mathbb{H}^d , $d \geq 4$. If each inner dihedral angle of Q is at least as large as the corresponding inner dihedral angle of P , then the d -dimensional hyperbolic volume of P is at least as large as that of Q .*

Remark 1. *Theorem 2 of the proof of Theorem 1 shows that Conjecture 1 holds when P and Q are nonobtuse-angled hyperbolic simplices of any dimension.*

Remark 2. *It is not hard to see via proper limit procedure (for details see [2]) that Theorem 1 extends to nonobtuse-angled convex polyhedra of finite volume in \mathbb{H}^3 .*

2. PROOF OF THEOREM 1

Case 1. *P and Q are simplices.*

Let X^n be the spherical Euclidean or hyperbolic space \mathbb{S}^n , \mathbb{E}^n or \mathbb{H}^n of constant curvature $+1$, 0 , -1 , and of dimension $n \geq 2$. By an n -dimensional simplex Δ^n in X^n we mean a compact subset with nonempty interior which can be expressed as an intersection of $n+1$ closed halfspaces. (In case of spherical space we require that Δ^n lies on an open hemisphere.) Let F_0, F_1, \dots, F_n be the $(n-1)$ -dimensional faces of the simplex Δ^n . Each $(n-2)$ -dimensional face can be described uniquely as an intersection $F_{ij} = F_i \cap F_j$. We will identify the collection of all inner dihedral angles of the simplex Δ^n with the symmetric matrix $\alpha = [\alpha_{ij}]$ where α_{ij} is the inner dihedral angle between F_i and F_j for $i \neq j$, and where the diagonal entries α_{ii} are set equal to π by definition. Then the Gram matrix $G(\Delta^n) = [g_{ij}(\Delta^n)]$ of the simplex $\Delta^n \subset X^n$ is the $(n+1) \times (n+1)$ symmetric matrix defined by $g_{ij}(\Delta^n) = -\cos \alpha_{ij}$. Note that all diagonal entries $g_{ii}(\Delta^n)$ are equal to one. Finally, let

$$\begin{aligned} G_+^n &= \{G(\Delta^n) \mid \Delta^n \text{ is an } n\text{-dimensional simplex in } \mathbb{S}^n\}, \\ G_0^n &= \{G(\Delta^n) \mid \Delta^n \text{ is an } n\text{-dimensional simplex in } \mathbb{E}^n\}, \\ G_-^n &= \{G(\Delta^n) \mid \Delta^n \text{ is an } n\text{-dimensional simplex in } \mathbb{H}^n\} \quad \text{and} \\ G^n &= G_+^n \cup G_0^n \cup G_-^n. \end{aligned}$$

The following lemma summarizes some of the major properties of the sets G_+^n , G_0^n , G_-^n and G^n that have been studied on several occasions including the papers of Coxeter [4], Milnor [7] and Vinberg [8].

Lemma 1.

- (1) The determinant of $G(\Delta^n)$ is either positive or zero or negative depending on whether the simplex Δ^n is spherical or Euclidean or hyperbolic.
- (2) G^n is a convex open set in \mathbb{R}^N with $N = \frac{n(n+1)}{2}$. (Note that the affine space consisting of all symmetric unidiagonal $(n+1) \times (n+1)$ matrices has dimension $N = \frac{n(n+1)}{2}$.)
- (3) G_0^n is an $(N-1)$ -dimensional topological cell that cuts G^n into two open subcells G_+^n and G_-^n .
- (4) G_+^n (resp. $G_+^n \cup G_0^n$) is a convex open (resp. convex closed) set in \mathbb{R}^N .

We will need the following property for our proof of Theorem 1 that seems to be a new property of G_+^n (resp., $G_+^n \cup G_0^n$) not yet mentioned in the literature. It is useful to introduce the notations $\mathbb{R}_{<0}^N = \{(x_1, x_2, \dots, x_N) \mid x_i < 0 \text{ for all } 1 \leq i \leq N\}$. and $\mathbb{R}_{\leq 0}^N = \{(x_1, x_2, \dots, x_N) \mid x_i \leq 0 \text{ for all } 1 \leq i \leq N\}$.

Lemma 2. $G_+^n \cap \mathbb{R}_{<0}^N$ (resp., $(G_+^n \cup G_0^n) \cap \mathbb{R}_{\leq 0}^N$) is a convex corner i.e. if $g = (g_1, g_2, \dots, g_N) \in G_+^n \cap \mathbb{R}_{<0}^N$ (resp., $g \in (G_+^n \cup G_0^n) \cap \mathbb{R}_{\leq 0}^N$), then for any $g' = (g'_1, g'_2, \dots, g'_N)$ with $g_1 \leq g'_1 < 0, \dots, g_N \leq g'_N < 0$ (resp., $g_1 \leq g'_1 \leq 0, \dots, g_N \leq g'_N \leq 0$) we have that $g' \in G_+^n \cap \mathbb{R}_{<0}^N$ (resp. $g' \in (G_+^n \cup G_0^n) \cap \mathbb{R}_{\leq 0}^N$).

Proof. Due to Lemma 1 it is sufficient to check the claim of Lemma 2 for the set $G_+^n \cap \mathbb{R}_{<0}^N$ only.

Let $g = (g_1, g_2, \dots, g_N) \in G_+^n \cap \mathbb{R}_{<0}^N$. Then it is sufficient to show that for any $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$ with $g_1 \leq \varepsilon_1 < 0, g_2 \leq \varepsilon_2 < 0, \dots, g_N \leq \varepsilon_N < 0$ we have that

$$(5) \quad \begin{aligned} g^1 &= (\varepsilon_1, g_2, \dots, g_N) \in G_+^n \cap \mathbb{R}_{<0}^N, \\ g^2 &= (g_1, \varepsilon_2, g_3, \dots, g_N) \in G_+^n \cap \mathbb{R}_{<0}^N, \\ &\vdots \\ g^N &= (g_1, \dots, g_{N-1}, \varepsilon_N) \in G_+^n \cap \mathbb{R}_{<0}^N. \end{aligned}$$

(Namely, it is easy to see that (5) and the convexity of $G_+^n \cap \mathbb{R}_{<0}^N$ imply that $G_+^n \cap \mathbb{R}_{<0}^N$ is indeed a convex corner. Although it is not needed here, for the sake of completeness we note that the origin of \mathbb{R}^N is in fact, an interior point of G_+^n .) Let Δ^n be the n -dimensional simplex of \mathbb{S}^n whose Gram matrix $G(\Delta^n) = [g_{ij}(\Delta^n)]$ corresponds to $g = (g_1, g_2, \dots, g_N)$ i.e.

$$(g_1, g_2, \dots, g_N) = (-\cos \alpha_{01}, -\cos \alpha_{02}, \dots, -\cos \alpha_{0n}, -\cos \alpha_{12}, \dots, -\cos \alpha_{(n-1)n}).$$

As $g \in G_+^n \cap \mathbb{R}_{<0}^N$ we have that $0 < \alpha_{01} < \frac{\pi}{2}, 0 < \alpha_{02} < \frac{\pi}{2}, \dots, 0 < \alpha_{0n} < \frac{\pi}{2}, 0 < \alpha_{12} < \frac{\pi}{2}, \dots, 0 < \alpha_{(n-1)n} < \frac{\pi}{2}$. In order to show that $g^1 = (\varepsilon_1, g_2, \dots, g_N) \in G_+^n \cap \mathbb{R}_{<0}^N$ we have to show the existence of an n -dimensional simplex Δ_1^n of \mathbb{S}^n with dihedral angles $\arccos(-\varepsilon_1), \alpha_{02}, \dots, \alpha_{0n}, \alpha_{12}, \dots, \alpha_{(n-1)n}$. (As the task left for the remaining parts of (5) is the same we do not give details of that here.) We will show the existence of Δ_1^n via polarity. Let $*\Delta^n = \{x \in \mathbb{S}^n \mid x \cdot y \leq 0 \text{ for all } y \in \Delta^n\}$ be the spherical polar of Δ^n , where $x \cdot y$ denotes the inner product of the unit vectors x and y . As it is well-known $*\Delta^n$ is an n -dimensional simplex of \mathbb{S}^n with edglength $\pi - \alpha_{01}, \pi - \alpha_{02}, \dots, \pi - \alpha_{0n}, \pi - \alpha_{12}, \dots, \pi - \alpha_{(n-1)n}$ each being larger than $\frac{\pi}{2}$.

Let F be the $(n-2)$ -dimensional face of $*\Delta^n$ disjoint from the edge of length $\pi - \alpha_{01}$ of $*\Delta^n$. Let v_0 and v_1 be the endpoints of the edge of length $\pi - \alpha_{01}$ of $*\Delta^n$. By assumption $\frac{\pi}{2} < \pi - \arccos(-\varepsilon_1) \leq \pi - \alpha_{01} < \pi$. Now, rotate v_1 towards v_0 about the $(n-2)$ -dimensional greatsphere \mathbb{S}^{n-2} of F in \mathbb{S}^n until the rotated

image \bar{v}_1 of v_1 becomes a point of the $(n-1)$ -dimensional greatsphere \mathbb{S}^{n-1} of the facet of ${}^*\Delta^n$ disjoint from v_1 . Obviously, the above rotation about \mathbb{S}^{n-2} decreases the (spherical) distance v_0v_1 in a continuous way. We claim via continuity that there is a rotated image say, v_{01} of v_1 such that the spherical distance v_0v_{01} is equal to $\pi - \arccos(-\varepsilon_1)$. Namely, the $n+1$ points formed by v_0, \bar{v}_1 and the vertices of F all belong to an open hemisphere of \mathbb{S}^{n-1} with the property that all pairwise spherical distances different from $v_0\bar{v}_1$ are larger than $\frac{\pi}{2}$. (Here we assume that v_0 and \bar{v}_1 are distinct since if they coincide, then the existence of v_{01} is trivial.) But, then a theorem of Davenport and Hajós [5] implies that $v_0\bar{v}_1 \leq \frac{\pi}{2}$ and so, the existence of v_{01} follows. Thus, the spherical polar of the n -dimensional simplex of \mathbb{S}^n spanned by v_0, v_{01} and F gives us Δ_1^n . This completes the proof of Lemma 2. \square

Now, we are in a position to show that $G_-^n \cap \mathbb{R}_{\leq 0}^N$ is monotone-path connected.

Lemma 3. *$G_-^n \cap \mathbb{R}_{\leq 0}^N$ is monotone-path connected in the following strong sense: if $g = (g_1, \dots, g_N) \in G_-^n \cap \mathbb{R}_{\leq 0}^N$ and $g' = (g'_1, \dots, g'_N) \in G_-^n \cap \mathbb{R}_{\leq 0}^N$ with $g'_1 \leq g_1, \dots, g'_N \leq g_N$, then $[\lambda g' + (1-\lambda)g] \in G_-^n \cap \mathbb{R}_{\leq 0}^N$ for all $0 \leq \lambda \leq 1$.*

Proof. Lemma 1 implies that $[\lambda g' + (1-\lambda)g] \in G^n$ for all $0 \leq \lambda \leq 1$ and so it is sufficient to prove that $[\lambda g' + (1-\lambda)g] \notin G_+^n \cup G_0^n$ for all $0 \leq \lambda \leq 1$. As $g \notin G_+^n \cup G_0^n$ and $G_+^n \cup G_0^n$ is convex moreover, $(G_+^n \cup G_0^n) \cap \mathbb{R}_{\leq 0}^N$ is a convex corner (Lemma 2) therefore there exists a supporting hyperplane H in \mathbb{R}^N that touches $G_+^n \cup G_0^n$ at some point $h \in G_0^n \cap \mathbb{R}_{\leq 0}^N$ and is disjoint from g and separates g from $G_+^n \cup G_0^n$. In fact, using again the convex corner property of $(G_+^n \cup G_0^n) \cap \mathbb{R}_{\leq 0}^N$ we get that H separates $h + \mathbb{R}_{\leq 0}^N$ from $G_+^n \cup G_0^n$ and therefore H separates $g + \mathbb{R}_{\leq 0}^N$ from $G_+^n \cup G_0^n$ as well. Finally, notice that $g' \in g + \mathbb{R}_{\leq 0}^N$ and $g + \mathbb{R}_{\leq 0}^N$ is disjoint from H and therefore $g + \mathbb{R}_{\leq 0}^N$ is disjoint from $G_+^n \cup G_0^n$. This finishes the proof of Lemma 3. \square

Now, we are ready to give a proof of the following volume monotonicity property of hyperbolic simplices.

Theorem 2. *Let P and Q be nonobtuse-angled n -dimensional hyperbolic simplices. If each inner dihedral angle of Q is at least as large as the corresponding inner dihedral angle of P , then the n -dimensional hyperbolic volume of P is at least as large as that of Q .*

Proof. By moving to the space of Gram matrices of n -dimensional hyperbolic simplices and then applying Lemma 3 we get that there exists a smooth one-parameter family $P(t)$, $0 \leq t \leq 1$ of nonobtuse-angled n -dimensional hyperbolic simplices with the property that $P(0) = P$ and $P(1) = Q$ moreover, if $\alpha_{01}(t), \alpha_{02}(t), \dots, \alpha_{0n}(t), \alpha_{12}(t), \dots, \alpha_{(n-1)n}(t)$ denote the inner dihedral angles of $P(t)$, then $\alpha_{ij}(t)$ is a monotone increasing function of t for all $0 \leq i < j \leq n$. Now, Schläfli's classical differential formula [7] yields that

$$(6) \quad \frac{d}{dt} \text{Vol}_n(P(t)) = \frac{-1}{n-1} \sum_{0 \leq i < j \leq n} \text{Vol}_{n-2}(F_{ij}(t)) \cdot \frac{d}{dt} \alpha_{ij}(t),$$

where $F_{ij}(t)$ denotes the $(n-2)$ -dimensional face of $P(t)$ on which the dihedral angle $\alpha_{ij}(t)$ sits and $\text{Vol}_n(\cdot)$, $\text{Vol}_{n-2}(\cdot)$ refer to the corresponding dimensional volume measures. Thus, as $\frac{d}{dt} \alpha_{ij}(t) \geq 0$ (6) implies that $\frac{d}{dt} \text{Vol}_n(P(t)) \leq 0$ and so indeed $P(0) \geq P(1)$, finishing the proof of Theorem 2. \square

Case 2. *The combinatorial type of P and Q is different from that of a tetrahedron.*

First, recall the following classical theorem of Andreev [1].

Andreev Theorem: A nonobtuse-angled compact convex polyhedron of a given simple combinatorial type, different from that of a tetrahedron and having given inner dihedral angles exists in \mathbb{H}^3 if and only if the following conditions are satisfied:

- (1) if 3 faces meet at a vertex, then the sum of the inner dihedral angles between them is larger than π ;
- (2) if 3 faces are pairwise adjacent but, not concurrent, then the sum of the inner dihedral angles between them is smaller than π ;
- (3) if 4 faces are cyclically adjacent, then at least one of the dihedral angles between them is different from $\frac{\pi}{2}$;
- (4) (for triangular prism only) one of the angles formed by the lateral faces with the bases must be different from $\frac{\pi}{2}$.

Second, observe that Andreev theorem implies that the space of the inner dihedral angles of nonobtuse-angled compact convex polyhedra of a given combinatorial type different from that of a tetrahedron in \mathbb{H}^3 is an open convex set. As a result we get that if P and Q are given as in Theorem 1 and are different from a tetrahedron, then there exists a smooth one-parameter family $P(t)$, $0 \leq t \leq 1$ of nonobtuse-angled compact convex polyhedra of the same simple combinatorial type as of P and Q with the property that $P(0) = P$ and $P(1) = Q$ moreover, if $\alpha_E(t)$ denotes the inner dihedral angle of $P(t)$ which sits over the edge corresponding to the edge E of P , then $\alpha_E(t)$ is a monotone increasing function of t for all edges E of P . Applying Schläfli's differential formula [7] to the smooth one-parameter family $P(t)$ we get that

$$(7) \quad \frac{d}{dt} \text{Vol}(P(t)) = -\frac{1}{2} \sum_{E_t} \text{length}(E_t) \cdot \frac{d}{dt} \alpha_E(t)$$

where E_t denotes the edge of $P(t)$ corresponding to the edge E of P and E (resp., E_t) runs over all edges of P (resp., $P(t)$). Hence, as $\frac{d}{dt} \alpha_E(t) \geq 0$ (7) implies that $\frac{d}{dt} \text{Vol}(P(t)) \leq 0$ and so indeed $P(0) \geq P(1)$, completing the proof of Theorem 1. \square

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