Edge-Antipodal 3-Polytopes

Károly Bezdek*†, Tibor Bisztriczky‡and Károly Böröczky*

August 30, 2004

Abstract

A convex 3-polytope in E^3 is called edge-antipodal if any two vertices, that determine an edge of the polytope, lie on distinct parallel supporting planes of the polytope. We prove that the number of vertices of an edge-antipodal 3-polytope is at most eight, and that the maximum is attained only for affine cubes.

Mathematical Subject Classification: 52A40, 52B10, 52C10, 52C17.

Key Words and Phrases: convex, polytope, edge-antipodal.

1 Introduction

Let X be a set of points in Euclidean d-space E^d. Then conv X and aff X denote, respectively, the convex hull and the affine hull of X.

Two points x and y are called *antipodal points* of X if there are distinct parallel supporting hyperplanes of conv X, one of which contains and the other contains y. We say that X is an *antipodal set* if any two points of X are antipodal points of X. In the case that X is a convex d-polytope P, a related notion was recently introduced by Talata [8]. P is an *edge-antipodal d-polytope* if any two vertices of P, that lie on an edge of P, are antipodal points of P.

According to a well-known result of Danzer and Grünbaum [2], conjectured independently by Erdős [3] and Klee [6], the cardinality of any antipodal set in E^d is at most 2^d. Talata in [8] conjectured that there exists a smallest positive integer m such that the cardinality of the vertex set of any edge-antipodal 3-polytope is at most m. In an elegant paper, Csikós [1] showed that $m \leq 12$. In this paper, we prove that $m = 8$.

*The authors were partially supported by the Hungarian National Science and Research Foundation OTKA T043556.

†Supported by a Natural Sciences and Engineering Research Council of Canada Discovery Grant.

‡Supported by a Natural Sciences and Engineering Research Council of Canada Discovery Grant.

Theorem *The number of vertices of an edge-antipodal* 3*-polytope* P *is at most eight, with equality only if* P *is an affine cube.*

We remark that with some additional case analysis, it can be deduced from the proof of the Theorem that the vertex set of P is in fact antipodal. This is not the case for edge-antipodal d-polytopes P_d when $d \geq 4$ (see [8] for $d = 4$), and thus, it seems highly challenging to determine the higher dimensional analogue of the Theorem. We note that Pór [7] showed recently that for each $d \geq 4$, there exists an integer $m(d)$, formula unknown, such that P_d has at most $m(d)$ vertices.

2 Proof of the Theorem

For sets $X_1, X_2, \ldots, X_n$ in E^3, let $[X_1, X_2, \ldots, X_n]$ $=\mathrm{conv}(X_1 \cup X_2 \cup \cdots \cup X_n)$ and $\langle X_1, X_2, \ldots, X_n\rangle = \mathrm{aff}(X_1 \cup X_2 \cup \cdots \cup X_n)$. For a point x, let $[x] = [\{x\}]$ and $\langle x\rangle = \langle\{x\}\rangle$.

For a point x, a line L and a plane H in E^3, let $\ell(x, L)$ ($h(x, H)$) denote the line (plane) through x that is parallel to L (H).

Let $P \subset E^3$ denote a (convex) 3-polytope with the set $\mathcal{V}(P)$ of vertices, the set $\mathcal{E}(P)$ of edges and the set $\mathcal{F}(P)$ of facets. We recall that by Euler's Theorem,

$$|\mathcal{V}(P)| - |\mathcal{E}(P)| + [\mathcal{F}(P)] = 2.$$

Let $v \in \mathcal{V}(P)$. Then v has *degree* k ($\deg v = k$) if v is incident with exactly k edges of P. It is a consequence of Euler's Theorem (cf. [4]) that the average degree of a vertex of P is less than six, and thus,

Remark 1 *Any* 3*-polytope contains a vertex of degree* k *with* $k \leq 5$.

Next, let $S = \{v_1, v_2, \ldots, v_n, v_{n+1} = v_1\} \subset \mathcal{V}(P)$, $n \geq 3$. We say that $[S]$ is a *contour section* of P if $\dim\langle S\rangle = 2$, $[S]$ is not a facet of P and $[v_i, v_{i+1}] \in \mathcal{E}(P)$ for $i = 1, \ldots, n$.

Finally, let v and w be antipodal vertices of P. When there is no danger of confusion, we denote by H_v^w and H_w^v, the distinct parallel supporting planes of P such that $v \in H_v^w$ and $w \in H_w^v$.

Henceforth, we assume that P is edge-antipodal. Thus, if $[v, w] \in \mathcal{E}(P)$ then v and w are antipodal.

We begin our arguments with some simple observations concerning a parallelogram $Q = [w, x, y, z]$ with sides $[w, x]$ and $[x, y]$:

Remark 2 *If* $\{[w, x], [x, y]\} \subset \mathcal{E}(P)$ *then* $\langle w, z\rangle$ *and* $\langle y, z\rangle$ *are supporting lines of* P, *and* $\langle Q\rangle \cap P \subset Q$.

Remark 3 *If* $[x, w, y, v] \subseteq Q \cap P$ *and* $[w, v] \in \mathcal{E}(P)$ *then* $v \in [y, z]$.

From these two remarks, we deduce

Remark 4 *Any facet or any contour section of* P *is a triangle or a parallelogram.*

We examine now P when it is non-simplicial or simplicial, and determine when a sub-polytope of P is necessarily edge-antipodal.

Lemma 1 *Let* $F = [w, x, y, z] \in \mathcal{F}(P)$ *be a parallelogram with sides* $[w, x]$ *and* $[x, y]$, H *be a plane such that* $H \cap F = [x, y]$ *and* $v \in (H \cap \mathcal{V}(P)) \setminus \{x, y\}$

1.1 If $H \cap P$ *is a contour section of* P *then* $H \cap P$ *is a parallelogram.*

1.2 If $H \cap P$ *is a facet of* P *then* $h(v, \langle F \rangle)$ *is a supporting plane of* P*.*

Proof. We suppose that $H \cap P = [x, y, v]$ is a contour section, and seek a contradiction.

Let $L = \langle y, z \rangle$ and $R = [F, v, p]$ where p is the point on $\ell(v, L)$ such that $Q = [v, y, z, p]$ and $Q' = [v, x, w, p]$ are parallelograms. Next, $H \cap P \notin \mathcal{F}(P)$ implies that there is a $u \in \mathcal{V}(P)$ such that H separates u and R, and $[u, y] \in \mathcal{E}(P)$. We have now a contradiction by Remark 2. On the one hand; $\langle Q \rangle \cap P \subseteq Q$ and $\langle Q' \rangle \cap P \subseteq Q'$, and so $\ell(u, L)$ meets the relative interior of $H \cap P$. On the other hand; $\ell(u, L)$ is a supporting line of P.

Let $H \cap P \in \mathcal{F}(P)$. Then $H \cap P$ is a parallelogram or a triangle by Remark 4.

If $H \cap P = [v, x, y, u]$ is a parallelogram with sides, say, $[v, x]$ and $[x, y]$ then $H_x^v \cap [v, x, y, u] = [x, y]$ and $H_v^x \cap [v, x, y, u] = [v, u]$ by Remark 2, and from this it follows that $h(v, \langle F \rangle)$ supports P. If $H \cap P = [v, x, y]$ then the assertion is immediate in the case that $H_x^v = \langle F \rangle$, and it is easy to check that $H_x^v \neq \langle F \rangle \neq H_y^v$ yields that $h(v, \langle F \rangle) \cap P \subseteq \ell(v, L)$. ■

Lemma 2 *Let* P *be simplicial and* $v \in \mathcal{V}(P)$*. Then* $\deg v \neq 5$.

Proof. We suppose that $[v, v_i, v_{i+1}] \in \mathcal{F}(P)$ for $i = 1, \ldots, 5$ with $v_6 = v_1$, and seek a contradiction.

Let $\tilde{P} = [v, v_1, \ldots, v_5]$. If $v_1, v_2, \ldots, v_5$ are coplanar then $[v_1, \ldots, v_5] \in \mathcal{F}(\tilde{P})$, $\mathcal{E}(\tilde{P}) \subset \mathcal{E}(P)$ and P is edge-antipodal; a contradiction by Remark 4.

Let, say, $[v_1, v_2, v_3, v_4] \in \mathcal{F}(\tilde{P})$. Then $H = \langle v_1, v_2, v_5 \rangle$ strictly separates v and $[v_3, v_4]$, and with $H \cap \langle v, v_j \rangle = \{u_j\}$ for $j \in \{3, 4\}$, $H \cap P$ is a pentagon with cyclically labelled vertices v_1, v_2, u_3, u_4, v_5. By Remark 2, $\ell(v_5, \langle v_1, v_2 \rangle)$ is a supporting line of $H \cap P$. Since v_1, v_2, v_3 and v_4 are coplanar, we obtain also from Remark 2 that $L' = \ell(v_3, \langle v_1, v_2 \rangle)$ is a supporting line of P. Then $\{[v, v_2, v_3], [v, v_3, v_4]\} \subset \mathcal{F}(P)$ yields that $H' = \langle v, L' \rangle$ is a supporting plane of P, and $H \cap H'$ is a supporting line of $H \cap P$. Since $u_3 \in H \cap H'$ and the lines $H \cap H'$ and $\ell(v_5, \langle v_1, v_2 \rangle)$ are parallel, we obtain that $\{u_3, u_4, v_5\} \subset H'$ and v, v_3, v_4 and v_5 are coplanar; a contradiction.

Since $\tilde{P}$ is simplicial, there is an edge among the $[v_i, v_{i+1}]$'s such that neither $[v_{i-1}, v_i, v_{i+1}]$ nor $[v_i, v_{i+1}, v_{i+2}]$ is a face of $\tilde{P}$. Let, say, $[v_2, v_3, v_5] \in \mathcal{F}(\tilde{P})$. Then each of $\langle v_1, v_2, v_3 \rangle$ and $\langle v_2, v_3, v_4 \rangle$ strictly separates v and v_5, and we may assume that $H = \langle v_1, v_2, v_3 \rangle$ separates v and v_4. Hence, with $H \cap \langle v, v_j \rangle = \{u_j\}$ for $j \in \{4, 5\}$, $H \cap \tilde{P}$ is a pentagon with cyclically labelled vertices v_1, v_2, v_3, u_4, u_5. We apply now Remark 2 with $\langle v_1, v_2, v_3 \rangle$ and $\langle v_2, v_3, v_4 \rangle$, and obtain that $\ell(v_1, \langle v_2, v_3 \rangle)$ and $\ell(v_4, \langle v_2, v_3 \rangle)$ are supporting lines of $\tilde{P}$. This yields directly that $\ell(v_1, \langle v_2, v_3 \rangle)$ and $\ell(u_4, \langle v_2, v_3 \rangle)$ are parallel supporting lines of the pentagon $H \cap \tilde{P}$. Then v_1, u_4 and u_5 are collinear, and v, v_1, v_4 and v_5 are coplanar; a contradiction. ■

Lemma 3 *Let* $\{w, v_1, v_2, v_3, v_4, v_5 = v_1\} \subset \mathcal{V}(P)$ *such that* $[w, v_i, v_{i+1}] \in \mathcal{F}(P)$ *for* $i = 1, 2, 3, 4$. *Then* $P_w = [\mathcal{V}(P) \setminus \{w\}]$ *is edge-antipodal.*

Proof. Since the assertion is immediate in the case that $\mathcal{E}(P_w) \subset \mathcal{E}(P)$, we may assume that the v_i's are not coplanar and that, say, $\mathcal{E}(P_w) \backslash \mathcal{E}(P) = \{[v_1, v_3]\}$.

Let $H = \langle w, v_1, v_3 \rangle$, $U = \langle v_2, v_4 \rangle$, $Q = [w, v_1, v_3, p]$ be the parallelogram with sides $[w, v_1]$ and $[w, v_3]$, and H_w and H_1 be distinct parallel supporting planes of P such that $w \in H_w$ and $v_1 \in H_1$. We assume that $v_3 \notin H_w$ and observe that with $(v_2, v_4) = [v_2, v_4] \setminus \{v_2, v_4\}$:

(i) $H \cap U \in H \cap P \subseteq Q$ by Remark 2,

(ii) $H_w \cap Q = \{w\}$ and H_1 strictly separates v_3 and p,

(iii) $\langle w, v_1, u \rangle$ and $\langle w, v_3, u \rangle$ are supporting planes of P for each $u \in U \backslash (v_2, v_4)$, and

(iv) $H \cap H_w$ and $H \cap H_1$ are supporting lines of the projection of P upon H along the direction of any line contained in H_w or H_1.

Let $H_w \cap U$ be the point $\bar{u}$, $\bar{U} = \langle w, \bar{u} \rangle$ and $\bar{P}$ be the projection of P upon H along $\bar{U}$.

Since $\bar{u} \in U \backslash (v_2, v_4)$, it follows from (iii) that $\langle w, v_1 \rangle$ and $\langle w, v_3 \rangle$ are supporting lines of $\bar{P}$. Since $\bar{U} \subset H_w$, it follows from (iv) that $H \cap H_1$ supports $\bar{P}$. But then $\langle v_1, p \rangle$ supports $\bar{P}$ by (ii), and consequently, $\langle w, v_3, \bar{u} \rangle$ and $\langle \ell(v_1, \bar{U}), p \rangle$ are parallel supporting planes of P, and hence of P_w.

In the case that $H_w \cap U = \emptyset$, letting figuratively $\bar{u} \in U$ tend to infinity yields that $\langle \ell(w, U), v_3 \rangle$ and $\langle \ell(v_1, U), p \rangle$ are parallel supporting planes of P, and hence of P_w. ■

Corollary 3.1 *Let* P *be simplicial and* $w \in \mathcal{V}(P)$ *such that* $\deg w \leq 4$. *Then* $P_w = [\mathcal{V}(P) \setminus \{w\}]$ *is edge-antipodal.*

We are now ready to proceed with the proof of the Theorem.

If P is not simplicial then by Remark 4, there is a parallelogram $F \in \mathcal{F}(P)$. By 1.2, there is a plane H, parallel to $\langle F \rangle$ and supporting P, that contains any vertex of $P \backslash F$ that is in an $F' \in \mathcal{F}(P)$ such that $F' \cap F \in \mathcal{E}(P)$. From this and Remark 2, it readily follows that H contains any vertex v of $P \backslash F$ such that $[v, x] \in \mathcal{E}(P)$ for some vertex x of F. Hence, $\mathcal{V}(P) \subset H \cup \langle F \rangle$ and $|\mathcal{V}(P)| \leq 8$ by Remark 4. We note that in this case, the degree of any vertex of P is at most four.

Let P be simplicial. If the degree of any vertex of P is at most four then $3\,|\mathcal{F}(P)| = 2\,|\mathcal{E}(P)| \leq 4\,|\mathcal{V}(P)|$, and it follows from Euler's Theorem that $|\mathcal{V}(P)| \leq 6$.

We suppose that there is a $w \in \mathcal{V}(P)$ such that $\deg w > 4$. Then $\deg w \geq 6$ by Lemma 2. From Remark 1, there is a $v_0 \in \mathcal{V}(P)$ such that $\deg v_0 \leq 4$. By Corollary 3.1, $P_0 = [\mathcal{V}(P) \setminus \{v_0\}]$ is edge-antipodal. We note that $w \in \mathcal{V}(P_0)$ and $\deg w \geq 5$. Thus, P_0 is simplicial by the preceding, and $\deg w \geq 6$ by Lemma 2.

Since each iteration of the above yields a simplicial edge-antipodal subpolytope of P with w as a vertex, we have a contradiction. ∎

Finally, we remark that if P is *strictly edge-antipodal*; that is, if $[v, w] \in \mathcal{E}(P)$ then there exist H_v^w and H_w^v such that $H_v^w \cap P = \{v\}$ and $H_w^v \cap P = \{w\}$ then $|\mathcal{V}(P)| \leq 5$. This follows from the Theorem (P is necessarily simplicial, $\mathcal{V}(P)$ is antipodal and $|\mathcal{V}(P)| \leq 6$) and the result of Grünbaum in [5] that there is no strictly antipodal set of six points in E^3.

References

[1] Csikós, B., *Edge-antipodal convex polytopes — a proof of Talata's conjecture*, Discrete Geometry, ed. A. Bezdek, Marcel Dekker, New York-Basel, 2003, 201–205.

[2] Danzer, L. and Grünbaum, B., *Über zwei Probleme bezüglich konvexer Körper von P. Erdős and V.L. Klee*, Math. Z. **79** (1962), 195–199.

[3] Erdős, P., *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.

[4] Fejes Tóth, L., *Lagerungen in der Ebene, auf der Kugel und in Raum*, Springer, Berlin, 1953.

[5] Grünbaum, B., Strictly antipodal sets, *Israel J. Math.*, **1** (1963), 5–10.

[6] Klee, V.L., *Unsolved problems in intuitive geometry*, Hectographical lecture notes, Seattle, 1960.

[7] Pór, A., *On e-antipodal polytopes*, Periodica Math. Hung., (submitted).

[8] Talata, I., *On extensive subsets of convex bodies*, Periodica Math. Hung., **38/3** (1999), 231–246.

K. Bezdek (bezdek@math.ucalgary.ca)
T. Bisztriczky (tbisztri@math.ucalgary.ca)
Department of Mathematics and Statistics
University of Calgary, Calgary, AB Canada T2N 1N4

K. Böröczky (boroczky@cs.elte.hu)
Department of Geometry
Eőtvős University, Budapest, Hungary, H-1117