From the Kneser-Poulsen conjecture to ball-polyhedra via Voronoi diagrams

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Abstract

A very fundamental geometric problem on finite systems of spheres was independently phrased by Kneser (1955) and Poulsen (1954). According to their well-known conjecture if a finite set of balls in Euclidean space is repositioned so that the distance between the centers of every pair of balls is decreased, then the volume of the union (resp., intersection) of the balls is decreased (resp., increased). In the first half of this paper we survey the state of the art of the Kneser-Poulsen conjecture in Euclidean, spherical as well as hyperbolic spaces with the emphases being on the Euclidean case. The methods of the proofs for many of the results are strongly relying on the underlying (truncated) Voronoi diagrams. Based on that it seems very natural and important to study the geometry of intersections of finitely many congruent balls say, of unit balls, from the viewpoint of discrete geometry in Euclidean space. We call these sets ball-polyhedra. In the second half of this paper we survey a selection of fundamental results known on ball-polyhedra. Besides the obvious survey character of this paper we want to emphasize our definite intention to raise quite a number of open problems to motivate further research.

1. Introduction

The Kneser-Poulsen conjecture raises an important fundamental problem on volume measure. This paper suveys the major developments regarding this problem and orients the attention of the reader towards a number of open questions in order to generate further research progress. Anyone interested in the history of the Kneser-Poulsen conjecture as well as in the many related references is refereed to the recent paper of Bezdek and Connelly [3].

2. The Kneser-Poulsen conjecture

Let $\|\ldots\|$ denote the standard Euclidean norm of the n-dimensional Euclidean space $\mathbf{E}^n$. So, if $\mathbf{p}_i, \mathbf{p}_j$ are two points in $\mathbf{E}^n$, then $\|\mathbf{p}_i - \mathbf{p}_j\|$ denotes the Euclidean distance between them. It will be convenient to denote the (finite) point configuration consisting of the points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$ in $\mathbf{E}^n$ by $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$. Now, if $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ are two configurations of N points in $\mathbf{E}^n$ such that for all $1 \le i < j \le N$ the inequality $\|\mathbf{q}_i - \mathbf{q}_j\| \le \|\mathbf{p}_i - \mathbf{p}_j\|$ holds, then we say that $\mathbf{q}$ is a *contraction* of $\mathbf{p}$. If $\mathbf{q}$ is a contraction of $\mathbf{p}$, then there may or may not be a continuous motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \ldots, \mathbf{p}_N(t))$, with $\mathbf{p}_i(t) \in \mathbf{E}^n$ for all $0 \le t \le 1$ and $1 \le i \le N$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$, and $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is monotone decreasing for all $1 \le i < j \le N$. When there is such a motion, we say that $\mathbf{q}$ is a *continuous contraction* of $\mathbf{p}$. Finally, let $B^n(\mathbf{p}_i, r_i)$ denote the closed n-dimensional ball centered at $\mathbf{p}_i$ with radius r_i in $\mathbf{E}^n$ and let $\mathrm{Vol}_n(\ldots)$ represent the n-dimensional volume (Lebesgue measure) in $\mathbf{E}^n$. In 1954 Poulsen [10] and in 1955 Kneser [8] independently conjectured the following for the case when $r_1 = \ldots = r_N$:

Conjecture 1. *If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbf{E}^n$, then*

$$\mathrm{Vol}_n[\cup_{i=1}^N B^n(\mathbf{p}_i, r_i)] \ge \mathrm{Vol}_n[\cup_{i=1}^N B^n(\mathbf{q}_i, r_i)].$$

Conjecture 2. *If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbf{E}^n$, then*

$$\mathrm{Vol}_n[\cap_{i=1}^N B^n(\mathbf{p}_i, r_i)] \le \mathrm{Vol}_n[\cap_{i=1}^N B^n(\mathbf{q}_i, r_i)].$$

3. Nearest and farthest point Voronoi diagrams

For a given point configuration $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbf{E}^n$ and radii $r_1, r_2, \ldots, r_N$ consider the following sets:

$$V_i = \{\mathbf{x} \in \mathbf{E}^n \mid \text{for all } j, \|\mathbf{x}-\mathbf{p}_i\|^2 - r_i^2 \leq \|\mathbf{x}-\mathbf{p}_j\|^2 - r_j^2\},$$

$$V^i = \{\mathbf{x} \in \mathbf{E}^n \mid \text{for all } j, \|\mathbf{x}-\mathbf{p}_i\|^2 - r_i^2 \geq \|\mathbf{x}-\mathbf{p}_j\|^2 - r_j^2\}.$$

The set V_i (resp., V^i) is called the *nearest (resp., farthest) point Voronoi cell* of the point $\mathbf{p}_i$. We now restrict each of these sets as follows:

$$V_i(r_i) = V_i \cap B^n(\mathbf{p}_i, r_i),$$

$$V^i(r_i) = V^i \cap B^n(\mathbf{p}_i, r_i).$$

We call the set $V_i(r_i)$ (resp., $V^i(r_i)$) the *nearest (resp., farthest) point truncated Voronoi cell* of the point $\mathbf{p}_i$. For each $i \neq j$ let $W_{ij} = V_i \cap V_j$ and $W^{ij} = V^i \cap V^j$. The sets W_{ij} and W^{ij} are the *walls* between the nearest point and farthest point Voronoi cells. Finally, it is natural to define the relevant *truncated walls* as follows:

$$W_{ij}(\mathbf{p}_i, r_i) = W_{ij} \cap B^n(\mathbf{p}_i, r_i) = W_{ij}(\mathbf{p}_j, r_j) = W_{ij} \cap B^n(\mathbf{p}_j, r_j),$$

$$W^{ij}(\mathbf{p}_i, r_i) = W^{ij} \cap B^n(\mathbf{p}_i, r_i) = W^{ij}(\mathbf{p}_j, r_j) = W^{ij} \cap B^n(\mathbf{p}_j, r_j).$$

4. Csikós's formula

The following formula discovered by Csikós [6] proves Conjecture 1 as well as Conjecture 2 for continuous contractions in a straighforward way in any dimension.

Theorem 1. *Let $n \geq 2$ and let $\mathbf{p}(t), 0 \leq t \leq 1$ be a smooth motion of a point configuration in $\mathbf{E}^n$ such that for each t, the points of the configuration are pairwise distinct. Then*

$$\frac{d}{dt}\mathrm{Vol}_n[\cup_{i=1}^N B^n(\mathbf{p}_i(t), r_i)] = \sum_{1 \leq i < j \leq N} (\frac{d}{dt} d_{ij}(t)) \cdot \mathrm{Vol}_{n-1}[W_{ij}(\mathbf{p}_i(t), r_i)],$$

$$\frac{d}{dt}\mathrm{Vol}_n[\cap_{i=1}^N B^n(\mathbf{p}_i(t), r_i)] = \sum_{1 \leq i < j \leq N} -(\frac{d}{dt} d_{ij}(t)) \cdot \mathrm{Vol}_{n-1}[W^{ij}(\mathbf{p}_i(t), r_i)],$$

where $d_{ij}(t) = \|\mathbf{p}_i(t) - \mathbf{p}_i(t)\|$.

Csikós [7] managed to generalize his formula to configurations of spheres called flowers also in hyperbolic as well as spherical spaces.

5. A short outline of the Bezdek-Connelly proof of the Kneser-Poulsen conjecture in the Euclidean plane

In the recent paper [3] Bezdek and Connelly proved Conjecture 1 as well as Conjecture 2 in the Euclidean plane. (In fact, the paper contains a proof of an extension of these conjectures to flowers as well.) In what follows we give an outline of the three step proof published in [3] by phrasing it through a sequence of three theorems each being higher dimensional. The proofs of these results are based on the underlying Voronoi diagrams.

Theorem 2. *Consider N moving closed n-dimensional balls $B^n(\mathbf{p}_i(t), r_i)$ with $1 \leq i \leq N, 0 \leq t \leq 1$ in $\mathbf{E}^n$. If $F_i(t)$ is the contribution of the ith ball to the boundary of the union $\cup_{i=1}^N B^n(\mathbf{p}_i(t), r_i)$ (resp., of the intersection $\cap_{i=1}^N B^n(\mathbf{p}_i(t), r_i)$), then*

$$\sum_{1 \leq i \leq N} \frac{1}{r_i} \cdot \mathrm{Vol}_{n-1}(F_i(t))$$

decreases (resp., increases) in t under any analytic contraction $\mathbf{p}(t)$ of the center points, where $0 \leq t \leq 1$.

Theorem 3. *Let the centers of the closed n-dimensional balls $B^n(\mathbf{p}_i, r_i)$, $1 \leq i \leq N$ lie in the $(n-2)$-dimensional (affine) subspace L of $\mathbf{E}^n$. If F_i stands for the contribution of the ith ball to the boundary of the union $\cup_{i=1}^N B^n(\mathbf{p}_i, r_i)$ (resp., of the intersection $\cap_{i=1}^N B^n(\mathbf{p}_i, r_i)$), then*

$$\frac{1}{2\pi} \sum_{1 \leq i \leq N} \frac{1}{r_i} \cdot \mathrm{Vol}_{n-1}(F_i)$$

is equal to the volume of $\cup_{i=1}^N B^{n-2}(\mathbf{p}_i, r_i)$ (resp., $\cap_{i=1}^N B^{n-2}(\mathbf{p}_i, r_i)$) lying L.

Theorem 4. *If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbf{E}^n$, then there is an analytic contraction of $\mathbf{p}$ onto $\mathbf{q}$ in $\mathbf{E}^{2n}$.*

Note that Theorem 2, 3 and 4 imply in a straighforward way that the Kneser-Poulsen conjecture holds in the Euclidean plane.

6. Further results obtained from the Bezdek-Connelly proof

It is worth listing two additional results obtained from the proof published in [3] in order to describe a more complete picture of the status of the Kneser-Poulsen conjecture. For more details see [3].

Theorem 5. *Let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ be two point configurations in $\mathbf{E}^n$ such* *that* $\mathbf{q}$ *is a piecewise-analytic contraction of* $\mathbf{p}$ *in* $\mathbf{E}^{n+2}$. *Then the conclusions of Conjecture 1 as well as Conjecture 2 hold in* $\mathbf{E}^n$.

Theorem 6. *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ *is an arbitrary contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ *in* $\mathbf{E}^n$ *and* $N \le n + 3$, *then both Conjecture 1 and Conjecture 2 hold.*

7. The Kneser-Poulsen conjecture for spherical polytopes

It is somewhat surprising that in spherical space for specific radius of balls (i.e. spherical caps) one can find a proof of Conjecture 1 and 2 in any dimension. The magic radius is $\frac{\pi}{2}$ and the following theorem describes the desired result in details. The proof of this result published by Bezdek and Connelly in [4] uses Schläfli formula combined with a projection argument that can be thaught of as a technique acting on the underlying spherical Voronoi diagram as well.

Theorem 7. *If a finite set of closed* n*-dimensional balls of radius* $\frac{\pi}{2}$ *(i.e. closed hemispheres) in the* n*-dimensional spherical space is rearranged so that the (spherical) distance between each pair of centers does not increase, then the (spherical)* n*-dimensional volume of the intersection does not decrease and the (spherical)* n*-dimensional volume of the union does not increase.*

8 Disk-polygons and ball-polyhedra

The previous sections indicate a good deal of geometry on unions and intersections of balls that is worth for studying. In particular, when we restrict our attention to intersections of balls the underlying convexity suggests a broad spectrum of new analytic and combinatorial results. To make the set up ideal for discrete geometry from now on we will look at intersections of finitely many congruent, closed n-dimensional balls with non-empty interior in $\mathbf{E}^n$. If $n = 2$, then we will call the sets in question *disk-polygons* and for $n \ge 3$ they will be called *ball-polyhedra*. This definition was introduced by Bezdek in a sequence of talks on ball-polyhedra at the University of Calgary in the fall of 2004. Based on that the paper [5] written by Bezdek, Lángi, Naszódi and Papez was the first article to suggest a systematic approach to the study of the geometry of disk-polygons and ball-polyhedra.

9 An analogue of Blaschke-Lebesgue theorem for disk-polygons

Perhaps the best known examples of disk-polygons are the Reuleaux polygons. Reuleaux polygons are convex domains of constant width and have been intensively studied (see for example [9]). The well-known Blaschke-Lebesgue theorem belonging to the core part of such investigations states that the Reuleaux triangle of width $w > 0$ has minimal area among the Reuleaux polygons of width w. The following theorem gives a generalization of that (see [1]).

Theorem 8. *Let* D *be a disk-polygon generated by disks of radius* $r > 0$ *such that the diameter of the center points of the disks is at most* r. *(The condition just posed means that the centers of the generating disks of* D *must all belong to* D.*) Then the area of* D *is at least as large as the area of a Reuleaux triangle of width* r.

It would be highly interesting to extend these investigations to higher dimensions.

10 Finding the shortest billiard trajectory in disk-polygons

Billiards have been around for quite some time in mathematics and generated a great deal of research. According to Birkhoff's well-known theorem if B is a strictly convex billiard table with smooth boundary (that is the boundary of B is a simple, closed, smooth and strictly convex curve), then for every positive integer $N > 1$ there exist at least two billiard trajectories in B. This motivates the following theorem that has just been proved ([2]). In order to state that theorem properly we need the following definition. Take a disk-polygon D with generating disks of radius $r > 0$. Then choose a positive ϵ "much smaller" than r and take the union of all disks of radius ϵ lying in D. The set obtained this way we call an ϵ*-rounded disk-polygon* and denote it by $D(\epsilon)$.

Theorem 9. *Let* D *be a disk-polygon generated by disks of radius* $r > 0$ *such that the diameter of the center points of the disks is at most* r. *(The condition just posed means that the centers of the generating disks of* D *must all belong to* D.*) Then for sufficiently small positive* ϵ *any of the shortest billiard trajectories in the* ϵ*-rounded disk-polygon* $D(\epsilon)$ *is a* 2*-periodic one.*

It would be natural and important to look for higher dimensional analogues of this theorem.

11 The illumination problem of ball-polyhedra

As we mentioned before [5] lays a rather broad ground for future study of ball-polyhedra by proving several new properties of them and raising open research problems as well. This list includes among many things analogues of the classical separation theorems of convex polytopes, a Kirchberger-type theorem, analogues of the Caratheodory and Steinitz theorems, and the Euler-Poincare formula for ball-polyhedra. Here we want to focus on another possible direction for research. Let K be a convex body (i.e. a compact convex set with nonempty interior) in the n-dimensional Euclidean space $\mathbf{E}^n, n \geq 2$. According to Hadwiger an exterior point $\mathbf{p} \in \mathbf{E}^n \setminus K$ of K illuminates the boundary point $\mathbf{q}$ of K if the half line emanating from $\mathbf{p}$ passing through $\mathbf{q}$ intersects the interior of K (at a point not between $\mathbf{p}$ and $\mathbf{q}$). Furthermore, a family of exterior points of K say, $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ illuminates K if each boundary point of K is illuminated by at least one of the point sources $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$. Finally, the smallest n for which there exist n exterior points of K that illuminate K is called the *illumination number* of K denoted by $I(K)$. In 1960, Hadwiger raised the following amazingly elementary but, very fundamental question. An equivalent but somewhat different looking concept of illumination was introduced by Boltyanski in the same year. There he proposed to use directions (i.e. unit vectors) instead of point sources for the illumination of convex bodies. Based on these circumstances the following conjecture we call the Boltyanski-Hadwiger illumination conjecture. According to this conjecture the illumination number $I(K)$ of any convex body K in $\mathbf{E}^n, n \geq 2$ is at most 2^n and $I(K) = 2^n$ if and only if K is an affine n-cube. This conjecture is easy to prove for $n = 2$ but, it is open for all $n \geq 3$. The following theorem is a bit more general than the related one proved in [5].

Theorem 10. *Let B_x be a ball-polyhedron in $\mathbf{E}^3$ having the property that the diameter of the centers of the generating balls is at most x, where $0 < x < 2r$ with r denoting the radius of the generating balls of B_x. Then for $0 < x \leq 0.57r$ we have that $I(B_x) = 4$; for $0.57r < x \leq 0.77r$ we have that $I(B_x) \leq 5$ and finally, for $0.77r < x \leq r$ it turns out that $I(B_x) \leq 6$.*

Actually, we believe that if B is an arbitrary ball-polyhedron of $\mathbf{E}^3$, then $I(B_x) \leq 5$. Last but not least it seems reasonable to conjecture that there exists a universal constant $c > 0$ such that the illumination number of any n-dimensional ball-polyhedron in $\mathbf{E}^n$ is smaller than $(2-c)^n$ for all $n \geq 3$.

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