On the Perimeter of the Intersection of Congruent Disks

K. Bezdek[*], University of Calgary
R. Connelly[†], Cornell University
B. Csikós[‡], Eötvös University, Budapest

Abstract

Almost 20 years ago, R. Alexander conjectured that, under an arbitrary contraction of the center points of finitely many congruent disks in the plane, the perimeter of the intersection of the disks cannot decrease. Even today it does not seem to lie within reach. What makes this problem even more important is the common belief that it would give a sharpening of the well-known Kneser-Poulsen conjecture for the special case of the intersection of congruent disks. Since the Kneser-Poulsen conjecture has just been proved in the plane, we feel that it is a good idea to call attention to this somewhat overlooked conjecture of Alexander. In this note, we prove Alexander's conjecture in some special cases.

1 Introduction

One of the most basic conjectures of discrete geometry is the longstanding conjecture of Kneser [K] and Poulsen [P]. The conjecture says that under any contraction of the center points of an arbitrary system of finitely many balls, the volume of the union (resp., intersection) of the balls cannot increase (resp., decrease) in any Euclidean space. This conjecture has just been proved

[*]Partially supported by the Hungarian National Science and Research Foundation OTKA T043556 and by a Natural Sciences and Engineering Research Council of Canada Discovery Grant.
[†]Research supported in part by NSF Grant No. DMS-0209595.
[‡]Supported by the Hungarian National Science and Research Foundation OTKA T047102.

for the plane in the recent paper [BeCo]. For an overview of the many partial results we refer the reader to [BeCo].

It seems that in the plane, for the case of the intersection of congruent disks, one can hope to sharpen the results of [BeCo]. That is, it is natural to conjecture the monotonicity of the perimeter of the intersection of congruent disks under any contraction of their centers. The analogous question for the union of congruent disks has a negative answer, as was observed by Habicht and Kneser long ago (for details see [BeCo]). In order to make the discussion below more precise, we introduce a bit of notation.

For a given distance $r > 0$ and a point P in the Euclidean plane E^2, denote by $D(P, r)$ the disk of radius r centered at P. If X is a point set in the plane, then denote by $I_r(X)$ the intersection of disks of radius r centered at points of X

$$I_r(X) = \bigcap_{P \in X} D(P, r).$$

It seems that Alexander ([A]) was the first who conjectured that if $f : X \to E^2$ is a contraction, then

$$\operatorname{per} I_r(X) \leq \operatorname{per} I_r(f(X)),$$

where $\operatorname{per} C$ stands for the perimeter of the convex set C.

We are not certain about the correctness of Alexander's conjecture, and, indeed, the supporting evidence that we collect in this note does not exclude the possibility of the existence of a counter-example for some specific system of at least five congruent disks. If Alexander's conjecture is true, it would be a rare instance of an asymmetry between intersections and unions for Kneser-Poulsen type questions.

2 Proof of some special cases of Alexander's conjecture

Definition 2.1. Suppose that the set X can be covered by a disk of radius r. Then we define the *r-convex hull* $C_r(X)$ of X as the intersection of all circular disks of radius r that contain X. X is said to be *r-convex* if it coincides with its r-convex hull. We say that a set X of points is in *r-convex position* if the r convex hulls of the proper subsets of X are strictly smaller than that of X.

The following equations are direct consequences of the definitions

$$C_r(X) = I_r(I_r(X)),\ I_r(X) = I_r(C_r(X)) = C_r(I_r(X)),\ C_r(C_r(X)) = C_r(X).$$

Proposition 1. *If X is a point set in the plane such that $I_r(X) \neq \emptyset$, then the Minkowski sum $I_r(X) + C_r(X)$ is a convex set of constant width $2r$.*

Proof. The width of the Minkowski sum of two sets in a given direction is the sum of the widths of the sets in that direction. Take an arbitrary direction and refer to it as the upward vertical direction, and call the orthogonal directions horizontal. Denote the horizontal supporting lines of $C_r(X)$ and $I_r(X) = I_r(C_r(X))$ by a, b and c, d respectively, where a is above b, c is above d (see Fig. 1). Since $C_r(X)$ and $I_r(X)$ are strictly convex, they have unique contact points A, B and C, D on a, b and c, d respectively $I_r(X)$ is contained in a disk of radius r centered around A, thus d is at most at distance r below a. We claim that the distance between a and d is exactly r. Consider the circle k of radius r which is tangent to a at A from below. If a point P at distance $\leq 2r$ from A is not contained in this circle, then the r-convex hull of the segment AP is not below the line a, therefore P cannot belong to $C_r(X)$, that is, k covers $C_r(X)$. This means that the center of k belongs to $I_r(X)$ and it lies at distance r below a, so the distance between a and d is exactly r. We remark that this argument also implies that the center of k is D. Analogously, c is exactly at distance r above b, which yields that the widths of $C_r(X)$ and $I_r(X)$ add up to $2r$. □

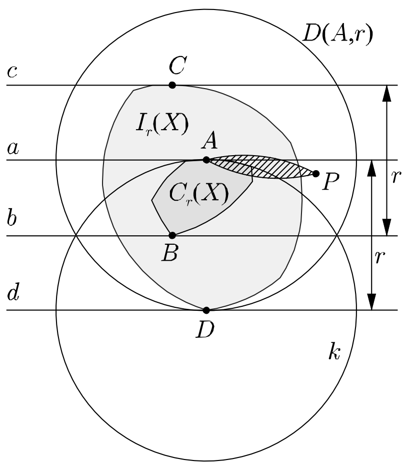


Figure 1

Corollary 1. *The sum of the perimeters of $C_r(X)$ and $I_r(X)$ is $2r\pi$.*

The corollary allows us to formulate the following statement, which is equivalent to Alexander's conjecture:

If $f : X \to E^2$ is a contraction, then $\operatorname{per} C_r(X) \geq \operatorname{per} C_r(f(X))$.

Fix the radius r. For a segment $0 \leq a \leq 2r$, denote by $l_r(a)$ the length of the shorter arc of a circle of radius r corresponding to a chord of length a. Obviously, $l_r(a) = 2r \arcsin(a/(2r))$ is a strictly increasing function of a. If A and B are two points lying at distance $\mathrm{d}(A, B) \leq 2r$ from one another, then $l_r(\mathrm{d}(A, B))$ will be denoted shortly by $l_r(A, B)$.

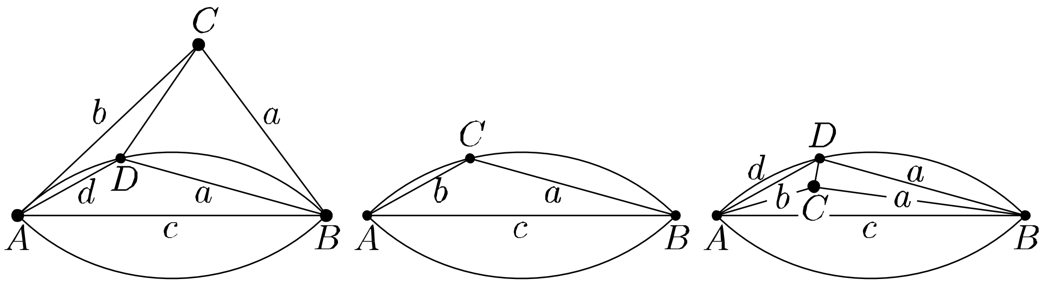
Lemma 1. *Suppose that the distances $a = \mathrm{d}(B, C)$, $b = \mathrm{d}(A, C)$, $c = \mathrm{d}(A, B)$ between the points A, B, C are all less than or equal to $2r$. Set $L = C_r(\{A, B\})$. Then*

- $l_r(a) + l_r(b) > l_r(c) \iff$ *C is in the exterior of L.*
- $l_r(a) + l_r(b) = l_r(c) \iff$ *C is on the boundary of L.*
- $l_r(a) + l_r(b) < l_r(c) \iff$ *C is in the interior of L.*

Proof. It is obvious that if C is on the boundary of L, then $l_r(a)+l_r(b) = l_r(c)$ (see Fig. 2.b).

Consider the case $C \in \operatorname{ext} L$, depicted in Fig. 2.a. If $a \geq c$, then $l_r(a) + l_r(b) > l_r(a) \geq l_r(c)$ and we are done. If $a < c$, then choose a point D on the boundary of L, for which $\mathrm{d}(B, D) = a$. Since C is in the exterior of L, $\angle CBA > \angle DBA$, thus, we have $d = \mathrm{d}(D, A) < b$ by the arm lemma. This implies $l_r(a) + l_r(b) > l_r(a) + l_r(d) = l_r(c)$.

The case $C \in \operatorname{int} L$ can be treated similarly, as shown in Fig. 2.c. Choose again a point D on the boundary of L, for which $\mathrm{d}(B, D) = a$. Since now C is in the interior of L, $\angle CBA < \angle DBA$, therefore $d = \mathrm{d}(D, A) > b$ and $l_r(a) + l_r(b) < l_r(a) + l_r(d) = l_r(c)$.



Figures 2.a, 2.b and 2.c

□

Lemma 2. *Suppose that the vertices of the convex quadrangle $ABCD$ are in r-convex position. Set $a = \mathrm{d}(A, B)$, $c = \mathrm{d}(C, D)$ $e = \mathrm{d}(B, D)$ and $f = \mathrm{d}(A, C)$. Then the following inequality holds*

$$l_r(a) + l_r(c) < l_r(e) + l_r(f).$$

Proof. The inequality $a + c < e + f$ is a simple corollary of the ordinary triangle inequality, so we may assume without loss of generality that $a < e$ and consequently $l_r(a) < l_r(e)$. The boundary of the r-convex hull of the diagonal BD consists of two arcs of radius r, one of which is on the same side of BD as the vertex A (see Fig. 3). Let E be the point on this arc such that $\mathrm{d}(E, B) = a$. Since A is not in the r-convex hull of the segment BD, we have $\angle ABD > \angle EBD$ and therefore $\angle ABC > \angle EBC$. Comparing the triangles ABC and EBC the arm lemma yields $\mathrm{d}(E, C) < \mathrm{d}(A, C) = f$.

E lies outside $C_r(\{D, C\})$, thus, by Lemma 1, we have

$$l_r(c) < l_r(D, E) + l_r(E, C) = (l_r(e) - l_r(a)) + l_r(E, C) < (l_r(e) - l_r(a)) + l_r(f).$$

□

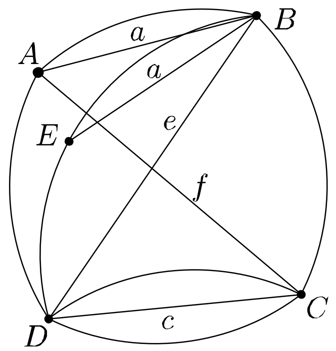


Figure 3

For a graph $\Gamma = (V, E)$, the vertex set of which is a point set $V = \{P_1, \dots, P_n\}$ in the plane such that each of the distances $\mathrm{d}(P_i, P_j)$ are at most $2r$, denote by $l_r(\Gamma)$ the sum $\sum_{\{P_i, P_j\} \in E} l_r(P_i, P_j)$.

Proposition 2. *Suppose that the points of the set $V = \{P_1, \dots, P_n\}$ are in r-convex position. Then $l_r(\Gamma) \geq \operatorname{per} C_r(V)$ for any Hamiltonian cycle Γ on the vertices V. Equality is achieved if and only if the Hamiltonian cycle goes through the points in the order of their appearance on the boundary of $C_r(V)$.*

Proof. There is only a finite number of Hamiltonian cycles, so there must be one, say Γ_{min}, for which $l_r(\Gamma_{min})$ is minimal. Represent the edges of Γ_{min} by straight line segments connecting the vertices. By Lemma 2 Γ_{min} cannot have intersecting edges, since if the edges P_iP_j and P_kP_l intersect, then we could decrease the value of $l_r(\Gamma_{min})$ by replacing these two edges either by P_iP_k and P_jP_l or by P_iP_l and P_jP_k (exactly one of the choices yields a Hamiltonian cycle). This means that a diagonal of the convex hull of V cannot be an edge of Γ_{min}. Indeed, if a diagonal P_iP_j were an edge, then removing the points P_i, P_j from the graph together with the three edges incident to them, we would get a path connecting points on different sides of P_iP_j, so this path should cross the segment P_iP_j somewhere. In conclusion, the Hamiltonian cycle Γ_{min} for which l_r attains its minimum is the one whose edges are the sides of the convex hull of V. The minimum $l_r(\Gamma_{min})$ is the perimeter of the r-convex hull of V. □

Proposition 3. *Let X be a compact subset of the plane such that $I_r(X) \neq \emptyset$ and X is contained in the boundary of $C_r(X)$. Then $\operatorname{per} C_r(X) \geq \operatorname{per} C_r(f(X))$ for any contraction $f : X \to E^2$.*

Proof. For any set X for which $C_r(X)$ is defined, the perimeter of $C_r(X)$ is the supremum of the perimeters of the sets $C_r(X')$, where X' runs over all finite subsets of X. Thus, it is enough to prove the statement when X is finite. Let Y be a minimal subset of X such that $C_r(f(Y)) = C_r(f(X))$, and let Γ be the Hamiltonian cycle which goes through the points of Y in the same order as they follow one another on the boundary of $C_r(X)$. The cyclic ordering on Y defined by Γ can be transmitted to a cyclic ordering of $f(Y)$, and therefore there is an induced Hamiltonian cycle $f(\Gamma)$ passing through the points of $f(Y)$. $f(Y)$ is in convex position; thus we can apply Lemma 2 to it and to the Hamiltonian cycle $f(\Gamma)$. This gives

$$\operatorname{per} C_r(X) \geq \operatorname{per} C_r(Y) = l_r(\Gamma) \geq l_r(f(\Gamma)) \geq \operatorname{per} C_r(f(Y)) = \operatorname{per} C_r(f(X)).$$

□

Proposition 4. *Let X be a compact subset of the plane such that $I_r(X) \neq \emptyset$, and let $f : X \to E^2$ be a contraction. Denote by Y the intersection of X and the boundary of $C_r(X)$. Then $C_r(f(Y)) = C_r(f(X))$ implies $\operatorname{per} C_r(X) \geq \operatorname{per} C_r(f(X))$.*

Proof. Proposition 3 can be applied to the compact set Y and gives

$$\operatorname{per} C_r(X) = \operatorname{per} C_r(Y) \geq \operatorname{per} C_r(f(Y)) = \operatorname{per} C_r(f(X)).$$

□

Let e be a straight line on the plane. A folding of the plane with respect to the line e is a mapping $f : E^2 \to E^2$ which fixes points on one side of e and reflects in e points on the other side. It is clear that every folding is a contraction of the plane. The following statement shows that Alexander's conjecture is true for foldings.

Proposition 5. *Let X be an arbitrary set of points on the plane such that $I_r(X) \neq \emptyset$. Then for any folding f, we have* $\operatorname{per}(C_r(X)) \geq \operatorname{per}(C_r(f(X))$.

Proof. Let e be the axis of the folding and Y denote the boundary of $C_r(X)$ Every point of $C_r(X)$ belongs to a segment parallel to e with endpoints in Y (the segment may degenerate to a point). This implies that every point of $f(C_r(X))$ is contained in a segment parallel to e with endpoints in $f(Y)$. In particular the ordinary convex hull of $f(Y)$ contains $f(C_r(X))$. Consequently $C_r(f(Y)) \supset f(C_r(X))$, from which $C_r(f(Y)) \supset C_r(f(C_r(X)))$. Applying Proposition 4 to the set $C_r(X)$ we obtain

$$\operatorname{per} C_r(X) = \operatorname{per} C_r(C_r(X)) \geq \operatorname{per} C_r(f(C_r(X))) \geq \operatorname{per} C_r(f(X)).$$

□

Definition 2.2. Let $f, g : X \to E^2$ be two mappings from a set X into the plane. We say that a map $\Phi : X \times [0,1]$ is a contracting (expanding) homotopy from f to g if $\Phi(P,0) = f(P)$, $\Phi(P,1) = g(P)$ for all $P \in X$ and the distance $\mathrm{d}(\Phi(P,t), \Phi(Q,t))$ is a weakly decreasing (weakly increasing) function of $t \in [0,1]$ for any $P, Q \in X$.

It is known that, for a contractive homotopy Φ and $P \in X$, the perimeter of the union of the disks $D(\Phi(P,t), r)$ is a weakly decreasing function of t (see [Bo], [Cs]. A slight modification of the ideas used in [Bo], [Cs] yields the following analogous result.

Proposition 6. *If $\Phi : X \times [0,1] \to E^2$ is a contractive homotopy, then the perimeter of $I_r(\Phi(X \times \{t\})$ is a weakly increasing, the perimeter of $C_r(\Phi(X \times \{t\})$ is a weakly decreasing function of $t \in [0,1]$.*

3 Alexander's conjecture for four circles

In this section we show that the special cases discussed above are enough to show Alexander's conjecture for four circles. Observe first that if the number of circles is at most three, then the conjecture follows from Proposition 6, since any contraction of at most three points can be obtained from the original configuration by a contracting homotopy.

Let X be a set of four points A, B, C, D and denote by A', B', C', D' the images of these points under a contraction $f : X \to E^2$.

It is enough to consider the case when f is injective and A', B', C', D' are in r-convex position. Indeed, if for example $D' \in C_r(\{A', B', C'\})$, then

$$\operatorname{per} C_r(X) \geq \operatorname{per} C_r(\{A, B, C\}) \geq \operatorname{per} C_r(\{A', B', C'\}) = \operatorname{per} C_r(f(X))$$

and we are done.

As for the r-convex hull of X we have three possibilities concerning the number of its vertices.

Case 1. If $C_r(X)$ is spanned by two points, say $C_r(X) = C_r(\{A, B\})$ then

$$l_r(A, B) \geq l_r(A, C) + l_r(C, B) \quad \text{and} \quad l_r(A, B) \geq l_r(A, D) + l_r(D, B)$$

by Lemma 1. Applying Proposition 2 to the Hamiltonian cycle $A'C'B'D'A'$ we obtain

$$\begin{aligned} \operatorname{per} C_r(X) = 2l_r(A, B) &\geq l_r(A, C) + l_r(C, B) + l_r(A, D) + l_r(D, B) \\ &\geq l_r(A', C') + l_r(C', B') + l_r(A', D') + l_r(D', B') \geq \operatorname{per} C_r(f(X)). \end{aligned}$$

Case 2. If the points of X are in r-convex position, then we get a special case of Proposition 4.

Case 3. The remaining case is when the r-convex hull $C_r(X)$ is spanned by three points in r-convex position, say A, B, C.

Case 3.1. If the fourth point D is in the r-convex hull of two of the points, for instance $D \in C_r(\{A, B\})$, then similar to Case 1, Lemma 1 and Proposition 2 yields

$$\begin{aligned} \operatorname{per} C_r(X) &= l_r(A, B) + l_r(B, C) + l_r(C, A) \\ &\geq l_r(A, D) + l_r(D, B) + l_r(B, C) + l_r(C, A) \\ &\geq l_r(A', D') + l_r(D', B') + l_r(B', C') + l_r(C', A') \geq \operatorname{per} C_r(X). \end{aligned}$$

Case 3.2. If D is not in the r-convex hull of any of the sides of the triangle ABC, then D is strictly inside the triangle ABC. The idea to handle this case is the following. We try to continuously contract the point set X as long as it remains an expansion of the system $f(X)$. If we are blocked for a contraction $f_0 : X \to E^2$, then we try to expand the system $f(X)$ continuously as long as it remains a contraction of $f_0(X)$. If this expansion deforms the contraction f to an expanded contraction $f_1(X)$ and it turns out that $\operatorname{per} C_r(f_0(X)) \geq \operatorname{per} C_r(f_1(X))$ by a previously verified case, then we are done since by Proposition 6, $\operatorname{per} C_r(X) \geq \operatorname{per} C_r(f_0(X))$ and $\operatorname{per} C_r(f_1(X)) \geq \operatorname{per} C_r(f(X))$. According to this idea it is enough to consider only configurations which belong to the single unsettled case 3.2 and which are not deformable toward one another in the above sense.

When we want to contract the system X, as long as it remains an expansion of $f(X)$, we are free to decrease the distances between the points of X until they reach the distance of the "target points" in $f(X)$. When the distance between two points becomes equal to the distance of their f-images, we lock their distance by putting a rigid rod between them. Going on with the continuous contraction of the system X, more and more rods will appear. We can contract the system continuously until the system of rods blocks any further continuous contraction.

Similarly, when after this procedure we start to expand the system $f(X)$, we are allowed to increase all the distances which have not reached the distance of the corresponding points in $f_0(X)$ yet. When a distance reaches this upper bound, we lock it by putting a rigid rod between the endpoints.

When the configurations X and $f(X)$ cannot be deformed closer to one another (i.e. $\mathrm{id}_X = f_0$, $f_1 = f$) we obtain two isomorphic graphs of rigid rods G and $f(G)$ on the vertices X and $f(X)$ respectively, such that the isomorphism is established by the map f, f preserves the lengths of the edges of G, X has no non-trivial continuous contraction preserving the edge lengths of G, and $f(X)$ has no non-trivial continuous expansion preserving the edge lengths of $f(G)$.

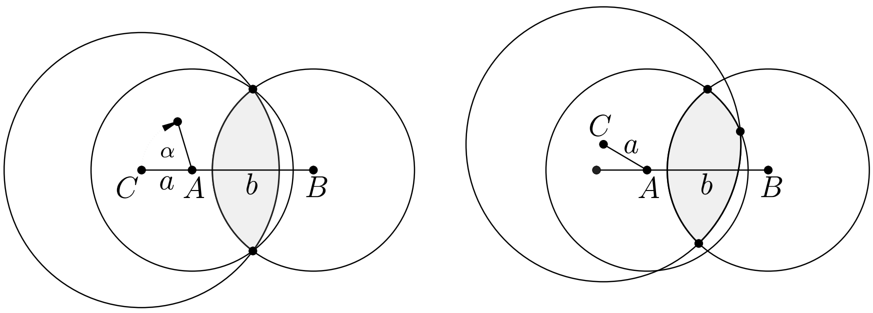
Let's study the possible structure of the graph of rods. Observe that if the two diagonals of the quadrangle $f(X)$ are rods, then X is congruent to $f(X)$, in other words, the tensegrity in which the neighboring vertices of a convex quadrangle are connected with cables and opposite vertices are connected with rods is globally rigid. Observe also that if a vertex of $f(X)$ is not connected to the opposite vertex, then it must be connected to both neighboring vertices, otherwise we can continuously expand the system $f(X)$. Thus, if a diagonal of the convex hull of $f(X)$ is not a rod, then all the sides of the convex hull must be rods. These two simple observations leave only three possibilities for the graph $f(G)$. $f(G)$ is either a complete graph or a graph consisting of a diagonal and the four sides, or it is the graph of the four sides. Alexander's inequality is obvious in the first case, it follows from Proposition 5 in the second case, and we claim that the third case cannot occur. The reason is that although making four sides of a convex quadrangle rigid blocks continuous contractions and expansions of the quadrangle, this is not true for concave quadrangles. On the contrary, according to the Carpenter's Ruler Theorem (although this is easy to see directly), a concave quadrangle made of rigid bars can be continuously expanded until its vertices are moved into convex position ([CoDR]). This means that if we change the shape of a concave quadrangle without changing the lengths of the sides, then the two diagonals increase or decrease simultaneously. Consequently, if G were a Hamiltonian cycle on the points of X, then we could continuously contract X. This completes the proof.

4 A counter-example for non-congruent circles

In [BeCo] an example is given, for three circles, only two of which are congruent, where, as their centers are continuously contracted, the perimeter of their union increases continuously even though the area of their union decreases.

The analogous statement for intersections is that there should be an example, where the perimeter of the intersection of three circles (not all congruent of course) decreases as their centers are contracted continuously. This is in fact the case in the following example.

Take two unit circles centered at A and B lying at distance $b < 2$ from one another. These two circles will be fixed. The center C of the third circle will rotate about A along a circle of radius a. The radius of the third circle is chosen in such a way that in the initial configuration, when C is at maximal distance $a + b$ from B, the three circles belong to a one parameter family of circles (see Fig. 4.a). As C moves away from its initial position, the angle α swept out by the half line AC increases and C gets closer to B.



Figures 4.a and 4.b

Figure 5 shows how the perimeter of the intersection of the three disks varies as α grows from 0 to π, choosing $a = 0.5$ and $b = 1.2$. From the evidence of the graph, the perimeter attains its minimum at a point $\alpha_{min} \approx 0.5492$ and it decreases on the interval $[0, \alpha_{min}]$. The perimeter of the intersection starts from the value ≈ 3.561 at $\alpha = 0$ and decreases to ≈ 3.4976 at $\alpha = \alpha_{min}$.

Figure 4.b depicts the configuration minimizing the perimeter of the intersection of the three disks.

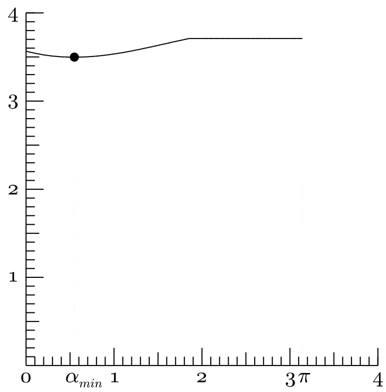


Figure 5

References

[A] R. Alexander, Lipschitzian mappings and total mean curvature of polyhedral surfaces. I, *Trans. Amer. Math. Soc.* **288** (1985), pp. 661–678.

[BeCo] K. Bezdek and R. Connelly, Pushing disks apart - the Kneser-Poulsen conjecture in the plane, *J. reine angew. Math.* **553** (2002), pp. 221–336.

[Bo] B. Bollobás, Area if the union of disks, *Elem. Math.* **23** (1968), pp. 60–61.

[CoDR] R. Connelly, E. D. Demaine and G. Rote, Straightening polygonal arcs and convexifying polygonal cycles, *Discrete Comput. Geom.* **30/2** (2003), pp. 205-239.

[Cs] B. Csikós, On the Hadwiger-Kneser-Poulsen conjecture, in *Bolyai Mathematical Studies 6, Intuitive Geometry* (Budapest 1995), Bolyai Society, Budapest, 1997, pp. 291–300.

[K] M. Kneser, Einige Bemerkungen über das Minkowskische Flächenmass, *Arch. Math.* **6** (1955), pp. 382–390.

[P] E. T. Poulsen, Problem 10, *Math. Scand.* **2** (1954), pp. 346.

Authors' addresses:
Károly Bezdek, Dept. of Math. and Stats., University of Calgary, 2500 University Drive N.W., Calgary, Alberta, Canada T2N 1N4
e-mail: bezdek@math.ucalgary.ca
Robert Connelly, Dept. of Math., Cornell University, Malott Hall, Ithaca, NY 14853 USA
email: connelly@math.cornell.edu
Balázs Csikós, Dept. of Geometry, Eötvös University, Budapest, P.O.B. 120, H-1518 Hungary
e-mail: csikos@cs.elte.hu