ANTIPODALITY IN HYPERBOLIC SPACE

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Abstract. An antipodal set in Euclidean n-space is a set of points with the property that through any two of them there is a pair of parallel hyperplanes supporting the set. In this paper we discuss the various possible ways to translate this notion to hyperbolic space and find the maximal cardinality of a hyperbolic antipodal set (according to the different definitions).

0. Introduction

The following definition is due V. Klee [5]. Let $X \subset \mathbb{R}^n$ be a set of points in Euclidean n-space, $\mathbb{R}^n$. Points $x_1 \in X$ and $x_2 \in X$ are in *antipodal position* if there is a pair of parallel hyperplanes through x_1 and x_2 supporting X. The set X is an *antipodal* set if any two points in it are in antipodal position.

A *strictly antipodal* set is defined in the same way with the additional condition that each of the supporting planes contains only one point of X.

In [1] it is shown that the maximum cardinality of an antipodal set in Euclidean n-space is 2^n with the vertices of an affine $n-$cube as the only configuration attaining this maximum. The sketch of the proof is the following. Let $X = \{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ be an antipodal set. Take the images of X under contractions by the factor $\frac{1}{2}$ from x_i, $i = 1, \ldots, k$. Call these images $X_1, \ldots, X_k$. The antipodality of X guarantees that $\mathrm{conv}(X_1), \ldots, \mathrm{conv}(X_k)$ are non-overlapping sets contained in $\mathrm{conv}(X)$. Then the theorem follows from computing volumes using the fact that $Vol(\mathrm{conv}(X_i)) = \frac{1}{2^n} Vol(\mathrm{conv}(X))$.

A different definition of antipodality was given by P. Erdős [2]. According to his definition a set in Euclidean space is antipodal if any three points form the vertices of a non-obtuse triangle. Though this definition is stricter than the one given by V. Klee, the maximum cardinality of such a set is the same, that is, 2^n as shown by an $n-$cube. And strict antipodality means, according to this definition, that all triangles are acute. In the following we are using Klee's definition when refering to Euclidean (strict) antipodality.

In section 1 we give two versions of hyperbolic antipodality, following Klee's and Erdős's idea. It turns out that according to the first version the maximum cardinality of an antipodal set is 7 in the hyperbolic plane. However, there is no upper bound on the cardinality of an antipodal set in three dimensions. In case of the second version of antipodality, arbitrarily large antipodal sets exist

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in the hyperbolic plane. Then, in section 2, we define hyperbolic antipodality in an independent third way and show that it is strongly related to Euclidean strict antipodality.

Finally, we remark that Erdős's notion of (strict) antipodality can be used in spherical n-space as well. It is not hard to see that an (a strictly) antipodal set of points in spherical n-space must lie on an open hemisphere. Moreover, the method of [1] yields that the cardinality of such a set is always strictly less than 2^n. The exact value of the maximum cardinality of an (a strictly) antipodal set, in the sense of Erdős, in spherical n-space seems as difficult as the problem of finding the maximum cardinality of strictly antipodal point sets in Euclidean n-space.

We denote the n-dimensional hyperbolic space by $\mathbb{H}^n$. For an introduction into hyperbolic geometry, see for example [8].

1. Extending Klee's and Erdős's notion of antipodality to hyperbolic space

Two hyperplanes in hyperbolic n-space are *parallel* if they "contain" exactly one common ideal point.

Definition 1. A *p-antipodal* set in hyperbolic n-space is a set of points with the property that through any two of them there is a pair of parallel hyperplanes supporting the set. Here "p" stands for parallel.

Claim 1.1. *There are arbitrarily large p-antipodal sets in* $\mathbb{H}^3$.

Proof. The proof uses the projective disk model of $\mathbb{H}^3$. In this model $\mathbb{H}^3$ is the set of points of the open unit ball B in $\mathbb{R}^3$ centered at the origin, and a plane of $\mathbb{H}^3$ is the intersection of a plane of $\mathbb{R}^3$ with B. Two planes are parallel if the Euclidean planes representing them intersect in a line tangent to B. Let $k > 2$ be an odd integer. We construct a p-antipodal set of cardinality k in $\mathbb{H}^3$.

Let D be the open disk which is the intersection of B with the "horizontal" plane $z = 0$ of $\mathbb{R}^3$. Take a regular k-gon on the horizontal plane $z = 0$ centered at the origin such that its vertices are exterior points of D and moreover, the midpoints of its sides are in D. Let X be the set of midpoints of the sides of this k-gon. X is a p-antipodal set. Indeed, for any two points of X take the Euclidean lines corresponding to their respective sides and then take their intersection. As this intersection point is an exterior point of D, we can take a tangent line to B through this point such that it is not in the same plane as the two side lines. Now, take the two planes spanned by this tangent line and either of the two side lines. The intersections of these two planes with B are the desired parallel supporting planes of X. □

Theorem 1.1. *The maximum cardinality of a p-antipodal set in* $\mathbb{H}^2$ *is* 7.

Proof. Here the proof uses the projective disk model of $\mathbb{H}^2$, i.e., the points of $\mathbb{H}^2$ are the points of the open unit disk D of $\mathbb{R}^2$ centered at the origin and the lines are open chords of D. Points of the unit circle are called ideal points, the unit circle is the circle at infinity. Two lines are parallel if the two chords representing them have one endpoint in common. Figure 1 shows that there is a p-antipodal set of 7 points on $\mathbb{H}^2$: take a regular $7-$gon the vertices of which are on the circle at infinity. Now take the 7 diagonals of the $7-$gon that connect vertices that are second neighbors

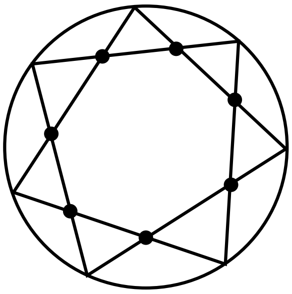

FIGURE 1. A set of 7 p-antipodal points on $\mathbb{H}^2$.

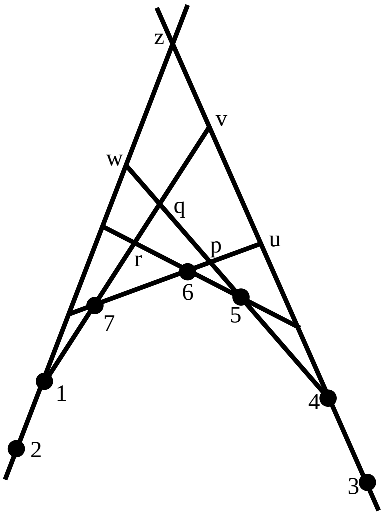

FIGURE 2

of each other. The configuration is the set of the pairwise intersections of these diagonals that are not on the circle at infinity.

If X is a p-antipodal set on $\mathbb{H}^2$, then the points in X are clearly in convex position, i.e., X is the set of vertices of a convex polygon, and so, it admits a natural cyclic ordering. Let $X = \{x_1, \ldots, x_7\}$ be a set of 7 p-antipodal points on $\mathbb{H}^2$ ordered in the natural way. We will show that no other point can be added to the configuration if they all have to form a p-antipodal set. First, we prove the following claim.

Claim 1.2. *For any* $i = 1, \ldots, 7$ *the Euclidean triangle bounded by the lines* $x_{i-1}x_i, x_ix_{i+1}$ *and* $x_{i+1}x_{i+2}$ *cannot contain the rest of the points of* X*. In other words, the two Euclidean lines* $x_{i-1}x_i$ *and* $x_{i+1}x_{i+2}$ *do not intersect on the same side of* x_ix_{i+1} *where the remaining points of* X *lie (indices are taken modulo* 7*).*

Proof. Suppose that the Euclidean lines x_1x_2 and x_3x_4 intersect on the same side of x_2x_3 where the remaining points of X lie. Now we have a configuration as

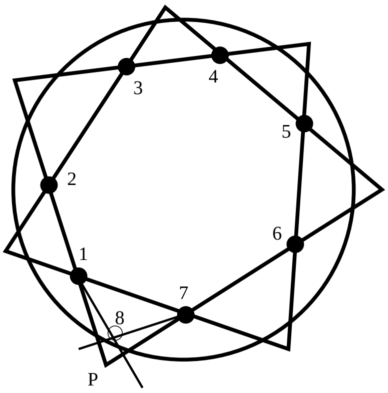

FIGURE 3

in Figure 2. Let $\{p\} := x_7x_6 \cap x_4x_5$, $\{q\} := x_1x_7 \cap x_4x_5$, $\{r\} := x_1x_7 \cap x_5x_6$, $\{u\} := x_7x_6 \cap x_3x_4$, $\{v\} := x_1x_7 \cap x_3x_4$, $\{w\} := x_2x_1 \cap x_4x_5$, $\{z\} := x_2x_1 \cap x_3x_4$.

Then, in order to have x_5 and x_6 in p-antipodal position, the circle at infinity has to pass through the triangle x_5px_6, while for having x_4 and x_7 in p-antipodal position the circle at infinity has to pass through the quadrangle $qpuv$. Since D contains x_5 and x_6, this is only possible if the circle at infinity goes through p.

Similarily, in order to have x_5 and x_7 in p-antipodal position, the circle at infinity has to pass through the quadrangle rx_6pq, while for having x_1 and x_4 in p-antipodal position, the circle at infinity has to pass through the quadrangle $qvzw$. Since the disk contains x_4 and x_5, this is only possible if the circle goes through q.

But, if the circle at infinity intersects the Euclidean line x_4x_5 in p and q, then x_4 and x_5 are outside D, a contradiction. This concludes the proof of Claim 1.2. □

Now, Claim 1.2 ensures that X is a configuration of 7 points as in Figure 3. Suppose that we can add an eighth point x_8 to the configuration. Since $X \cup \{x_8\}$ must be a convex polygon, x_8 must be in a "triangle" bounded by the lines $x_{i-1}x_i$, x_ix_{i+1} and $x_{i+1}x_{i+2}$ for some $i = 1, \dots, 7$ (indices are taken modulo 7). Actually, here we allow the Euclidean lines $x_{i-1}x_i$ and $x_{i+1}x_{i+2}$ to be parallel in the Euclidean sense, in which case their "intersection" is a point at infinity of the Euclidean plane. Let $\{p\} := x_2x_1 \cap x_6x_7$. We can assume that x_8 is in the triangle x_1x_7p. Now, in order to have x_8 and x_7 (resp., x_8 and x_1) in p-antipodal position the Euclidean line x_1x_8 (resp., x_7x_8) must intersect the circle at infinity in the "triangle" x_1x_7p. In this case x_1 and x_5 are clearly *not* in p-antipodal position, a contradiction. This finishes the proof of Theorem 1.1. □

Definition 2. A set of points in $\mathbb{H}^n$ is *a-antipodal* if any angle determined by any three of them is acute (i.e., less than $\frac{\pi}{2}$). Here, "a" stands for acute.

Somewhat surprisingly one can construct a-antipodal sets of arbitrarily large cardinality in $\mathbb{H}^2$. We use the conformal disk model to show this. In this model, points of $\mathbb{H}^2$ are again points of the open Euclidean unit disk centered at the origin, and lines are the diameters and the circular arcs in the disk that are perpendicular to the unit circle (the circle at infinity). In this model the angle of two intersecting lines is the Euclidean angle of the circular arcs representing them.

Now, for any integer $N > 2$ take the vertices of a regular N-gon in the disk centered at the origin such that these vertices are "almost" on the circle at infinity. One can easily see that the angles of this convex polygon can be arbitrarily small and thus any three vertices determine a small angle.

2. Horospherical antipodality

Now we introduce horospherical antipodality in $\mathbb{H}^n$. Most importantly, it turns out that the maximum cardinality of a horospherically antipodal set in $\mathbb{H}^n$ is equal to the maximum cardinality of a strictly antipodal set in $\mathbb{R}^n$.

Definition 3. An *h-antipodal* set in hyperbolic n-space is a set of points $X \subset \mathbb{H}^n$ with the property that for any two of them $(x_1, x_2 \in X)$ X is contained in the intersection of the horoballs H_1 and H_2, where H_1 is the horoball bounded by the horosphere that passes through x_1, contains x_2 inside and is perpendicular to the hyperbolic line $\overline{x_1x_2}$, and H_2 is defined similarily. "h" stands for horospherical in the definiton.

In other words, for any two points $x_1, x_2 \in X$ we take the line connecting them and take the horosphere with centre at the ideal point $\overrightarrow{x_1x_2}$ that goes through x_1 and the horosphere with centre at the ideal point $\overrightarrow{x_2x_1}$ that goes through x_2. Then X has to be in the intersection of the two horoballs bounded by these horospheres.

Remark 2.1. One can easily verify that a sufficiently small copy of a Euclidean strictly antipodal set in the center of the Poincaré disk model of $\mathbb{H}^n$ is h-antipodal.

The connection between the problem of determining the maximum cardinality of a strictly antipodal set in Euclidean n-space and the maximum cardinality of an h-antipodal set in hyperbolic n-space is even more tight.

Theorem 2.2. *An h-antipodal set $X = \{x_1, \dots, x_k\} \in \mathbb{H}^n$ in hyperbolic n-space can be mapped bijectively onto a Euclidean strictly antipodal set $X' = \{x'_1, \dots, x'_k\} \in \mathbb{R}^n$ in Euclidean n-space.*

Proof. To define the projection, we take the Poincaré half-space model of $\mathbb{H}^{n+1}$. More precisesely, we regard $\mathbb{H}^{n+1}$ as a half-space in $\mathbb{R}^{n+1}$ with a "horizontal" bounding hyperplane, the floor F, and we choose the open unit hemisphere in $\mathbb{H}^{n+1}$ with centre at the origin. We denote this hemisphere by $\mathbb{H}^n$ as it is a model of hyperbolic n-space. Now, we project $\mathbb{H}^n$ centrally from the origin onto the horizontal Euclidean n-plane that touches the hemisphere on the top. We denote this n-dimensional Euclidean plane by T, and the image of the set X by X'. See Figure 4.

Now, let $X = \{x_1, \dots, x_k\} \in \mathbb{H}^n$ be an h-antipodal set. We prove that X' is a Euclidean strictly antipodal set in T. Given two points of X', say x'_1 and x'_2, we will find two parallel Euclidean hyperplanes in T (i.e., $(n-1)$-dimensional affine subspaces in $\mathbb{R}^{n+1}$) that support X' in x'_1 and in x'_2 , respectively. Since X is an h-antipodal set in $\mathbb{H}^n$, there are two horospheres h_1 and h_2 that support X in x_1 and x_2, respectively, and are perpendicular to the hyperbolic line $\overline{x_1x_2}$. In our hemisphere model h_1 and h_2 are $(n-1)$-dimensional spheres on the n-dimensional hemisphere $\mathbb{H}^n$ that touch $\mathbb{S}^{n-1}$, the sphere at infinity. The hyperbolic line $\overline{x_1x_2}$ is a vertical semicircle in $\mathbb{H}^n$, i.e., it is the intersection of $\mathbb{H}^n$ with a 2-dimensional plane that is orthogonal to the floor F. So the center of the semi-circle, that represents the hyperbolic line $\overline{x_1x_2}$, is on F.

Now, since X is supported by h_1 and h_2, X is also supported by the n-dimensional Euclidean hyperplanes (L_1 and L_2) in $\mathbb{R}^{n+1}$ that are perpendicular to the semi-circle $\overline{x_1x_2}$ at x_1 and x_2, since these hyperplane are tangent to h_1 and h_2, respectively. Moreover, L_1 and L_2 contains no other points of X than x_1 and x_2.

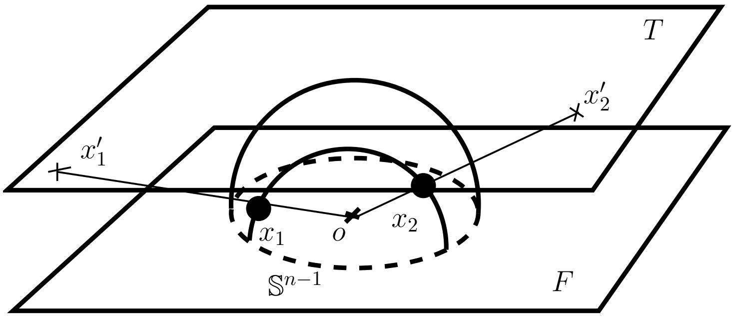

Figure 4. The projection of the conformal hemi-sphere model onto Euclidean space.

Since the center of this semi-circle is on F, L_1 and L_2 intersect in an $(n-1)$-dimensional affine subspace in F that contains the origin. Hence, $L_1 \cap T$ and $L_2 \cap T$ are parallel $(n-1)$-dimensional affine subspaces of T, and according to the definition of the projection from X to X' they support X' and contain only x'_1 and x'_2 from X'. This completes our proof. □

As a straightforward corollary of Remark 2.1 and Theorem 2.2 we get the following

Corollary 2.2.1. *The maximum cardinality of an h-antipodal set in the n-dimensional hyperbolic space is the same as the maximum number of strictly antipodal points in the n-dimensional Euclidean space.*

Corollary 2.2.2. *The maximum cardinality of an h-antipodal set in* $\mathbb{H}^2$ *is* 3*, in* $\mathbb{H}^3$ *it is 5.*

Proof. The planar result is obvious. For the 3-dimensional part, we refer to [4] where it is shown that the maximal cardinality of a strictly antipodal set in $\mathbb{R}^3$ is 5. □

For the sake of completeness we remark here that the method of [1], as outlined in the introduction, does not give any of the results proved above.

3. LOCAL UPPER BOUND ON THE CARDINALITY OF P- AND A-ANTIPODAL SETS IN $\mathbb{H}^n$

In this section we prove that there is a real number $\delta(n) > 0$ such that, if the diameter of a p-antipodal (resp.,an a-antipodal) set in $\mathbb{H}^n$ is smaller than δ, then its cardinality cannot exceed 2^n. We first need a stability version of the Danzer–Grünbaum result in [1] that is interesting on its own.

Theorem 3.1. *For any dimension $n \in \mathbb{Z}_+$ there is a real number $R(n) > 0$ depending on n such that, if $X \subset \mathbb{R}^n$ is a set of diameter at most one with the property that*

(*) *for any two points $x, y \in X$ there is a pair of hyperplanes H_x and H_y supporting X at x and y, respectively, such that the $(n-1)-$dimensional balls $H_x \cap \mathbf{B}(x, R(n))$ and $H_y \cap \mathbf{B}(y, R(n))$ are disjoint,*

then the cardinality of X is at most 2^n.

Proof. The statement clearly holds for $n = 1$. We say that a set "has the property (*) with R" if the balls involved in (*) can be of radius R. Suppose that for some $n > 1$ there is a sequence of sets $X_1, X_2, \ldots$ in $\mathbb{R}^n$ of diameter at most one such that X_i has the property (*) with $R = i$. Contradicting the claim of the theorem we suppose that the cardinality of each set X_i is $2^n + 1$.

We assume that n is the smallest dimension for which the theorem fails. So every X_i is $n-$dimensional, i.e., they span $\mathbb{R}^n$. For every i we can furthermore assume that the *John ellipsoid* of $K_i := \operatorname{conv} X_i$ (i.e., the ellipsoid of maximal volume contained in the polytope K_i, see [6], p.13) is centered at the origin. Since $\dim(X_i) = n$, this ellipsoid E_i is non-denegerate. According to John's theorem $E_i \subseteq K_i \subseteq nE_i$.

We fix i and act on X_i with a linear transformation that carries E_i into an ellipsoid that contains the unit ball around the origin and that preserves property (*) for X_i without making diam X_i too large. The linear transformation is constructed as follows.

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ in the directions of the axes of E_i. Let the widths of E_i in these directions be $2c_1, \ldots, 2c_n$, respectively. Since diam $X \leq 1$ and $E_i \subset \operatorname{conv} X$, we have $c_1, \ldots, c_n \leq \frac{1}{2}$. We take the linear transformation ϕ with the matrix $\operatorname{diag}(\frac{1}{c_1}, \ldots, \frac{1}{c_n})$ in the basis $e_1, \ldots, e_n$. We observe three properties of $\phi(X_i)$.

First, $\phi(X_i)$ has property (*) with $R = i$ as ϕ does not contract any vector.

Second, since $\phi(E_i)$ contains the unit ball, so does $\phi(K_i) = \operatorname{conv}(\phi(X_i))$.

Third, $\operatorname{diam}(\phi(X_i)) \leq 2n^{\frac{3}{2}}$. This is true since $X_i \subset nE_i$. So, for $j \in \{1, \ldots, n\}$, the width of $\phi(X_i)$ in the direction e_j is not greater than the width of $n\phi(E_i)$ in the same direction, i.e, $2n$. Hence, $\phi(X_i)$ is contained in a box with side length not greater than $2n$, which proves that the third observation holds.

So, after acting with a suitable linear transformation on each set X_i we have the sets $Y_i := \phi_i(X_i)$ that have property (*) with $R = i$ and that are of diameter not greater than $2n^{\frac{3}{2}}$. Moreover, for all $i = 1, \ldots, n$ we have $\mathbf{B}(\mathbf{0}, 1) \subset \operatorname{conv} Y_i$.

A compactness argument shows that it is possible to select a subsequence of the sequence of sets $\{Y_i\}$ such that they converge to a set Z in $\mathbb{R}^n$. To avoid the introduction of multiple indices we assume that $\{Y_i\}$ itself is a convergent sequence of sets in $\mathbb{R}^n$. Here convergence is meant in a natural way, one can define it as convergence in the Hausdorff-metric.

It is clear that Z is an antipodal set in $\mathbb{R}^n$. So, according to [1] the cardinality of Z is not greater than 2^n. Hence, there are two sequences of points $\{x_i\}$ and $\{y_i\}$ with $x_1, y_1 \in Y_1$; $x_2, y_2 \in Y_2; \ldots$ such that $x_i \neq y_i$ for all $i = 1, 2, \ldots$, and the sequences $\{x_i\}$ and $\{y_i\}$ converge to the same point $z \in Z$. Using the (*) property of the sets Y_i we get that there is a pair of hyperplanes H_x^i and H_y^i through x_i and y_i supporting Y_i in x_i and in y_i, respectively, and containing the disjoint $(n-1)-$dimensional balls $H_x^i \cap B(x_i, i)$ and $H_y^i \cap B(y_i, i)$. By another compactness argument one can select a subsequence of the sequence of sets $\{Y_i\}$ such that both subsequences of the sequences of unit normal vectors of the hyperplanes $\{H_x^i\}$ and $\{H_y^i\}$ converge. To avoid the introduction of multiple indices we assume that $\{Y_i\}$ itself has this property. Let the two hyperplanes with the two limiting normal vectors through z be H_x and H_y. H_x and H_y are clearly the same yielding that Z is contained in a hyperplane. However, Z was obtained as the limit of the sequence of sets $\{Y_i\}$; since $\mathbf{B}(\mathbf{0}, 1) \subset \operatorname{conv} Y_1$, this is a contradiction that finishes the proof of the theorem. $\square$

Theorem 3.2. *For every dimension $n \in \mathbb{Z}_+$ there is a $\delta(n) > 0$ such that any p-antipodal (resp., a-antipodal) set of diameter at most $\delta(n)$ in $\mathbb{H}^n$ is of cardinality at most 2^n.*

Proof. Taking the projective disk model of $\mathbb{H}^n$ as a subset of $\mathbb{R}^n$ it is easy to see that a p-antipodal (resp., a-antipodal) set of a sufficiently small diameter has property (*) of the previous theorem with the desired radius $R(n)$. $\square$

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