C*-algebras associated with topological relations

Berndt Brenken*

Abstract

A Hilbert bimodule is associated with certain closed relations on a locally compact Hausdorff space. The Cuntz-Pimsner C*-algebra of this bimodule is a C*-algebra associated with the relation. Various examples of relations, including some that are not locally homeomorphic with the base space, and their resulting C*-algebras are considered.

*Acknowledgement: The author acknowledges support, in connection with this research, from the Natural Sciences and Engineering Research Council of Canada.

C*-algebras associated with topological relations

In [13] Pimsner associated a remarkable C*-algebra with a Hilbert bimodule in a manner generalizing the construction, going back to Dixmier and Roberts, of the Cuntz C*-algebras as C*-algebras generated by a Hilbert space of isometries. Pimsner showed this C*-algebra possesses universal mapping properties determined by the bimodule structure, with various bimodule structures giving rise to different known families of universal C*-algebras. For example, the universal C*-algebra associated with an automorphism acting on a C*-algebra, as well as the Cuntz, and Cuntz-Krieger algebras, arise from this unifying viewpoint. In this paper we describe a method to associate a certain Hilbert bimodule with a closed relation on a locally compact topological space. The Cuntz-Pimsner C*-algebra of this bimodule then yields a C*-algebra for such a relation. The notion of a relation encompasses homeomorphisms and continuous proper self-mappings of a locally compact Hausdorff space, as well as their inverse relations. Since there is an interplay involving the structure of C*-algebras associated with homeomorphisms and surjections of topological spaces and dynamical properties of these maps, the C*-algebras of relations described here should provide an interesting class of algebras whose structure reflects the dynamical aspects of relations.

The second section describes examples of bimodules and resultant C*-algebras for specific relations. Some examples of relations not locally homeomorphic to the base space are also considered. The case of a closed relation coming from a continuous map is considered first, followed by relations on discrete spaces. The full relation on a compact space is also considered, and in the case of the torus for example, one obtains a classifiable unital Kirchberg algebra. In Example (d) we consider the relation which is the inverse of the a- fold covering map of the torus. There we find a natural way in which the C*-algebra of this relation is a particular limit of Cuntz algebras resulting in one of the solenoidal algebras considered in [1]. In the last example a simple branched covering over the compact one-torus is considered, yielding an algebra with generating relations akin to basic trigonometric identities. Although a general groupoid approach as in [7] should be within reach to arrive at some of the conclusions presented here, the explicit structure of these algebras should also be of independent interest.

The following description of a bimodule associated with a relation was partially motivated by the Hilbert bimodules of a directed graph [8, 3]. Indeed, the case of relations on discrete topological spaces reduces to the Cuntz, or graph C*-algebra situation so the C*-algebras considered here may be viewed as C*-algebras of continuous graphs. Various approaches to this have been considered by others; cf. [6], [7]. Also, at the 2002 Satellite ICM Conference where the results contained below were reported, I received the preprint [9]. The bimodules considered below are a generalization of the 'continuous diagrams' considered in [17]. They are also specific examples of the topological quivers briefly described in [15]. Higher multiplicity topological relations will be examined in subsequent papers.

Notation: If V is a subset of a topological space X, $\overline{V}$ is the closure of V, and $X \setminus V$ or V^c is the complement of V. For α a relation, $\mathrm{Dom}(\alpha) = \{\ x \mid (x, y) \in \alpha$ for some $y \in Y\ \}$. For h a continuous function, or υ a regular Borel measure on a locally compact space, $\mathrm{supp}(h)$ or $\mathrm{supp}(\upsilon)$ will denote the support of h or υ. The domain and range of h are $\mathrm{dom}(h)$ and $\mathrm{ran}(h)$ respectively. For the product space $X \times Y$ let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the natural projection maps. For X a locally compact space, $C_0(X)$ is the C*-algebra of continuous functions vanishing at infinity on X, while $C_b(X)$ and $C_c(X)$ are respectively the bounded continuous functions and the continuous functions with compact support. The space X_∞ is the one-point compactification of X, and $M(X)_+$ denotes the positive regular Borel measures on X. We use the w^*-topology on $M(X)_+$, noting that $M(X) \cong C_c(X)^*$ where $C_c(X)$ carries the inductive limit topology. For $\tau : Y \to Z$ a map of spaces, $\tau_\#$ denotes the dual map $Map(Y,\mathbb{C}) \leftarrow Map(Z,\mathbb{C})$ given by composition, $\tau_\#(h) = h \circ \tau$. If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on $\mathcal{H}$, and for $n \in \mathbb{N}$, $\mathcal{M}_n$ denotes the $n \times n$ matrix algebra over $\mathbb{C}$. For $\mathcal{X}$ a right Hilbert module over a C*-algebra A, denote by $\mathcal{L}(\mathcal{X})$, or by $\mathcal{L}_A(\mathcal{X})$ if A needs to be emphasized, the C*-algebra of adjointable operators on $\mathcal{X}$. If $\mathcal{X}$ is a bimodule over A then the left action of A on $\mathcal{X}$ is defined by a *-homomorphism φ of A to $\mathcal{L}(\mathcal{X})$. The ideal of 'compact' operators on $\mathcal{X}$ is denoted by $\mathcal{K}(\mathcal{X})$, and for $a, b \in \mathcal{X}$, denote by $\Theta_{a,b}$ or $\Theta(a,b)$ the element of $\mathcal{K}(\mathcal{X})$ that maps $x \in \mathcal{X}$ to $a\langle b, x\rangle$.

Parts of this work were initiated while visiting the Wilhelms Universitaet, Muenster. I am grateful to Prof. J. Cuntz for conversations regarding aspects of this work, and also to the group there for their gracious hospitality.

1 The bimodule associated with a topological relation

For X, Y locally compact spaces a relation α is a subset of the locally compact Hausdorff space $X \times Y$. We shall consider certain closed relations here, namely, those subsets α of $X \times Y$ that are closed, and equipped with the subspace topology. We describe a Hilbert bimodule structure starting with $C_c(\alpha)$, namely $C_c(\alpha)$ has a right pre-Hilbert module structure over the C*-algebra $C_0(X)$, and $C_0(Y)$ acts on the left as adjointable operators on the completion of $C_c(\alpha)$. Since continuous maps from X to Y, say for X and Y compact, may be described as *-homomorphisms from the C*-algebras $C(Y)$ to $C(X)$, we broaden the situation and consider positive linear maps r from $C_0(Y)$ to $C_0(X)$ for Y, Y locally compact. This is equivalent to considering maps μ: $X \to C_0(Y)^*$ that are continuous in the w^*-topology, that vanish at infinity – so $\mu_x \to 0$ w^* for $x \to \infty$ – , and are positive. The positive elements of $C_0(Y)^*$ correspond to finite, positive, regular Borel measures on Y. Starting with such a map μ we can define a relation α on $X \times Y$ by setting $\alpha(x) = \{\, y \in Y \mid (x, y) \in \alpha \,\}$ equal to the closed subset $\mathrm{supp}(\mu_x)$ of Y. This is not enough to ensure that the subset α is itself closed, but we only consider such closed relations here.

For X, Y locally compact Hausdorff spaces we briefly describe the correspondence between positive linear maps r: $C_0(Y) \to C_0(X)$ and positive continuous maps $\mu \in C_0(X, C_0(Y)^*)$, where $C_0(Y)^*$ is given the w^*-topology. Since these are abelian C* -algebras, the positive linear map r must be continuous, and on using self adjoint approximate units for C*-algebras it follows that r is also a *-map. The dual map r^*: $C_0(X)^* \to C_0(Y)^*$ is continuous with respect to the w^*-topology, and since the map $X \to C_0(X)^*$ given by $x \to \delta_x$ where $\delta_x(f) = f(x)$ is a continuous map we obtain a w^*- continuous, uniformly bounded map $\mu : X \to C_0(Y)^*$ with image in the positive cone of $C_0(Y)^*$ satisfying $\mu_x \to 0$ w^* as $x \to \infty$. Conversely, given such a map μ define r by $r(f)x = \mu_x(f)$ for $f \in C_0(Y)$ and $x \in X$. This yields a positive linear map r with image in $C_0(X)$.

We further broaden our setting here by considering positive w^*-continuous maps $\mu : X \to M(Y)_+$ to construct our bimodules, so that μ_x is not required to be a finite measure.

Lemma 1.1 *Let $\alpha \subseteq X \times Y$ be closed with X, Y locally compact Hausdorff spaces . If $f \in C_c(\alpha)$ and V is a given open set in $X \times Y$ containing supp(f) then there is a $g \in C_c(X \times Y)$ with $g|_\alpha = f$ and supp$(g) \subseteq V$. If f is positive then g may be chosen with $\|g\|_\infty = \|f\|_\infty$.*

Proof. Set $\mathrm{supp}(f) = K$, a compact subset of α, so also compact and therefore closed in $X \times Y$. Choose a real valued function h in $\mathrm{C_c}(X \times Y)$ with $0 \leq h \leq 1$, $h|_K = 1$, $\mathrm{supp}(h) \subset V$ with V an open neighbourhood of K with compact closure. Since $\overline{V}$ is compact and Hausdorff it is normal, and the Tietze extension theorem yields a continuous function $k : \overline{V} \to \mathbb{C}$ that agrees with f on the closed set $\overline{V} \cap \alpha$. Define $g = 0$ on V^c, and $k \cdot h|_{V^-}$ on $\overline{V}$. Clearly $\mathrm{supp}(g) \subseteq \mathrm{supp}(h|_{V^-}) = \mathrm{supp}(h) \subset V$, so g has compact support. To conclude that g is continuous, note that $k \cdot h|_{V^-}$ is zero on $\overline{V} \cap V^c = \partial V$ since $h = 0$ on V^c.

It remains to show that g agrees with f on α. Write α as a disjoint union of the three sets $\alpha \cap K, \alpha \cap V \cap K^c$, and $\alpha \cap V^c$. On the first set $h = 1$, and so $f = k = h \cdot k = g$ since $K \subset \overline{V} \cap \alpha$. Since f is zero on K^c, we need to show g is also zero on the second and third sets. As noted, $\mathrm{supp}(g) \subset V$, while on the second set k agrees with f which is zero there. ■

Proposition 1.2 *Let $\mu : X \to M(Y)_+$ be a positive w^*-continuous map and $\alpha \subseteq X \times Y$ a closed relation with $\alpha(x) \supseteq$ supp(μ_x) for $x \in X$. For $f \in C_c(\alpha)$ let f_x be the function on $\alpha(x)$ given by $f_x(y) = f(x, y)$. There is a positive linear * map $\Psi : C_c(\alpha) \to C_c(X)$ given by $(\Psi f)(x) = \mu_x(f_x)$, $x \in X$.*

Proof. Note that if $\alpha(x)$ is the empty set interpret $\mu_x(f_x)$ as zero. Let $g \in \mathrm{C_c}(X \times Y)$ with $g|_\alpha = f$ and denote $\mathrm{supp}(g) = K$, the closure of an open set in $X \times Y$. Then since $\mathrm{supp}(\mu_x) \subseteq \alpha(x)$ and $g_x \in \mathrm{C_c}(Y)$ we have $\mu_x(g_x) = \int_{\mathrm{Y}} g_x d\mu_x = \int_{\alpha(x)} g_x d\mu_x = \int_{\alpha(x)} f_x d\mu_x = \mu_x(f_x)$ for $x \in X$. Since $g_x = 0$ for $x \notin \pi_1(K)$ and $\mu_x(f_x) = \mu_x(g_x)$ we have $\{x | (\Psi f)(x) \neq 0\} \subseteq \pi_1(K)$ and Ψf has compact support. Let V be a neighbourhood of K with compact closure.

We show that Ψf is continuous. Since μ is w^*-continuous and $\pi_X(\overline{V})$ is compact, the set $\{\mu_x(h) | x \in \pi_X(\overline{V})\}$ is a bounded set for each $h \in \mathrm{C_c}(Y)$. The principle of uniform boundedness then implies that there is an M such that whenever $\mathrm{supp}(h) \subseteq \pi_{\mathrm{Y}}(\overline{V})$, $|\mu_x(\mathrm{h})| \leq M \|\mathrm{h}\|_\infty$ for all $x \in \pi_1(\overline{V})$ (cf. [4] 12.3 p213). The map g is uniformly continuous on $\pi_1(\overline{V}) \times \pi_2(\overline{V})$. Thus there is a uniform open cover $\mathcal{U}$ of $\pi_1(\overline{V}) \times \pi_2(\overline{V})$ consisting of basic open rectangles formed from uniform open covers of $\pi_1(\overline{V})$ and $\pi_2(\overline{V})$, and g differs on any element of $\mathcal{U}$ by at most ε/M. Thus there is a cover of $\mathcal{W}$ of $\pi_1(K)$, each element of which is a subset of $\pi_1(\overline{V})$, so that if x, $\widetilde{x}$ belong to an element of $\mathcal{W}$ then $|g_x(y) - g_{\widetilde{x}}(y)| \leq \varepsilon/M$ for $y \in \pi_2(\overline{V})$. This actually holds for all $y \in Y$ since for $a \in X$ each g_a is zero on $\pi_2(K)^c$. Note that g is defined at these points even though f need not be. Thus, given x in $\pi_1(V)$, there is an open set O containing x and contained in $\pi_1(\overline{V})$ so that $\|g_x - g_{\widetilde{x}}\|_\infty \lneqq \varepsilon/M$ whenever $\widetilde{x} \in O$. By w^*-continuity, there is an open set U containing x such that $|\mu_x(g_x) - \mu_{\widetilde{x}}(g_x)| \lneqq \varepsilon$ whenever $\widetilde{x} \in U$. Noting that g_x and $g_{\widetilde{x}}$ both have support in $\pi_2(\overline{V})$ we have for $\widetilde{x} \in O \cap U \subset \pi_1(\overline{V})$ that $|\Psi(f)(x) - \Psi(f)(\widetilde{x})| = |\mu_x(g_x) - \mu_{\widetilde{x}}(g_{\widetilde{x}})| \leq \varepsilon + M\|g_x - g_{\widetilde{x}}\|_\infty \lneqq 2\varepsilon$. Since $g_x = 0$ for $x \in$ in $\pi_1(K)^c$ the claim follows. ■

With a positive w^*-continuous map $\mu : X \to M(Y)_+$ and a closed relation $\alpha \subseteq X \times Y$ satisfying $\alpha(x) \supseteq \mathrm{supp}(\mu_x)$ for $x \in X$ we proceed to associate a Hilbert bimodule $\mathcal{C}(\alpha)$. First define a right action of the C*-algebra $\mathrm{C_0}(X)$ on $\mathrm{C_c}(\alpha)$ by $f \cdot h = f((\pi_1)_\#(h))$ and a left action via φ of the C*-algebra $\mathrm{C_0}(Y)$ on $\mathrm{C_c}(\alpha)$ by $\varphi(k)f = ((\pi_2)_\#(k)) f$ where $f \in \mathrm{C_c}(\alpha)$, $h \in \mathrm{C_0}(X)$, $k \in \mathrm{C_0}(Y)$ and multiplication is in $\mathrm{C_c}(\alpha)$.

Proposition 1.3 *For X, Y locally compact Hausdorff spaces, $\mu : X \to M(Y)_+$ a positive w^*-continuous map, and $\alpha \subseteq X \times Y$ a closed relation with $\alpha(x) \supseteq supp(\mu_x)$ for $x \in X$, the map Ψ:$C_c(\alpha) \to C_c(X)$ is a conditional expectation. It is faithful if $\alpha(x) = supp(\mu_x)$ for $x \in X$.*

Proof. We see that $\Psi(f)h = \Psi(f \cdot h)$ for $f \in C_c(\alpha)$, $h \in C_0(X)$ since the function $(f \cdot h)_x$ defined on Y is the function $h(x)f_x$ in $C_0(Y)$ and thus $\Psi(f \cdot h)(x) = \mu_x((f \cdot h)_x) = \mu_x(h(x)f_x) = h(x)\mu_x(f_x) = h(x)\Psi(f)(x)$ for each $x \in X$. If $\alpha(x) = \mathrm{supp}(\mu_x)$ for $x \in X$ and $f \in C_c(\alpha)$ is positive with $\Psi(f) = 0$ we have $\mu_x(f_x) = 0$, $x \in X$. Since $\mu_x(f_x) \gneq 0$ if $f_x(y) \neq 0$ for some $y \in \mathrm{supp}(\mu_x)$ $= \alpha(x)$, we have $f(x, y) = 0$ for all $y \in \alpha(x)$, and $f = 0$. ■

If the conditional expectation Ψ is faithful we obtain an inner product $C_0(X)$ module structure on $C_c(\alpha)$ by $\langle f, g\rangle = \Psi(f^*g)$ for $f, g \in C_c(\alpha)$. Completing $C_c(\alpha)$ in the induced norm $\|\|_\alpha$ yields a right Hilbert $C_0(X)$ module $\mathcal{C}(\alpha)$. We continue to denote the resulting positive linear *-map of $\mathcal{C}(\alpha) \to C_0(X)$ as Ψ . Note that $\|f\|_\alpha^2 = \left\|\Psi(|f|^2)\right\|_\infty = \sup_{x\in X}\left|\mu_x(|f_x|^2)\right|$, and since $(\varphi(k)f)_x = kf_x$ for $k \in C_0(Y)$ and $f \in C_c(\alpha)$ we have that $\|\varphi(k)f\|_\alpha^2 \leq \|k\|_\infty^2 \sup_{x\in X}\left|\mu_x(|f_x|^2)\right| = \|k\|_\infty^2\,\|f\|_\alpha^2$. Thus the left action of $C_0(Y)$ on $C_c(\alpha)$ extends by continuity to an action φ of $C_0(Y)$ on $\mathcal{C}(\alpha)$. We denote Ψ by Ψ_μ and $\mathcal{C}(\alpha)$ by $\mathcal{C}_\mu(\alpha)$ if there is a need to emphasise the dependence on the map μ.

The *-subalgebra of $C_c(\alpha)$ consisting of functions with support contained in a given compact subset K of α is closed in the sup norm, so is a C*-algebra contained in $C_c(\alpha)$. The restriction of the positve linear map Ψ to this C*-algebra must therefore be continuous and $\Psi : C_c(\alpha) \to C_c(X)$ is a continuous map when $C_c(\alpha)$ carries the inductive limit topology, i.e., for K compact in α there is a constant c_K so that $\|f\|_\alpha \leq c_K \|f\|_\infty$ for all $f \in C_c(\alpha)$ with $\mathrm{supp}(f) \subseteq K$. In particular the inclusion of $C_c(\alpha)$ in $\mathcal{C}(\alpha)$ is a continuous injection with dense range and so any dense (in the inductive limit topology) subset of $C_c(\alpha)$ is also dense in $\mathcal{C}(\alpha)$.

Proposition 1.4 *If X, Y are locally compact Hausdorff spaces with $\mu : X \to M(Y)_+$ a positive w^*-continuous map, and $\alpha \subseteq X \times Y$ a closed relation with $\alpha(x) = supp(\mu_x)$ for $x \in X$, then $\mathcal{C}(\alpha)$ is an essential $C_0(Y)$ - $C_0(X)$ Hilbert bimodule.*

Proof. We first note that that the left action of $C_0(Y)$ is by adjointable operators. Since $(k^* \cdot f)_x = k^*f_x$ it follows that the adjoint of left multiplication by $k \in C_0(Y)$ is left multiplication by k^*. To show that the left action φ of $C_0(Y)$ is essential it is enough to see that $\varphi(C_0(Y))\mathcal{C}(\alpha)$ is dense in $\mathcal{C}(\alpha)$. For a given $f \in C_c(Y)$, choose $g \in C_c(X \times Y)$ with $g|_\alpha = f$ and $k \in C_c(Y)$ with $0 \leq k \leq 1$, $k = 1$ on π_2 (supp(f)). Then clearly $\varphi(k)f = f$ in $C_c(\alpha)$. ■

We may also consider the pre-Hilbert module $C_c(\alpha)$ over $C_0(X)$ as a module over $C(X_\infty)$, or over the multiplier algebra $C(\beta X)$ for example. The completion $\mathcal{C}(\alpha)$ remains unchanged as does the algebra $\mathcal{L}(\mathcal{C}(\alpha))$. Since $C_c(\alpha)$ is an ideal in $C_b(\alpha)$ the left action φ of $C_0(Y)$ on $C_c(\alpha)$ defined above may also be extended to an action of $C(\widetilde{Y})$ on $C_c(\alpha)$ where $\widetilde{Y}$ is any compactification of Y.

Given a relation α say that a map μ with the properties stated in Proposition 1.4 is associated with α. The conditions on α and μ place some restrictions on the set α. For example, as remarked in [17], we have that α cannot have "vertical" boundary segments.

Proposition 1.5 *For X, Y locally compact Hausdorff spaces, $\mu : X \to M(Y)_+$ a positive w^*-continuous map, and $\alpha \subseteq X \times Y$ a closed relation with $\alpha(x) = supp(\mu_x)$ for $x \in X$, the restriction of π_1 to α is an open map.*

Proof. Suppose $U = \alpha \cap (U_1 \times U_2)$ with U_1, U_2 open in X, Y respectively and $\pi_1(U)$ is not open in X. There is a net x_η in U_1 with limit x in $\pi_1(U)$ and $x_\eta \notin \pi_1(U)$. Clearly $x \in U_1$ which is open, so the x_η are in U_1. Since $x_\eta \notin \pi_1(U)$ we have $\alpha(x_\eta) \cap U_2 = \phi$, so $\text{supp}(\mu_x) \subseteq \alpha(x)$ implies that $\mu_{x_\eta}(U_2) = 0$ for all η. Continuity of μ implies $\mu_x\ (U_2) = 0$ so that $U_2 \cap \alpha(x) = U_2 \cap \text{supp}(\mu_x)$ $= \phi$. Thus $(x \times \alpha(x)) \cap (U_1 \times U_2) = \phi$, contradicting the choice of x. ∎

2 The C*-algebra associated with a topological relation

Given a closed relation $\alpha \subseteq X \times Y$ with X, Y locally compact Hausdorff spaces we may form the disjoint union $X + Y$ and view α as a closed relation on a locally compact space X. Similarly, if a positive map μ is associated with a closed relation we may extend μ to be a map defined on the disjoint union, which is then associated with the relation defined on this disjoint union. Thus the development in section 1 could have proceeded without loss of generality under the assumption that $X = Y$, so that $\mathcal{C}(\alpha)$ is a $\mathrm{C}_0(X)$- Hilbert bimodule. However it is sometimes helpful to distinguish the domain and range spaces of a relation. The comments after Proposition 1.4 show that we can also view $\mathcal{C}(\alpha)$ as a Hilbert bimodule over $\mathrm{C}(X_\infty)$, or over $\mathrm{C}(\widetilde{X})$ for some compactification $\widetilde{X}$ of X.

For a closed relation α and associated map μ define the C*-algebra of the relation α to be the Cuntz -Pimsner C*-algebra ([13]) of the $\mathrm{C}_0(X)$- Hilbert bimodule $\mathcal{C}(\alpha)$. Denote this C*-algebra by $\mathrm{C}^*(\alpha)$ - this is generated as a universal C*-algebra both by the linear space $\mathcal{X}$ and the algebra A.

For a given Hilbert bimodule $\mathcal{X}$ over a C*-algebra A there are actually various Cuntz-Pimsner C*-algebras that one can form. For example, for each ideal J_0 contained in the ideal $J = \varphi^{-1}(\mathcal{K}(\mathcal{X}))$ of A there is a relative Cuntz-Pimsner C*-algebra $\widetilde{\mathcal{O}}(J_0, \mathcal{X})$ which is generated as a universal C*-algebra by both $\mathcal{X}$ and the algebra A. One can also choose to form the unaugmented C*-algebra $\mathcal{O}(J_0, \mathcal{X})$ which is generated by $\mathcal{X}$ alone. If φ is injective on the ideal J_0 of J then the relative Cuntz-Pimsner C*-algebra $\widetilde{\mathcal{O}}(J_0, \mathcal{X})$ is the unique C*-algebra satisfying the following universal property: For T: $\mathcal{X} \to \mathcal{B}(\mathcal{H})$ a linear map and σ :A $\to \mathcal{B}(\mathcal{H})$ a nondegenerate *-homomorphism with

1. $T(fa) = T(f)\sigma(a)$,
2. $T(\varphi(a)f) = \sigma(a)T(f)$
3. $T^*(f)T(g) = \sigma(\langle f, g\rangle)$ for $f, g \in \mathcal{X}$, $a \in$ A
4. $\sigma^1(\varphi(a)) = \sigma(a)$ for $a \in J_0$, where $\sigma^1 : \mathcal{K}(\mathcal{X}) \to \mathcal{B}(\mathcal{H})$ is defined by $\sigma^1(f \otimes g) = T(f)T(g)^*$

there is a unique nondegenerate representation $\pi : \widetilde{\mathcal{O}}(J_0, \mathcal{X}) \to \mathcal{B}(\mathcal{H})$ with $\pi(qT_f) = T(f)$ for $f \in \mathcal{X}$ and $\pi(q\varphi_\infty(a)) = \sigma(a)$ for $a \in$ A. A pair (T, σ) satisfying the first two conditions above is referred to as a covariant pair for the Hilbert bimodule $\mathcal{X}$ over A. Such a pair is called isometric if it also satisfies condition 3 ([18]). Note that this condition implies $\|T(f)\| \leq \|f\|_\alpha$. If, in addition, the pair (T, σ) satisfies condition 4 say that the pair is an ideal, or a J_0-ideal, isometric covariant representation. The map q is determined by J_0 and is the quotient map of the Toeplitz C*-algebra associated with the bimodule $\mathcal{X}$. This Toeplitz C*-algebra is generated by the creation operators T_f and the diagonal action φ_∞ of A on the Fock space over $\mathcal{X}$. The representation π

is denoted by $T \times \sigma$ and any representation of $\widetilde{\mathcal{O}}(J_0, \mathcal{X})$ arises in this manner. The unaugmented C*-algebra $\mathcal{O}(J_0, \mathcal{X})$ is described in a similar fashion where A is replaced by the closed two-sided *-ideal $\langle \mathcal{X}, \mathcal{X} \rangle$ of A, J_0 by $J_0 \cap \langle \mathcal{X}, \mathcal{X} \rangle$,and the nondegenerate condition on σ is replaced with the condition that image of T acts nondegenerately on $\mathcal{H}$. If J_0 is zero denote the algebras by $\widetilde{\mathcal{O}}(\mathcal{X})$ or $\mathcal{O}(\mathcal{X})$. The C*-algebra C*(α) is then $\widetilde{\mathcal{O}}(J,\mathcal{X})$.

We describe some examples of these relations with their associated diffusions μ.

Example (a) Let f: dom(f) $\to X$ be a continuous map with $\alpha =$ graph(f) closed, where X is a locally compact Hausdorff space and dom(f) $\subseteq X$ a subspace of X .

The next lemma gives examples of such maps f.

Lemma 2.1 *If f :dom(f) $\to X$ is continuous and proper with X locally compact Hausdorff and dom(f) $\subseteq X$ then $\alpha =$ graph(f) is closed in $X \times X$.*

Proof. Assume that $x_\eta \to x$ and $f(x_\eta) = y_\eta \to y$ for a net x_η in dom(f). If K is a compact neighourhood of y, then $f^{-1}(K)$ is a compact neighourhood in dom(f) containing x_η for all $\eta \geq \eta_0$ some η_0. Thus $f^{-1}(K)$ must contain any accumulation point of these x_η, of which there is at least one. Since x is the only possible accumulation point we have $x \in$ dom(f). Thus $f(x) = y$ by continuity of f and so $(x, y) \in$ graph(f). ■

If f :dom(f) $\to X$ is continuous with closed graph, with X locally compact Hausdorff and dom(f) open in X there is a continuous map $\mu : X \to M(X)^+$ with supp($\mu(x)$) $= f(x)$. For example define μ_x to be $\delta_{f(x)}$ for $x \in$ dom(f) and the zero measure for $x \notin$ dom(f). To see that this map is continuous first note that $\delta_{f(x_\eta)} \to \delta_{f(x)}$ in the w^* topology if $x_\eta \to x$ with $x \in$ dom(f), since dom(f) is open and $x_\eta \in$ dom(f) eventually. For $x \notin$ dom(f) we need to check that $\mu(x_\eta) \to 0$ in $M(X)_+$, or that $k(f(x_{\tilde{\eta}})) \to 0$ for $k \in$ C$_c(X)$ if $x_{\tilde{\eta}} \in$ dom(f) for a subnet of x_η. However if K is a compact subset of X then eventually all the points $f(x_{\tilde{\eta}}) \in K^c$ since otherwise the $f(x_{\tilde{\eta}})$ would have an accumulation point y in K and $(x, y) \in \overline{\alpha} = \alpha$. Thus $x \in$ dom(f), a contradiction. Thus eventually $k(f(x_{\tilde{\eta}})) = 0$ for $k \in$ C$_c(X)$. We therefore have a diffusion μ associated with the map f, and since the elements μ_x are actually in C$_0(Y)^*$ we may view this map as a map $\mu : X \to$ C$_0(Y)^*_+$. A slight variation of the continuity argument above shows that this map remains w^* continuous. If dom(f) $= X$ this map is the unique μ satisfying the support condition along with the condition that μ_x is bounded and of norm 1 for each x.

For f : dom(f) $\to X$ a continuous map with closed graph defined on an open subset of a locally compact space X, the locally compact space $D =$ dom(f) is homeomorphic to the locally compact space $\alpha =$ graph(f) via the map $x \to (x, f(x))$, so the pre-Hilbert C$_0(X)$ bimodule C$_c$(graph(f)) can be identified with the bimodule C$_c(D)$ where the left and right actions are given by $\varphi(k)h = f_\#(k)h$ and $h \cdot g(x) = h(x)g(x)$ for $k, g \in$ C$_0(X)$, $h \in$ C$_c(D)$, and $x \in D$. Here $f_\#$ is the *-homomorphism of C$_0(X)$ to C$_b(D)$ given by $f_\#(k)(x) = k(f(x))$. Using the map $\mu : D \to$ C$_0(X)^*_+$ associated with f, the inner product on C$_c(D)$ is given by $\langle h, g\rangle (x) = \overline{h}(x)g(x)$ for $x \in D$. Under this identification the norm $\| \|_\alpha$ on C$_c(D)$ is just $\| \|_\infty$, so the completed Hilbert bimodule $\mathcal{C}(\alpha)$ over C$_0(X)$ is C$_0(D)$.

Lemma 2.2 *The kernel of the *-homomorphism $\varphi : C_0(X) \to \mathcal{L}(C_0(D))$ is $C_0(X \setminus \overline{f(D)})$. For U open in X the map φ is injective on the ideal $C_0(U)$ if and only if $U \subseteq \overline{f(X)}$.*

Proof. For $k \in \mathrm{C}_0(X)$, the restriction of k to $f(D)$ is zero if and only if it is zero on $\overline{f(D)}$; in other words k is in the ideal $\mathrm{C}_0(X \setminus \overline{f(D)})$. The second statement follows from $\mathrm{C}_0(U) \cap \mathrm{C}_0(X \setminus \overline{f(D)}) = \mathrm{C}_0(U \cap (X \setminus \overline{f(D)})$ which is zero if and only if $U \subseteq \overline{f(D)}$. ■

Note that the maximal ideal of $\mathrm{C}_0(X)$ on which φ is injective is $\mathrm{C}_0(\mathrm{Int}\ \overline{f(D)})$, which is zero if and only if $f(D)$ is nowhere dense in X, so when $\ker(\varphi)$ is an essential ideal of $\mathrm{C}_0(X)$.

Lemma 2.3 *If in addition f is a proper map then the ideal $\varphi^{-1}(\mathcal{K}(C_0(D)))$ is $C_0(X)$.*

Proof. If A is a C*-algebra viewed as a Hilbert module over itself, the C*-algebra of compact operators on A is isomorphic to A, with the operator corresponding to multiplication by an element of A ([12]). Thus with $\mathrm{A} = \mathrm{C}_0(D)$, $\varphi^{-1}(\mathcal{K}(\mathrm{C}_0(D))) = \{k \in \mathrm{C}_0(X) \mid f_\#(k) \in \mathrm{C}_0(D)\}$. However, if f is proper then $f_\#\colon \mathrm{C}_0(X) \to \mathrm{C}_0(D)$. ■

If $f : \mathrm{dom}(f) \to X$ is proper with X locally compact Hausdorff and $\mathrm{dom}(f)$ open in X, so that $\mathrm{dom}(f)$ is also locally compact, then there is an extension of f to a continuous map $\mathrm{F} : X_\infty \to X_\infty$, so that $\mathrm{F}|_{dom(f)} = f$. For example define F on $X_\infty \setminus \mathrm{dom}(f)$ by mapping it to ∞ in X_∞. If U is open in X then $\mathrm{F}^{-1}(U) = f^{-1}(U)$ is open in $\mathrm{dom}(f)$, so open in X, and thus also in X_∞. If U is open in X_∞ and of the form K^c for some compact set K of Y, then $\mathrm{F}^{-1}(U) = X_\infty \setminus \mathrm{F}^{-1}(K) = X_\infty \setminus f^{-1}(K)$ which is an open neighbourhood of ∞ in X_∞ since f is proper. Thus F is continuous, and this extension is continuous if and only if $\mathrm{dom}(f)$ is open in X. The range of the continuous map $\widetilde{\mu} : X_\infty \to \mathrm{C}(X_\infty)^*_+$. associated with this map F can be altered by restricting each $\widetilde{\mu}_x$ to the ideal $\mathrm{C}_0(Y)$, so $\widetilde{\mu}_\infty = 0$ for example. The map $\widetilde{\mu} : X_\infty \to \mathrm{C}_0(X)^*_+$ then becomes an extension of the map $\mu : X \to \mathrm{C}_0(X)^*_+$ associated with f. Thus we may instead focus attention on base point preserving continuous maps $g : X_\infty \to X_\infty$ defined on the one point compactification of a locally compact Hausdorff space X. The original proper map f is then the restriction of g to a domain which is an open subset of X and with range in X. Of course, if X is actually compact and $\mathrm{dom}(f) = X$ then we need not restrict ourselves to base point preserving maps.

We now restrict attention to a proper map f, and let $\mathrm{F}\colon X_\infty \to X_\infty$ be an extension of f as above. The bimodule $\mathcal{X} = \mathrm{C}(X_\infty)$ and $\mathrm{A} = \mathrm{C}(X_\infty)$. Let J_0 be the ideal $\mathrm{C}_0(\mathrm{Int}(\mathrm{F}(X_\infty)))$ of $\mathrm{C}(X_\infty) = \varphi^{-1}(\mathcal{K}(\mathcal{X}))$, the maximal ideal of A on which φ is injective. Note that φ is injective on A if and only if F is surjective, so in this case we take J_0 to be A. If (T, σ) is a covariant pair for the bimodule then condition 3 states that $T(k) = T(1)\sigma(k)$ is an isometry for every unitary $k \in \mathrm{A}$, where 1 is the constant function with value 1 on X_∞. Condition 2 may be restated as $\mathrm{T}\,\sigma(\mathrm{F}_\#(h)) = \sigma(h)\,\mathrm{T}$ for $h \in \mathrm{A}$, where T denotes the isometry $T(1)$. Condition 2 is also equivalent to the condition that $\mathrm{T}^*\sigma(h)\mathrm{T} = \sigma(\mathrm{F}_\#(h))$ and $\mathrm{T}\,\mathrm{T}^*\sigma(h)\,\mathrm{T} = \sigma(h)\,\mathrm{T}$ for $h \in \mathrm{A}$. For condition 4 note that if $l \in J_0$ then $\varphi(l)$ is the compact operator $\Theta_{F_\#(l),1}$ on X. Thus $\sigma^1(\varphi(l)) = T(\mathrm{F}_\#(l))T(1)^* = \sigma(l)\mathrm{T}\,\mathrm{T}^*$ by condition 2, so condition 4 states that $\sigma(l) = \sigma(l)\mathrm{T}\,\mathrm{T}^*$ for $l \in J_0$. We also have that $\sigma(l) = \mathrm{T}\,\mathrm{T}^*\sigma(l)$ for $l \in J_0$ since the right hand side is equal to $\mathrm{T}\,\sigma(\mathrm{F}_\# l)\mathrm{T}^* = T(\mathrm{F}_\#(l))\mathrm{T}^* = \sigma(l)\mathrm{T}\,\mathrm{T}^*$. In general if P is a projection in a C*-algebra containing an abelian C*-algebra A then $\{x \in \mathrm{A} \,|\, x\mathrm{P} = \mathrm{P}x = x\}$ is an ideal of A determined by P. Condition 4 thus states that the ideal of $\mathrm{C}(X_\infty)$ determined by the projection $\mathrm{T}\,\mathrm{T}^*$ contains the ideal J_0. The relative Cuntz-Pimsner C*-algebra $\widetilde{\mathcal{O}}(J_0,\mathcal{X})$ is then the universal C* -algebra generated by $\mathrm{A} = \mathrm{C}(X_\infty)$ and an isometry T with $\mathrm{T}\,\mathrm{F}_\#(h) = h\,\mathrm{T}$ for $h \in \mathrm{A}$ and $\mathrm{T}\,\mathrm{T}^*\,h$ (or $h\,\mathrm{T}\,\mathrm{T}^*$) $= h$ for $h \in J_0$. If F is a surjective map of X_∞ then J_0 is all of A and the C*-algebra $\widetilde{\mathcal{O}}(J_0,\mathcal{X})$ is the universal C*-algebra generated by $\mathrm{C}(X_\infty)$ and a unitary U with $\mathrm{U}^*h\,\mathrm{U} = \mathrm{F}_\#(h)$ for $h \in \mathrm{C}(X_\infty)$. This algebra is $\mathrm{C}(\mathrm{Z_F}) \times_{\sigma_F} \mathbf{Z}$ where $\mathrm{Z_F}$ is the compact inverse limit space $\{(x_n) \in \prod_{\mathbb{N}} \mathrm{X}_\infty \mid x_n = \mathrm{F}(x_{n+1}),\ n \geq 0\}$ and σ_F is the homeomorphism of $\mathrm{Z_F}$ given by the shift to the left (lemma 12 of [14], [19]).

Example (b) Let E $= (E^0, E^1,\ r, s)$ be a directed graph where the edge set E^1, and the vertex set E^0 are countable and r, s are the range and source maps of E^1 to E^0. There are various graph C*-algebras one can associate with these graphs (see for example [10] , [8], [2] and the references therein) with the Cuntz algebras $\mathcal{O}_n$ and Cuntz-Krieger algebras $\mathcal{O}_A$ arising as special cases when E^0 is a finite set. Given such a graph which is multiplicity free define a closed relation α on E^0 by $(v, w) \in \alpha$ if and only if there is an edge $e \in E^1$ with $r(e) = v$ and $s(e) = w$. Then α may be identified with E^1, and $\mathrm{C_c}(\alpha) = \mathrm{C_c}(E^1)$ is a pre-Hilbert bimodule over $\mathrm{A} = \mathrm{C}_0(E^0)$ where the diffusion $\mu : E^0 \to M(E^0)_+$ is given by μ_v = counting measure on the discrete set $\alpha(v) = \{w \in E^0 | \,(v, w) \in \alpha\} \cong \{e \in E^1 | \,r(e) = v\}$ and the right and left actions of A on $\mathrm{C_c}(E^1)$ take the form given by $f \cdot h = f\, r_\#(h)$ and $\varphi(h)f = s_\#(h)f$ for $h \in$ A. This is the same bimodule treated in [3] where it is shown, slightly extending results of [8] to the case where E has sinks, that for a suitable ideal J_0 the relative Cuntz-Pimsner algebra $\widetilde{\mathcal{O}}(J_0,\mathcal{X})$ is the graph C*-algebra C*(E) considered in [8]. In [3] it is shown that the unaugmented Cuntz-Pimsner algebra $\mathcal{O}(J_0,\mathcal{X})$ is the graph C*-algebra G^*(E) considered in [2]. Since the C*-algebra G^*(E) of an arbitrary graph E that is not multiplicity free can be described in terms of a multiplicity free graph ([2]) we can obtain all the algebras G^*(E) in this manner.

Example (c) Consider the full relation $\alpha = X \times X$where X is a compact topological Hausdorff space with a countable base, so that $\mathrm{A} = \mathrm{C}(X)$ is separable. Let μ be a normalized regular Borel measure on X and assume that it has no atomic part. If $\mu_x = \mu$ for all $x \in X$ then the pre-Hilbert $\mathrm{C}(X)$- module $\mathrm{C_c}(\alpha)$ is $\mathrm{C}(X \times X) \approxeq \mathrm{C}(X) \otimes \mathrm{C}(X)$ with inner product on simple tensors given by $\langle g_1 \otimes h_1, g_2 \otimes h_2\rangle_{\mathrm{A}} = \overline{g_1} g_2 \langle h_1, h_2\rangle$, where $\langle, \rangle$ denotes the inner product in the Hilbert space $\mathrm{L}^2(X, \mu)$. The right action of A on simple tensors is given by $(g \otimes h)l = gl \otimes h$ for $l \in$ A. If we view A as a Hilbert module over itself in the usual way then the completion $\mathcal{X}$ of the above pre-Hilbert module is just the exterior tensor product ([12]) $\mathrm{A} \otimes \mathrm{L}^2(X, \mu)$ of the Hilbert A-module A with the Hilbert $\mathbb{C}$-module $\mathrm{L}^2(X,\mu)$. The left action φ of A on simple tensors is described by $\varphi(l)(g \otimes h) = g \otimes lh, (l \in \mathrm{A})$. Since the C*-algebra of compact operators on $\mathcal{X}$ is isomorphic to $\mathrm{A}\otimes\mathcal{K}(\mathrm{L}^2(X, \mu)$ ([12]), and since the conditions on X ensure that the range of the multiplication representation $\mathrm{M}{:}\mathrm{C}(X) \to \mathcal{B}(\mathrm{L}^2(X,\mu))$ has zero intersection with the compact operators on $\mathrm{L}^2(X,\mu)$, we conclude that $\varphi^{-1}(\mathcal{K}(\mathcal{X}))= 0$. Thus condition 4 describing the relative Cuntz-Pimsner C*-algebras plays no role here and the algebra can basically be described as a Toeplitz type Pimsner C*-algebra. For (T,σ) a covariant representation of $\mathcal{X}$ on a Hilbert space $\mathcal{H}$, condition 3 becomes $T^*(g_1 \otimes h_1)T(g_2 \otimes h_2) = \overline{g_1} g_2 \langle h_1, h_2\rangle$ on simple tensors. If (T, σ) is an isometric covariant representation of $\mathcal{C}(\alpha) = \mathrm{A}\otimes\ \mathrm{L}^2(X, \mu)$, the restriction $\widetilde{T}$ of T to the infinite dimensional Hilbert space $\mathrm{L}^2(X, \mu)$ is then a linear map of this Hilbert space into $\mathcal{B}(\mathcal{H})$, yielding a representation, and so a unital injection, of the Cuntz algebra $\mathcal{O}_\infty$ in $\mathcal{B}(\mathcal{H})$. For U a unitary in A, multiplying $\widetilde{T}$ on the left by the element $\sigma(U)$ in $\mathcal{B}(\mathcal{H})$ is equal to the quasifree automorphism of $\mathcal{O}_\infty$ described by the map$\widetilde{T}\circ\mathrm{M}_U$ of $\mathrm{L}^2(X,\mu)$ to $\mathcal{B}(\mathcal{H})$. Conversely, given an isometric linear map $\widetilde{T}$ of $\mathrm{L}^2(X,\mu)$ in $\mathcal{B}(\mathcal{H})$ and a unital representation σ of A in $\mathcal{B}(\mathcal{H})$ satisfying $\sigma(a)\widetilde{T} = \widetilde{T}\circ \mathrm{M}_a$ then one obtains an isometric covariant representation (T, σ) of $\mathcal{C}(\alpha)$ by setting $T(g \otimes h) = \widetilde{T}(h)\sigma(g)$ on simple tensors $g \otimes h$ in $\mathrm{C}(X)\otimes\mathrm{C}(X)$.

If we set $X = \mathbb{T}$ and μ to be Haar measure then some of the structure involving the quasifree automorphisms simplifies. This is also the case for X a compact abelian group with diffuse Haar measure. Let $\{\chi_n \mid n \in \mathbb{Z}\}$ be the character group of $\mathbb{T}$. We have for the unitary U of A defined by the generating character χ_1 on $\mathbb{T}$, that $U\widetilde{T}(\chi_n) = \widetilde{T}(M_U(\chi_n)) = \widetilde{T}(\chi_{n+1})$ for $n \in \mathbb{Z}$. Thus $\mathcal{O}(\mathcal{X})$ is the universal C*-algebra generated by a family $\{T_n \mid n \in \mathbb{Z}\}$of isometries with orthogonal ranges and a unitary U with $UT_n = T_{n+1}$. Equivalently, it is the C*-algebra generated by an isometry T and a unitary U with $T^*U^nT = 0$ for $n \gneqq 0$.

Theorem 2.4 *The C*-algebra $\mathcal{O}(\mathcal{X})$ is isomorphic to the crossed product of $\mathcal{O}_\infty$ by an automorphism σ.*

Proof. The Cuntz algebra $\mathcal{O}_\infty$ is isomorphic to the C*-algebra generated by a family of isometries $\{S_n \mid n \in \mathbb{Z}\}$ with orthogonal range projections. Define an automorphism of $\mathcal{O}_\infty$ by $\sigma(S_n) = S_{n+1}$ for all n. Since $\{U^nTU^{-n} \mid n \in \mathbb{Z}\}$ are isometries in $\mathcal{O}(\mathcal{X})$ with orthogonal ranges there is a *-homomorphism $\varphi : \mathcal{O}_\infty \to \mathcal{O}(\mathcal{X})$ mapping S_n to U^nTU^{-n}. By the universal property of a crossed product there is a *-homomorphism $\theta : \mathcal{O}_\infty \times_\sigma \mathbb{Z} \to \mathcal{O}(\mathcal{X})$ with $\theta(S_n) = U^nTU^{-n}$ which is clearly surjective. The universal property of $\mathcal{O}(\mathcal{X})$ implies that there is a *-homomorphism $\eta : \mathcal{O}(\mathcal{X}) \to \mathcal{O}_\infty \times_\sigma \mathbb{Z}$ mapping U to the unitary V in $\mathcal{O}_\infty \times_\sigma \mathbb{Z}$ implementing σ, and T to S_0. Since $\eta \circ \theta(V) = V$ and $\eta \circ \theta(S_n) = \eta(U^nTU^{-n}) = \sigma^n(S_0) = S_n$ we have that $\eta \circ \theta = I$ on $\mathcal{O}_\infty \times_\sigma \mathbb{Z}$, so θ is injective and thus a *-isomorphism. ∎

Proposition 2.5 *Let $\{S_n \mid n \in \mathbb{Z}\}$ be isometries with orthogonal range projections generating the Cuntz algebra $\mathcal{O}_\infty$. The automorphism of $\mathcal{O}_\infty$ defined by $\sigma(S_n) = S_{n+1}$ for $n \in \mathbb{Z}$ is outer.*

Proof. Suppose U is a unitary in $\mathcal{O}_\infty$ with $US_nU^* = S_{n+1}$ for $n \in \mathbb{Z}$. Then $Up_nU = US_nS_n^*U^* = S_{n+1}S_{n+1} = p_{n+1}$ where p_n is the range projection of S_n. Set W_k equal to the set of $k-$ tuples of elements from $\mathbb{Z}$ and $W = (\cup_{k \gneqq 0} W_k) \cup \{\phi\}$. For $\mu = (l_1, ..., l_k) \in W_k$ let $\pi(\mu) = l_1$, set $S_\mu = S_{l_1}...S_{l_k}$, and define $S_\phi = I$. Since any nonzero word in S_k and S_k^* is of the form $S_\mu S_\nu^*$ for some unique $\mu, \nu \in W$ ([5]) there is a finite linear combination of words, $x = \sum_{i \in J} \alpha_i S_{\mu_i} S_{\nu_i}^*$, with $\|U - x\| \lneqq \epsilon$. In particular $\|UaU^* - xax^*\| \lneqq \epsilon(2+\epsilon)$ for a in the unit ball of $\mathcal{O}_\infty$. Since the partial isometry S_ν^* has initial space contained in p_l where $l = \pi(\nu)$ and $\nu \neq \phi$, we have that $S_\nu^* p_k$ is equal to 0 if $k \neq \pi(\nu)$, p_k if $\nu = \phi$, and S_ν^* if $k = \pi(\nu)$. By taking adjoints $p_k S_\mu$ is 0 if $k \neq \pi(\mu)$, S_μ if $k = \pi(\mu)$, and p_k if $\mu = \phi$. Choose $r \in \mathbb{N}$ with both r and $r+1$ not in the finite subset F of $\mathbb{Z}$ where $\mathrm{F} = \cup_{i \in J} \{\pi(\mu_i) \cup \pi(\nu_i)\}$. Then $xp_rx^* = \sum \left\{ \alpha_i \overline{\alpha}_j S_{\mu_i} p_r S_{\mu_j}^* \mid i, j \in J \text{ such that } \nu_i = \phi, \nu_j = \phi \right\}$. Since $S_{\mu_i} S_{\nu_j}$ is a partial isometry from a subspace of p_k into p_l where $k = \pi(\mu_j)$ and $l = \pi(\mu_i) \in$ F, we have that the range of xp_rx^* is perpendicular to the range of $Up_rU^* = p_{r+1}$ in any representation of $\mathcal{O}_\infty$ on a Hilbert space. Thus $\|Up_rU^* - xp_rx^*\| \geq 1$, contradicting the choice of x to be within ϵ of U. ∎

Corollary 2.6 *The C*-algebra of the relation $\mathbb{T} \times \mathbb{T}$, $\mathcal{O}(\mathcal{X})$, is a classifiable unital Kirchberg algebra with $K_0(\mathcal{O}(\mathcal{X}))$ and $K_1(\mathcal{O}(\mathcal{X}))$ both isomorphic to $\mathbb{Z}$.*

Proof. Since the automorphism σ is outer by the previous proposition, and $\mathcal{O}_\infty$ is a Kirchberg algebra satisfying the UCT, so is the crossed product algebra $\mathcal{O}_\infty \times_\sigma \mathbb{Z}$ (see Prop. 4.3.2 and other results in [16] for example). Since $\mathrm{K}_1(\mathcal{O}_\infty)$ is 0, $\mathrm{K}_0(\mathcal{O}_\infty)$ is $\mathbb{Z}$, and the induced map σ_* is the identity on $K_0(\mathcal{O}_\infty)$ the Pimsner-Voiculescu six term exact sequence yields the stated K-groups of $\mathcal{O}(\mathcal{X})$. ■

Note that the K-groups of $\mathcal{O}(\mathcal{X})$ also follow from [13] where it is shown that the Toeplitz algebra is K-equivalent to A, which in this example is C($\mathbb{T}$) . We point out that the bimodule considered

here is an example of the bimodule discussed in [11] where the conclusions of the last Corollary are drawn in a general context.

Example (d) Consider the closed relations on $X = \mathbb{T}^d$ arising in connection with the solenoidal C*-algebras of [1]. Let α be the connected component of the identity in $\left\{(x, y) \in \mathbb{T}^d \times \mathbb{T}^d \mid ay = Fx\right\}$ where $a \in \mathbb{N}$ with $a > 1$, and F $\in \mathfrak{m}_d(\mathbb{Z})$ has nonzero determinant (with $\mathbb{T}$ considered as an additive group). For our present purposes we only illustrate the situation with the special case $d = 1$ and $F = I$. We will later consider elsewhere the more general situation which can be obtained by tensoring bimodules from Example (a) above with this situation. The situation considered here is also a special case of a relation obtained as the inverse of a covering map $c : \mathbb{T} \to \mathbb{T}$, $c(t) = at$. The C*-algebra considered here is also an example of a continuous graph C*-algebra defined in [6] and in more general form in [7]. Thus let $\alpha = \{(x, y) \in \mathbb{T} \times \mathbb{T} \,|x = ay\}$. For $x \in \mathbb{T}$, the subset $\alpha(x)$ is a finite discrete set, and if μ_x is normalized counting measure on $\alpha(x)$ we obtain a positive ω^*-continuous map $\mu : T \to$ C($\mathbb{T}$)* where $\mu_t(f) = a^{-1} \sum_{k=0}^{a-1} \{f(t_k) \mid at_k = t\}$ for $f \in$ C($\mathbb{T}$) . The subset α is homeomorphic with $\mathbb{T}$ via the parametrization $h : t \to (at, t)$ for $t \in \mathbb{T}$: this is true in the general case if $(a, \det(F)) = 1$. Thus we may identify $C_c(\alpha)$ with C($\mathbb{T}$) and the right action of A = C($\mathbb{T}$) on C($\mathbb{T}$) is given by $(fh)(t) = f(t)h(at)$ while the left action φ is given by $\varphi(h)f(t) = h(t)f(t)$ for $f \in$ C($\mathbb{T}$) and $h \in$ A. For $f \in$ C($\mathbb{T}$) the norm $\|f\|_\alpha^2 = \left\|\Psi(|f|^2)\right\|_\infty = \sup_{t\in\mathbb{T}} a^{-1} \sum_{k=0}^{a-1} \left\{|f|^2(t_k) \mid at_k = t\right\} \le \|f\|_\infty^2$, so for $\{\chi_n \mid n \in \mathbb{Z}\}$ the character group of $\mathbb{T}$, it follows that $Span\{\chi_n \mid n \in \mathbb{Z}\}$ is a dense subspace of the Hilbert bimodule $\mathcal{X} = \mathcal{C}(\alpha)$, the completion of C($\mathbb{T}$) in the $\|\|_\alpha$ norm. Letting $U \in$ A denote the unitary element $U(t) = t$, $(t \in T)$ of A, the actions of A on C($\mathbb{T}$) are described by $\chi_n U = \chi_{n+a}$ and $\varphi(U)\chi_n = \chi_{n+1}$, $n \in \mathbb{Z}$. Writing $t \in \mathbb{T}$ as $e^{2\pi ir}$ we have that the conditional expectation satisfies $\Psi(\chi_n)(t) = a^{-1} \sum_{k=0}^{a-1} \chi_n\left(e^{2\pi i(r+k)/a}\right) = a^{-1}e^{2\pi inr/a} \sum_{k=0}^{a-1} e^{2\pi ikn/a} = U^{n/a}(t)$ if $n \in a\mathbb{Z}$ and zero otherwise. Thus the element $\Theta(\chi_n, \chi_k^*)$ of $\mathcal{K}(\mathcal{X})$ applied to χ_l is equal to $\chi_n\psi(\chi_{l-k})$, which is zero unless $l - k \in a\mathbb{Z}$ in which case it is $\chi_n U^{(l-k)/a} = \chi_{n+l-k.}$ It follows that for any $n \in \mathbb{Z}$, all but one of the terms in the following sum is zero and $\sum_{k=0}^{a-1} \Theta(\chi_{n+k}, \chi_{n+k}^*)(\chi_l) = \chi_l$. Therefore the identity operator on X is equal to $\sum_{k=0}^{a-1} \Theta(\chi_k, \chi_k^*) \in \mathcal{K}(\mathcal{X})$ and the ideal $J = \varphi^{-1}(\mathcal{K}(\mathcal{X})$ is all of A. The map φ defining the left action is clearly injective on J.

If (T, σ) is an ideal isometric covariant representation of the bimodule $\mathcal{C}(\alpha)$ on a Hilbert space $\mathcal{H}$ it satisfies the conditions

1. $T_n\sigma(U) = T_{n+a}$
2. $\sigma(U^k)T_n = T_{n+k}$
3. $T_n^*T_m = \sigma(U^{(m-n)/a})$ if $m - n \in a\mathbb{Z}$, and 0 otherwise
4. $\sum_{k=0}^{a-1} T_kT_k^* = I_{\mathcal{H}}$

for $k,n \in \mathbb{Z}$, where T_n =T(χ_n). Note that the third and fourth conditions imply that for each $n \in \mathbb{Z}$ the elements $\left\{T_n, T_{n+1}, ...T_{n+(a-1)}\right\}$ are isometries defining a representation of the Cuntz algebra $\mathcal{O}_a$. Setting T to be the isometry T_0 we see that condition 2) allows one to define T_n as $\sigma(U^n)T$ for $n \in \mathbb{Z}$, while condition1) is then equivalent to $\sigma(U^a)T = T\sigma(U)$. It follows that $T_n^* T_m = T^*\sigma(U^{m-n})T$ which by condition 1) is $T^*T\sigma(U^{(m-n)/a}) = \sigma(U^{(m-n)/a})$ if $m-n \in aZ$, and 0 otherwise since condition 4) implies that the range projections of the T_k , $k = 0$,...,$a-1$, are orthogonal. Thus conditions 3) and 4) may be replaced with the stipulation that T is an isometry with $\sum_{k=0}^{a-1} U^k TT^* U^{-k} = I_{\mathcal{H}}$.

Conversely, if T is an isometry and V a unitary on a Hilbert space $\mathcal{H}$ with $V^a T = TV$ and $\sum_{k=0}^{a-1} V^k TT^* V^{-k} = I_{\mathcal{H}}$ then this gives rise to an isometric covariant representation (T, σ) of $\mathcal{C}(\alpha)$ on $\mathcal{H}$ by setting T$(\chi_n) = V^n T$ and $\sigma(U) = V$ for $n \in \mathbb{Z}$. Note that by condition 3) the linear map T may be extended to $\mathcal{C}(\alpha)$ by continuity from the dense subalgebra *Span* $\{\chi_n \mid n \in \mathbb{Z}\}$.

Theorem 2.7 *For $a \in \mathbb{N}, a > 1$, the C*-algebra $C^*(\alpha) = \widetilde{\mathcal{O}}(A,\mathcal{C}(\alpha))$ of the relation $\alpha = \{(x,y) \in \mathbb{T}\times\mathbb{T} \mid ay = x\}$ is the universal C*-algebra generated by an isometry T and a unitary U with $U^a T = TU$ and $\sum_{k=0}^{a-1} U^k TT^* U^{-k} = I$.*

Lemma 2.8 *Let U be a unitary and T an isometry in a C*-algebra satisfying the conditions of Theorem 2.7. Then any word in U, U^*, T, and T^*may be expressed in the form $U^g T^b T^{*c} U^{-h}$ where $b,c \in \mathbb{N}$ and $g,h \in \mathbb{Z}$. Furthermore h may be chosen with $0 \le h < a^c$.*

Proof. First note that multiplying $U^a T = TU$ on the left by U^{-a} and by U^{-1}on the right implies $U^{-a}T = TU^{-1}$, from which it follows that $T^c U^b = U^{ba^c} T^c$ for $b \in \mathbb{Z}, c \in \mathbb{N}$. In particular any power of U to the right of a positive power of T may be moved to the left of this power of T. Similarly,by taking adjoints, any power of U to the left of a positive power of T^* may be moved to the right of this power of T^*. The sum condition implies orthogonality conditions on the range projections of the isometries $U^k T$, so that $T^* U^b T$ is zero if $b \notin a\mathbb{Z}$, otherwise it $= U^k$ for $b = ak$. Thus $T^{*c} U^b T^c = U^k$ if $b = a^c k$, and zero if $b \notin a^c\mathbb{Z}$. It follows , for $c, d \in \mathbb{N}$ and $b \in \mathbb{Z}$, that

$$T^{*c} U^b T^d = \begin{cases} U^k T^{d-c} & \text{if } c = c \wedge d \text{ and } b = a^c k \text{ some } k, \\ T^{*(c-d)} U^k & \text{if } d = c \wedge d \text{ and } b = a^d k \text{ some } k, \\ 0 & \text{otherwise.} \end{cases}$$

Applying this identity to the inner three terms formed from the product of two terms of the stated form $U^g T^b T^{*c} U^{-h}$ again yields a term of the stated form, after possibly moving U terms to the right or left of powers of T or T^*. Since any word in U, U^*, T, and T^* may be viewed as a product of terms of the standard form this finishes the initial claim. Writing $h = a^c k + r$ with $0 \le r \lneq a^c$ and noting that $T^{*c} U^{-a^c k} = U^{-k} T^{*c}$ shows the final claim. ■

Proposition 2.9 *For each $n \in \mathbb{N}$, $\mathcal{A}_n = Span\left\{U^g T^n T^{*n} U^{*h} \mid g,h \in \mathbb{Z}\right\}$ is a unital *-subalgebra of $C^*(\alpha)$ with $\mathcal{A}_n \subset \mathcal{A}_{n+1}$.*

Proof. The product $(U^g T^n T^{*n} U^{*h})(U^r T^n T^{*n} U^{*s}) = U^g T^n (T^{*n} U^{r-h} T^h) T^{*n} U^{*s}$, and since $T^{*n} U^{r-h} T^n = U^k$ if $r - h = a^n k$ for some $k \in \mathbb{Z}$, and otherwise is zero, the product, if not zero must be $= U^g U^{a^n k} T^n T^{*n} U^{*s}$ which is an element of $\mathcal{A}_n$. The element $U^g T^n T^{*n} U^{*h} = U^g T^n (\sum_{k=0}^{a-1} U^k TT^* U^{*k}) T^{*n} U^{*h} = \sum_{k=0}^{a-1} U^{g+ka^n} T^{n+1} T^{*(n+1)} U^{*(h+ka^n)} \in \mathcal{A}_{n+1}$, so $\mathcal{A}_n \subset \mathcal{A}_{n+1}$. Since $I \in \mathcal{A}_0$, the algebras $\mathcal{A}_n$ are unital. ■

Lemma 2.10 *Let* $p_0, ..., p_{b-1}$ *be projections in* $\mathcal{B}(\mathcal{H})$ *with sum* I, *and* W *a unitary in* $\mathcal{B}(\mathcal{H})$ *with* $ad(W)(p_k) = p_{\sigma(k)}$ *for* σ *a cyclic permutation of order* b. *Then* $spec(W)$ *contains the* b *-th roots of unity.*

Proof. Let $\mathcal{D}$ be the C*-algebra generated by W and $\{p_k \mid k = 0, ..., b-1\}$. Defining $e_{0,k} = p_0 W^{*k}$ for $k = 0, ..., b-1$ the elements $e_{j,k} = (e_{0,j})^* e_{0,k}$ form a system of matrix units for (a representation of) $\mathcal{M}_b$ in $\mathcal{B}(\mathcal{H})$. The element W^b lies in the commutant of $\mathcal{D}$, so is in the commutant of $\mathcal{M}_b$. Since $W = \sum_{k=0}^{b-1} W e_{kk} = \sum_{k=0}^{b-2} e_{k+1,k} + W^b e_{o,b-1}$ it follows that $\mathrm{C}^*(W^b) \otimes \mathcal{M}_b \cong \mathcal{D}$, with the element $(W^b)^n \otimes e_{j,k}$ of $\mathrm{C}^*(W^b) \otimes \mathcal{M}_b$ corresponding to $W^{j+bn} p_0 W^{*k}$ for $n \in \mathbb{N}$. If π is the quotient *-homomorphism of $\mathcal{D}$ to $\mathcal{M}_n$ defined via a quotient map of $\mathrm{C}^*(W^b)$ to $\mathbb{C}$, then $spec(W) \supseteq spec(\pi(W))$ where $\pi(W)$ is the unitary $\sum_{k=0}^{b-2} e_{k+1,k} + e_{0,b-1}$ which has spectrum equal to the b^{th} roots of unity. ■

For $0 \le i, j \le a^n - 1$ the elements $\mathrm{E}^n_{i,j} = (U^i T^n)(U^j T^n)^*$ are a system of matrix units for $\mathcal{M}_{a^n}$ in $\mathcal{A}_n$. The unitary U^{a^n} satisfies $U^{a^n} T^n = T^n U$ so it commutes with the elements $\mathrm{E}^n_{i,j}$.

Proposition 2.11 *Let* $C^*(\alpha)$ *be the universal* C^**-algebra generated by an isometry* T *and a unitary* U *with* $U^a T = TU$ *and* $\sum_{k=0}^{a-1} U^k T T^* U^{-k} = I$. *We have that* $spec(U) = \mathbb{T}$ *and the subalgebra* $\mathcal{A}_n \cong C(\mathbb{T}) \otimes \mathcal{M}_{a^n}$.

Proof. Applying Lemma 2.10 to the unitary $W = U$ and the projections $p_k = \mathrm{E}^n_{k,k}$ where $b = a^n$ we conclude that $spec(U)$ contains the a^n -th roots of unity. As n is arbitrary, $spec(U) = \mathbb{T}$. Proposition 2.9 and part of Lemma 2.8 imply that the C*-subalgebra $\mathcal{A}_n$ is isomorphic to the C*-algebra generated by U and the projections p_k. The argument in Lemma 2.10 thus shows $\mathcal{A}_n$ isomorphic to $\mathrm{C}^*(U^{a^n}) \otimes \mathcal{M}_{a^n} \cong \mathrm{C}(\mathbb{T}) \otimes \mathcal{M}_{a^n}$. ■

The proof of Proposition 2.9 shows that the matrix unit $\mathrm{E}^n_{i,j}$ in $\mathcal{A}_n$ is identified with the element $\sum_{k=0}^{a-1} \mathrm{E}^{n+1}_{i+ka^n, j+ka^n}$ of $\mathcal{A}_{n+1}$. This is just the inclusion of $\mathcal{M}_{a^n}$ into $\mathcal{M}_{a^{n+1}}$ which maps a matrix A to the block diagonal matrix in $\mathcal{M}_{a^{n+1}}$ with A repeated a times down the diagonal. Also, for each $n \in \mathbb{N}$, we have that the unitary $U \in \mathcal{A}_n$, where U is viewed as the element $\sum_{k=0}^{a^n-2} \mathrm{E}^n_{k+1,n} + U^{a^n} \mathrm{E}^n_{0,a^n-1}$. Comparing this element of $\mathcal{A}_n$ with with the corresponding element of $\mathcal{A}_{n+1}$ and using the isomorphism of $\mathcal{A}_n$ with $\mathrm{C}(\mathbb{T}) \otimes \mathcal{M}_{a^n}$ shows that the inclusion maps of $\mathcal{A}_n$ in $\mathcal{A}_{n+1}$ are maps which allow one to identify the closure of the *-subalgebra $\cup_{n \ge 0} \mathcal{A}_n$ of $\mathrm{C}^*(\alpha)$ as the Bunce-Deddens algebra$\mathfrak{B} = \mathfrak{B}(\{a^n\})$, a simple, unital, nuclear, separable C*-algebra with a unique faithful trace ([5]). It is also clearly in the UCT class and an AH-algebra of bounded dimension growth, so it is approximately divisible ([16], pg 48).

Theorem 2.12 *The* C^**-algebra* $C^*(\alpha)$ *is isomorphic to the semidirect product* C^**-algebra* $\mathfrak{B} \times_\rho \mathbb{N}$ *where* $\mathfrak{B}$ *is the Bunce-Deddens algebra* $B(\{a^n\})$ *and* ρ *is a corner endomorphism scaling the unique trace on* $\mathfrak{B}$. *It is a classifiable unital Kirchberg algebra with both K-groups* $K_0(C^*(\alpha))$ *and* $K_1(C^*(\alpha))$ *isomorphic to* $\mathbb{Z}/(a-1)\mathbb{Z}$.

Proof. The isometry T in $\mathrm{C}^*(\alpha)$ defines a corner *-endomorphism $\rho(x) = TxT^*$ for $x \in \mathfrak{B}$. The trace τ on $\mathfrak{B}$ restricts to the usual normalized trace τ_n on $\mathcal{A}_n \cong \mathcal{M}_{a^n}(\mathrm{C}(\mathbb{T}))$, so $\tau(\mathrm{E}^n_{k,k}) = \tau_n(E^n_{k,k}) = a^{-n}$ while $\tau \circ \rho(E^n_{k,k}) = \tau_{n+1}(E^{n+1}_{ak,ak}) = a^{-(n+1)}$ and ρ scales the trace by a^{-1}. Lemma 2.8 and the comments above imply that $\mathrm{C}^*(\alpha)$ is generated as a C*-algebra by the isometry T and the C*-subalgebra $\mathfrak{B}$, so there is a surjection of $\mathfrak{B} \times_\rho \mathbb{N}$ onto $\mathrm{C}^*(\alpha)$ which must be an isomorphism since $\mathfrak{B} \times_\rho \mathbb{N}$ is simple ([16]). We also have that $\mathfrak{B} \times_\rho \mathbb{N}$ is a unital, purely infinite, separable, nuclear, and in the UCT class, so a classifiable Kirchberg algebra ([16]).

The Pimsner Voiculescu six term exact sequence may be used to compute the K-groups as in the manner for the Cuntz algebras (cf. [16]). Using that $\mathrm{K}_0(\mathfrak{B}) \cong \mathbb{Z}[a^{-1}]$ and $\mathrm{K}_1(\mathfrak{B}) \cong \mathbb{Z}$ we obtain the following exact sequence of abelian groups:

$$\begin{array}{ccccc} \mathbb{Z}[a^{-1}] & \overset{I-K_0(\overline{\rho})}{\rightarrow} & \mathbb{Z}[a^{-1}] & \rightarrow & \mathrm{K}_0(\mathrm{C}^*(\alpha)) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(\mathrm{C}^*(\alpha)) & \leftarrow & \mathbb{Z} & \underset{I-K_1(\overline{\rho})}{\leftarrow} & \mathbb{Z} \end{array}$$

Since ρ scales the trace by a^{-1} it follows that the first horizontal map is the injection $(a-1)/a$, so the left hand vertical map is 0. If p_0 is the projection $\rho(I)$ then $p_0 = \mathrm{E}^1_{0,0}$ and $\rho(U) = \rho(I)\rho(U)\rho(I) = p_0 TUT^* p_0 = p_0 U^a p_0 = p_0 M^a p_0$ where M is the matrix $\sum_{k=0}^{a-2} \mathrm{E}^1_{k+1,n} + U^a \mathrm{E}^1_{0,a-1}$ in $\mathcal{A}_1$ or the element $\sum_{k=0}^{a-2} e_{k+1,k} + Ue_{o,a-1}$ of $\mathrm{C}(\mathbb{T}) \otimes \mathcal{M}_a$. It follows from this that the lower right horizontal arrow is the injection $1-a$ on $\mathbb{Z}$, and so the right vertical map is also 0. Exactness yields the stated K-groups. ∎

For V a unitary in a unital C*-algebra A and β a unital $*$-endomorphism, $\beta(V)$ is a unitary in A. If π is a representation of A in $\mathcal{B}(\mathcal{H})$ and $\{S_k \mid k = 0, ..., b-1\}$ are isometries in $\mathcal{B}(\mathcal{H})$ determining a representation of the Cuntz algebra $\mathcal{O}_b$ then $\psi(x) = \sum_{k=0}^{b-1} S_k x S_k^*$ is a unital $*$-endomorphism of $\mathcal{B}(\mathcal{H})$. By definition $(\pi, \{S_k\}_{k=0}^{b-1})$ is a covariant representation of (A, β) in the sense of Stacey ([19]) if $\pi \circ \beta = \psi \circ \pi$. We note that this is the case if and only if $\pi(\beta(V))S_l = S_l \pi(V)$ for each unitary $V \in A$ and each $l = 0, ..., b-1$. For if $\pi(\beta(V)) = \psi(\pi(V))$ then $\pi(\beta(V))S_l = \psi(\pi(V))S_l = \sum_{k=0}^{b-1} S_k \pi(V) S_k^* S_l = \sum_{k=0}^{b-1} S_k \pi(V) \delta_{k,l} I = S_l \pi(V)$ for each l and unitary V. Conversely we have $\psi(\pi(V)) = \sum_{k=0}^{b-1} S_k \pi(V) S_k^* = \sum_{k=0}^{b-1} \pi(\beta(V)) S_k S_k^* = \pi(\beta(V)) \sum_{k=0}^{b-1} S_k S_k^* = \pi(\beta(V))$ for each unitary $V \in A$, so $\pi \circ \beta = \psi \circ \pi$.

Now consider $A = \mathrm{C}(\mathbb{T})$ and the unital $*$-endomorphism β of A dual to the map $t \to at$ on $\mathbb{T}$ (written additively). In the C*-algebra $\mathrm{C}^*(\alpha)$ the isometries $S_k = U^k T$ for $k = 0, ..., a-1$, and the unitary U satisfy $U^a S_k = S_k U$, so $\beta(U) S_k = S_k U$ for each k. By the preceding discussion, since U generates the C*-algebra $\mathrm{C}(\mathbb{T})$, there is a representation of the (Stacey) crossed product algebra $\mathrm{C}(\mathbb{T}) \times_\beta^a \mathbb{N}$ with image all of $\mathrm{C}^*(\alpha)$. Thus $\mathrm{C}^*(\alpha)$ is a simple quotient of $\mathrm{C}(\mathbb{T}) \times_\beta^a \mathbb{N}$. Since there are covariant representations $(\pi, \{S_k\}_{k=0}^{a-1})$ of $(\mathrm{C}(\mathbb{T}), \beta)$ that do not satisfy the relations in Theorem 2.7 – for example the unitary $V = I$ satisfies $V^a S_k = S_k V$ for each k – $\mathrm{C}^*(\alpha)$ must be a proper quotient algebra.

Example (e) Consider an elementary closed relation on the compact space $X = \mathbb{T}$ which is not locally homeomorphic with X, namely $\alpha = \{(x, y) \mid x \in \mathbb{T} \text{ and } y = x \text{ or } x^{-1}\}$. Here view $\mathbb{T}$ multiplicatively as the unit circle on the complex plane. The relation α provides an example of a branched covering ([7]) via the map $\pi_1 : \alpha \to \mathbb{T}$ with branch sets $S = \{(1,1), (-1,-1)\}$ and $S' = \{1, -1\}$, however this is not a covering of a space by itself, so the approach considered in [7] associating a C*-algebra to a branched covering does not immediately apply here. If μ_x is normalized counting measure on $\alpha(x) = \{x, x^{-1}\}$ for $x \notin \{1, -1\}$ then one must have that $\mu_1 = \delta_1$ and $\mu_{-1} = \delta_{-1}$ if $\mu : \mathbb{T} \to M(\mathbb{T})_+$ is to be ω^*–continuous. A continuous function F on α may equivalently be viewed as a pair (f, g) of continuous functions on $\mathbb{T}$ where $f(t) = F(t,t)$ and $g(t) = F(t, t^{-1})$, so that the space $\mathrm{C}(\alpha)$ may be identified with the subspace $\{(f, g) \mid f - g \in B\}$ of

C($\mathbb{T}$) $\oplus$ C($\mathbb{T}$) where B is the ideal of functions in C($\mathbb{T}$) vanishing at $\{1,-1\}$. The right action of C($\mathbb{T}$) on C(α) may then be written as $(f,g)\cdot h=(fh,gh)$, while the (clearly injective) left action φ of C($\mathbb{T}$) on C(α) becomes $\varphi(h)(f,g)=(hf,\gamma_{\#}(h)g)$ where γ is the map $t\to t^{-1}$ on $\mathbb{T}$ and $h\in$ C($\mathbb{T}$). Under this identification the conditional expectation Ψ :C(α) $\to$ C($\mathbb{T}$) is $\Psi(f,g)=\frac{1}{2}(f+g)$, and the norm on C($\alpha$) is $\|(f,g)\|_{\alpha}^{2}=\frac{1}{2}\left\||f|^{2}+|g|^{2}\right\|_{\infty}$. We have that $\|f\|^{2}\vee\|g\|^{2}\leq 2\|(f,g)\|_{\alpha}^{2}$ and $\|(f,g)\|_{\alpha}\leq\|f\|_{\infty}\vee\|g\|_{\infty}$. Thus the norm on C($\alpha$) is equivalent to the l^{∞} norm on C($\mathbb{T}$) $\oplus$ C($\mathbb{T}$) and so the space C(α) is complete and forms a Hilbert bimodule over A = C($\mathbb{T}$) .

Theorem 2.13 *The ideal* $\varphi^{-1}(\mathcal{K}(\mathcal{C}(\alpha)))$ *of* C($\mathbb{T}$) *is* $B=\left\{h\mid h\mid_{\{1,-1\}}=0\right\}$.

Proof. If $\varphi(l)\in\mathcal{K}(\mathcal{C}(\alpha))$ there is a sequence K_n with $\|(\varphi(l)-K_n)H\|_{\alpha}\to 0$ as $n\to\infty$ uniformly on the unit ball $\{H\mid\|H\|_{\alpha}\leq 1\}$ of $\mathcal{C}(\alpha)$, where each K_n is a finite sum $\sum\Theta(F_i,G_i)$. Writing H as (h_1,h_2), the element K_nH has the form (ah_1+bh_2,ch_1+dh_2) for some $a,b,c,d\in$ C($\mathbb{T}$) with $a(1)=b(1)=c(1)=d(1)$ and $a(-1)=b(-1)=c(-1)=d(-1)$. Since $\varphi(l)H=(lh_1,\gamma_{\#}(l)h_2)$ we need that both $\|lh_1-(ah_1+bh_2)\|_{\infty}=\|(l-a)h_1-bh_2\|_{\infty}$ and $\left\|\gamma_{\#}(l)h_2-(ch_1+dh_2)\right\|_{\infty}=\left\|(\gamma_{\#}(l)-d)h_2-ch_1\right\|_{\infty}$ are small, say less than ϵ, in C($\mathbb{T}$) . Choosing $h_1=h_2\equiv 1$ and evaluating at 1 and -1 we have that $l(1)$ must be within ϵ of $2a(1)=2b(1)$ and that $l(-1)$ must be within ϵ of $2d(-1)=2b(-1)$. For each $t\in\mathbb{T}\setminus\{1,-1\}$ consider those $H=(h_1,h_2)$ in the unit ball of $\mathcal{C}(\alpha)$ so that the values h_1 and h_2 at t are either 0 and 1 respectively, or 1 and 0. It follows that $l-a$, b, $\gamma_*(l)-d$, and c must all lie within ϵ of 0 on $\mathbb{T}\setminus\{1,-1\}$. Since b is continuous we have that $\|b\|_{\infty}\leq\epsilon$, so $|l(1)|\leq|l(1)-2b(1)|+2|b(1)|\leq 3\epsilon$, and similarly $l(-1)$ is within 3ϵ of 0. Thus $l\in B$.

Conversely, let $l\in B$ be a function with $l(e^{2\pi i\theta})\geq 0$ for $\theta\in[0,\frac{1}{2}]$ and $l(e^{2\pi i\theta})=0$ for $\theta\in[\frac{1}{2},1]$. Set $F=(f_1,f_2)$, $G=(g_1,g_2)$ with $f_1=\sqrt{l}=g_1$ and $f_2=g_2=\gamma_{\#}(f_1)$. Then $f_1\overline{g_2}=f_2\overline{g_1}=0$ and $f_1\overline{g_1}=l$, $f_2\overline{g_2}=\gamma_{\#}(l)$, so $2\Theta(F,G)H=\varphi(l)H$. Thus $\varphi(l)\in\mathcal{K}(\mathcal{C}(\alpha))$, and similarly if $l\in B$ with $l(e^{2\pi i\theta})=0$ for $\theta\in[0,\frac{1}{2}]$ and $l(e^{2\pi i\theta})\geq 0$ for $\theta\in[\frac{1}{2},1]$. Since linear combinations of such elements yield all of B, and since φ is linear, we have the reverse inclusion. ∎

Setting $v\in$ C($\mathbb{T}$) to be the function $v(t)=t,(t\in\mathbb{T})$, define elements $W,V\in$ C(α) by $W=(v,v^*)$ and $V=(v,v)$. The functions W and V together separate the points of α so $Span_{\mathbb{C}}\{W^mV^n\mid m,n\in Z\}$ is norm dense in C(α) with the usual sup norm $\|\ \|_{\infty}$. Since $\|(f,g)\|_{\infty}=\|f\|_{\infty}\vee\|g\|_{\infty}$ and since $\|\ \|_{\alpha}$ is equivalent to $\|\ \|_{\infty}$, this subalgebra is also dense in the Hilbert module $\mathcal{C}(\alpha)$. Letting U $\in$ A = C($\mathbb{T}$) denote the unitary element U$(t)=t,(t\in T)$, the actions of A on $\mathcal{C}(\alpha)$ are described by (W^mV^n)U$=W^mV^{n+1}$ and U$(W^mV^n)=W^{m+1}V^n$. The conditional expectation satisfies $\Psi(W^mV^n)=\frac{1}{2}\mathrm{U}^n(\mathrm{U}^m+\mathrm{U}^{-m})$.

Lemma 2.14 *For* $a,n,r\in\mathbb{Z}$ *the following relationship between adjointable operators on* $\mathcal{C}(\alpha)$ *holds*

$$\Theta\left(W^r,W^{-(r+a)}\right)-\Theta\left(V^n,V^nW^{-a}\right)=\frac{1}{2}\varphi((U^{2r}-1)\,U^a)$$

Proof. For $H=(h_1,h_2)\in\mathcal{C}(\alpha)$ we have $(\Theta\left(W^r,W^{-(r+a)}\right)-\Theta\left(V^n,V^nW^{-a}\right))(h_1,h_2)=W^r\Psi(W^{(r+a)}(h_1,h_2))-V^n\Psi(W^aV^{-n}(h_1,h_2))=W^r\frac{1}{2}(v^{a+r}h_1+v^{-(r+a)}h_2)-V^n\frac{1}{2}(v^{a-n}h_1+v^{-(a+n)}h_2)=(\frac{1}{2}(v^{a+2r}h_1+v^{-a}h_2),\frac{1}{2}(v^ah_1+v^{-(2r+a)}h_2))-(\frac{1}{2}(v^ah_1+v^{-a}h_2),\frac{1}{2}(v^ah_1+v^{-a}h_2))=(\frac{1}{2}(v^{a+2r}-v^a)h_1,\frac{1}{2}(v^{-(2r+a)}-v^{-a})h_2)=\frac{1}{2}\varphi((\mathrm{U}^{2r}-1)\mathrm{U}^a)(h_1,h_2)$. ∎

Thus an ideal isometric covariant representation (T,σ) of the bimodule $\mathcal{C}(\alpha)$ on a Hilbert space $\mathcal{H}$ satisfies

1. $\mathrm{T}(W^mV^n)\sigma(U) = \mathrm{T}(W^mV^{n+1})$
2. $\sigma(U)\mathrm{T}(W^mV^n) = \mathrm{T}(W^{m+1}V^n)$
3. $\mathrm{T}(W^mV^n)^*\mathrm{T}(W^rV^s) = \frac{1}{2}\sigma(\mathrm{U}^{s-n}(\mathrm{U}^{r-m} + \mathrm{U}^{m-r}))$
4. $\mathrm{T}(W^r)\mathrm{T}(W^{-(r+a)})^* - \mathrm{T}(V^n)\mathrm{T}(W^{-a}V^n)^* = \frac{1}{2}\varphi((\mathrm{U}^{2r}-1)\mathrm{U}^a)$

for $m, n, a, r, s \in \mathbb{Z}$.

The third condition shows that $\mathrm{T}(W^mV^n)^*\mathrm{T}(W^mV^n) = \sigma(I) = I_{\mathcal{H}}$, so the $\mathrm{T}(W^mV^n)$ are isometries. The first two relations indicate that $\sigma(U^m)T\sigma(U^n) = \mathrm{T}(W^mV^n)$ where T is the isometry $\mathrm{T}(I)$. Relation 3 then reads $T^*\sigma(\mathrm{U}^a)T = \frac{1}{2}\sigma(\mathrm{U}^a + \mathrm{U}^{-a})$ for $a \in \mathbb{N}$, while the fourth states $\sigma(\mathrm{U}^r)TT^*\sigma(\mathrm{U}^r\mathrm{U}^a) - TT^*\sigma(\mathrm{U}^a) = \frac{1}{2}\sigma((\mathrm{U}^{2r}-1)\mathrm{U}^a)$, or after multiplying by $\sigma(\mathrm{U}^{-a})$ on the left, just $\sigma(\mathrm{U}^r)TT^*\sigma(\mathrm{U}^r) - TT^* = \frac{1}{2}\sigma(\mathrm{U}^{2r}-1)$ for $r \in \mathbb{Z}$. This is equivalent to the single condition $\sigma(\mathrm{U})TT^* - TT^*\sigma(\mathrm{U}^*) = \frac{1}{2}\sigma(\mathrm{U} - \mathrm{U}^*)$ since for r positive the left side $\sigma(\mathrm{U}^r)TT^*\sigma(\mathrm{U}^r) - TT^*$ is the collapsing series $\sum_{k=0}^{r-1}\sigma(\mathrm{U}^k)(\sigma(\mathrm{U})TT^* - TT^*\sigma(\mathrm{U}^*))\sigma(\mathrm{U}^{k+1}) = \sum_{k=0}^{r-1}\sigma(\mathrm{U}^k)\frac{1}{2}(\sigma(\mathrm{U} - \mathrm{U}^*)\sigma(\mathrm{U}^{k+1}) = \frac{1}{2}(\sigma(\mathrm{U}^{2r}-1)$.

One still needs to check that the fourth relation implies the general form of relation 4, namely $\sigma^1 \circ \varphi = \sigma$ on the ideal J of A. Note that both $\sigma^1 \circ \varphi$ and σ are *-homomorphisms, so it is sufficient to show $\sigma^1 \circ \varphi = \sigma$ on a set of generators of the ideal J. Since the two functions $\mathrm{U} - 1$ and $\mathrm{U} - \mathrm{U}^*$ separate the points of $\mathbb{T}$ except for the points ± 1, where both are zero, it is enough to state relation 4) at these two elements. We have just seen that these two conditions are equivalent, so it is enough to impose the condition only, for example, on the element $\mathrm{U} - \mathrm{U}^*$.

Conversely, if T is an isometry and U a unitary on a Hilbert space $\mathcal{H}$ with $T^*U^aT = \frac{1}{2}(U^a + U^{-a})$ for $a \in \mathbb{N}$ and $UTT^* - TT^*U^* = \frac{1}{2}(U - U^*)$, then this gives rise to an isometric covariant representation (T,σ) of $\mathcal{C}(\alpha)$ on $\mathcal{H}$ by setting $\mathrm{T}(W^mV^n) = U^mTU^n$ and $\sigma(\mathrm{U}) = U$. Note that the set $\{W^mV^n \mid m, n \in \mathbb{Z}\}$ is linearly independent in $\mathcal{C}(\alpha)$: this may be shown by considering the inner product $\langle f, g\rangle = \int_{\mathbb{T}} \Psi(f^*g)(t)dt$ on $\mathcal{C}(\alpha)$ and checking that this set is orthonormal. Thus T may be extended by linearly without ambiguity to a dense subspace of $\mathcal{C}(\alpha)$. Conditions 1 and 2 are then true by definition while conditions 3 and 4 follow easily. We have shown the following Theorem.

Theorem 2.15 *The Cuntz-Pimsner C*-algebra associated to the relation* $\alpha = \{(x, y) \mid x \in \mathbb{T}$ *and* $y = x^{\pm 1}\}$ *is the universal C*-algebra generated by an isometry* T *and a unitary* U *satisfying*

$$T^*U^aT = \frac{1}{2}(U^a + U^{-a}) \quad \textit{for} \ \ a \in \mathbb{N} \tag{1}$$

$$UTT^* - TT^*U = \frac{1}{2}(U - U^*). \tag{2}$$

There are representations of this algebra with abelian image, so this algebra is not simple. For example, if U, T are operators satisfying the conditions of Theorem 2.15 and if in addition T is a unitary, then U must be a self-adjoint unitary: $U - U^* = UTT^* - TT^*U = \frac{1}{2}(U - U^*)$, so $U - U^* = 0$. Furthermore U and T commute since $U = \frac{1}{2}(U - U^*) = T^*UT$, so $TU = UT$. Thus $U = I - 2q$ for some projection q commuting with T , and $T = T_1 \oplus T_2$ where $T_1 = Tq$ and $T_2 = T(I - q)$. Conversely, if T and U are commuting unitaries with U self-adjoint then since U^a is either U or I we have the conditions of Theorem 2.15 satisfied. The universal such C*-algebra generated by U

and T is isomorphic to the abelian C*-algebra generated by the projection $q = (I - U)/2$ and T, which is isomorphic to the C*-algebra generated by two commuting unitaries T_1 and T_2, namely $C(\mathbb{T}) \oplus C(\mathbb{T})$.

References

[1] Brenken, B., The Local Product Structure of Expansive Automorphisms of Solenoids and their Associated C*-algebras, Canad. J. Math. 48, (1996), 692 - 709.

[2] Brenken, B. C*-algebras of infinite graphs and Cuntz-Krieger algebras, Canad. Math. Bull.45, (2002), 321-336.

[3] Brenken, B. Endomorphisms of Type I von Neumann algebras with discrete center, J. Operator Theory, to appear.

[4] Choquet, G., Lectures on Analysis, Vol. I., W.A. Benjamin, Reading, Massachusetts, 1969.

[5] Davidson, K., C*-algebras by Example, Fields Institute Monographs, Amer. Math. Soc. Providence, Rhode Island, 1996.

[6] Deaconu, V., Generalized Solenoids and C*-algebras, Pacific J. Math.190, (1999), 247 - 260.

[7] Deaconu, V., Muhly, P.S., C*-algebras associated with branched coverings, Proc.Amer. MAth. Soc. 129, (2001),1077 - 1086.

[8] Fowler, N., Laca, M., Raeburn, I., The C*-algebras of Infinite Graphs, Proc. Amer. Math. Soc. 128, (2000), 2319 - 2327.

[9] Katsura, K., A class of C*-algebras generalizing both graph algebras and homeomorphism C*-algebras I, fundamental results, preprint.

[10] Kumjian, A., Pask, D., Raeburn, I., Renault, J., Graphs, Groupoids, and Cuntz-Krieger algebras. J. Funct. Anal. 144, (1997), 505 - 541.

[11] Kumjian, A. On certain Cuntz-Pimsner algebras, preprint.

[12] Lance, E. C., Hilbert C*-modules-A toolkit for operator algebraists, London Math. Soc. Lecture Note Series, Vol. 210, Cambridge University Press, Cambridge, 1995.

[13] Pimsner, M., A class of C*-algebras generalizing both Cuntz - Krieger algebras and crossed products by $\mathbb{Z}$, Fields Institute Communications 12 (1997), 189 - 212.

[14] Muhly, P.S., and Solel, B. On the simplicity of some Cuntz-Pimsner algebras, Math. Scand. 83 (1998), 53 - 73.

[15] Muhly, P.S., and Solel, B., On the Morita Equivalence of Tensor Algebras, Proc. London Math. Soc., 81, (2000), 361 - 377.

[16] Rordam, M., Classification of Nuclear C*-algebras, Encyclopaedia Math. Sci. Vol 126, Springer Verlag, Berlin, Heidelberg 2002.

[17] Schweizer, J., Crossed Products by C*-Correspondences and Cuntz-Pimsner Algebras, in 'C*-algebras, Proceedings of the SFB-Workshop on C*-algebras, Muenster, 1999', J. Cuntz, S. Echterhoff (Eds.), Springer Verlag, Berlin, 2000.

[18] Solel, B., Operator algebras over C*-correspondences, in Operator Algebras and Applications (Samos, 1996), NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci., Vol. 495, pp. 429 - 448, Kluwer Acad. Publ., Dordrecht, 1997.

[19] Stacey, P. J., Crossed products of C*-algebras by *-Endomorphisms, J. Australian Math. Soc., Ser. A 54 (1993), 204 - 212.

Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N 1N4
e-mail: bbrenken@ucalgary.ca